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On Covers[1]

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In this note a sufficient condition for the existence of flat covers of modules is given. In a special case when flat modules have "enough" flat submodules (especially if the class of flat modules is hereditary), the necessity of the condition is also proved. All results are proved in a more general setting.

In what follows R stands for an associative ring with identity and by the word "module" we shall always mean a unital left R-module. Dualizing the notion of the injective envelope of a module [ES], H. Bass [B] investigated the projective cover of a module and he characterized the class of rings R over which every module has a projective cover. By a projective cover of a module M we mean an epimorphism $\varphi : F \to M$ with F projective and such that the kernel K of φ is superfluous in F in the sense that $K + L = F$ implies $L = F$ whenever L is a submodule of F. Recently, the general theory of covers has been studied intensively. If $\mathscr{G}$ is an abstract class of modules (i.e., $\mathscr{G}$ is closed under isomorphic copies) then a

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homomorphism $\varphi : G \to M$ with $G \in \mathcal{G}$ is called a *$\mathcal{G}$-precover* of the module M if for each homomorphism $f : F \to M$ with $F \in \mathcal{G}$ there is $g : F \to G$ such that $\varphi g = f$. A $\mathcal{G}$-precover of M is said to be a *$\mathcal{G}$-cover* if every endomorphism f of G with $\varphi f = \varphi$ is the automorphism of G. It is well known (see, e.g., [Xu]) that an epimorphism $\varphi : F \to M, F$ projective, is a projective cover of the module M if and only if it is a $\mathcal{P}$-cover of M, where $\mathcal{P}$ denotes the class of all projective modules. Denoting by $\mathcal{F}$ the class of all flat modules, Enoch's conjecture [E], whether every module over any associative ring has an $\mathcal{F}$-cover, remains still open.

In this note we shall show that the condition (P) below ensuring the existence of "enough" pure submodules in a flat module is sufficient for the existence of flat covers. As for the converse we shall prove only a partial result assuming the "ubiquity" of flat submodules of flat modules (the condition is satisfied especially if the class of flat modules is hereditary). All the results will be reformulated in the general setting working with the abstract class of modules $\mathcal{G}$.

As usual, λ^+ denotes the successor of the cardinal λ and $|M|$ denotes the cardinality of the set M. Setting $\mu_0 = \max(|R|, \aleph_0)$, we take any complete co-abstract set $\mathcal{K}$ of modules M with $|M| \le \mu_0^+$ and we define the cardinal μ as the first one such that $\mu > |E(M)|$ for each $M \in \mathcal{K}$ ($E(M)$ is the injective envelope of M). Recall that the *co-abstractness* of $\mathcal{K}$ means that the members of $\mathcal{K}$ are pairwise non-isomorphic, while the *completeness* means that every module M with $|M| \le \mu_0^+$ is isomorphic to a (unique) member of the set $\mathcal{K}$. By an *abstract class* $\mathcal{G}$ of modules we mean any class of modules closed under isomorphic images. A submodule N of the module M is called *$\mathcal{G}$-pure in M*, if the factor-module M/N belongs to $\mathcal{G}$. Further, the class $\mathcal{G}$ is called *inductive* if it is closed under unions of chains.

We start with the formulation of the purity condition and then we proceed to the sufficiency of this condition for the existence of some precovers.

DEFINITION 1. Let $\mathcal{G}$ be an abstract class of modules. We say that $\mathcal{G}$ satisfies the *condition* (P) if to each infinite cardinal λ there exists a cardinal $\kappa > \lambda$ such that for every $F \in \mathcal{G}$ with $|F| \ge \kappa$ and every $K \le F$ with $|F/K| \le \lambda$, the submodule K contains a non-zero $\mathcal{G}$-pure submodule of F.

THEOREM 2. *Let $\mathcal{G}$ be an abstract class of modules closed under arbitrary direct sums. If $\mathcal{G}$ satisfies the condition (P) and for each $F \in \mathcal{G}$ the set of all $\mathcal{G}$-pure submodules of F is inductive, then every module has a $\mathcal{G}$-precover.*

Proof. Let M be an arbitrary module, $\lambda = \max(|M|, \aleph_0)$ and let $\mathcal{G}_M$ be any complete co-abstract subset of $\mathcal{G}$ consisting of modules of cardinalities less than κ. If $\{G_\alpha \mid \alpha \in A\}$ is any list of all the members of $\mathcal{G}_M$, then for each homomorphism $g \in \mathrm{Hom}(G_\alpha, M)$ we take an isomorphic copy $G_{\alpha g}$ of G_α together with the isomorphism $\psi_{\alpha g} : G_\alpha \to G_{\alpha g}$. We are going to show that the module

$$G = G_M = \bigoplus_{\alpha \in A} \left(\bigoplus_{g \in \mathrm{Hom}(G_\alpha, M)} G_{\alpha g} \right)$$

together with the natural evaluation map $\varphi = \varphi_M : G \to M$ induced by the homomorphisms $g\psi_{\alpha g}^{-1} : G_{\alpha g} \to M$ is a $\mathcal{G}$-precover of the module M.

So, let $F \in \mathcal{G}$ and $f : F \to M$ be arbitrary. By hypothesis, there is a maximal $\mathcal{G}$-pure submodule K of F contained in $\mathrm{Ker}\, f$. We denote $\bar{F} = F/K$, $\pi : F \to F/K$ the canonical projection, and $\bar{f} : \bar{F} \to M$ the unique homomorphism such that $\bar{f}\pi = f$. Now if $\bar{H} = H/K \subseteq \mathrm{Ker}\, \bar{f}$ is $\mathcal{G}$-pure in $\bar{F}$, then $H \subseteq \mathrm{Ker}\, f$ and H is $\mathcal{G}$-pure in F since the factor-module $F/H \cong F/K/H/K$ belongs to the class $\mathcal{G}$. So, the maximality of K yields that $\bar{F}$ has no non-zero $\mathcal{G}$-pure submodule contained in $\mathrm{Ker}\, \bar{f}$. Moreover, $|\bar{F}/\mathrm{Ker}\, \bar{f}| = |F/\mathrm{Ker}\, f| = |\mathrm{Im}\, f| \le \lambda$ and so the condition (P) yields that $|\bar{F}| < \kappa$. Hence there is an index $\alpha \in A$ and an isomorphism $\varphi_\alpha : \bar{F} \to G_\alpha$ and if we denote $\iota_{\alpha g} : G_{\alpha g} \to G$ the canonical embedding of $G_{\alpha g}$, $g : G_\alpha \to M$, into G, then for $h = \iota_{\alpha, \bar{f}\varphi_\alpha^{-1}} \psi_{\alpha, \bar{f}\varphi_\alpha^{-1}} \varphi_\alpha \pi : F \to G$ we have $\varphi h = \varphi \iota_{\alpha, \bar{f}\varphi_\alpha^{-1}} \psi_{\alpha, \bar{f}\varphi_\alpha^{-1}} \varphi_\alpha \pi = \bar{f}\varphi_\alpha^{-1} \psi_{\alpha, \bar{f}\varphi_\alpha^{-1}}^{-1} \psi_{\alpha, \bar{f}\varphi_\alpha^{-1}} \varphi_\alpha \pi = \bar{f}\pi = f$ and we are through. ∎

Corollary 3. *If, in addition, the class $\mathcal{G}$ in Theorem 2 is closed under direct limits, then every module has a $\mathcal{G}$-cover.*

Proof. Every module has a $\mathcal{G}$-precover by Theorem 2 and [Xu, Theorem 2.2.8] applies. ∎

Theorem 4. *If the class $\mathcal{F}$ of all flat modules satisfies the condition (P), then every module has a flat cover.*

Proof. It is well known (see, e.g., [Xu]) that all the hypotheses of the preceding corollary are satisfied. ∎

To show that the cardinality condition of Theorem 1 is meaningful we are going to prove a partial converse of Theorem 1 and to point out several situations in which this condition is satisfied. One of the natural requirements is that the class $\mathcal{G}$ is hereditary. However, the following weak version is also good enough.

Definition 5. Let $\mathcal{G}$ be an abstract class of modules. We say that a module F is *$\mathcal{G}$-prehereditary* if for every submodule $K \le F$ with $|K| > \mu_0$ and every submodule L of K with $|L| < |K|$ there is a submodule H of F such that $L \le H \le K$, $H \in \mathcal{G}$, and $|H| < |K|$. The class $\mathcal{G}$ itself is said to be *$\mathcal{G}$-prehereditary* if every module $F \in \mathcal{G}$ is $\mathcal{G}$-prehereditary.

LEMMA 6. *Let K be a submodule of a module F such that $|F| \geq \mu$ and $|F/K| \leq \mu_0$. Then K contains a submodule L such that $\mu_0 < |F/L| < \mu$ and consequently $|L| = |K| = |F|$.*

Proof. Let N be any submodule of K of the cardinality μ_0^+ and let L be any submodule of K maximal with respect to $L \cap N = 0$ (N-high submodule of K). It is well known that the submodule $\frac{L \oplus N}{L}$ is essential in K/L. Hence $|\frac{L \oplus N}{L}| = |N| = \mu_0^+$ yields $\mu_0^+ \leq |K/L| < \mu$. Looking at the short exact sequence $0 \to K/L \to F/L \to F/K \to 0$ we see that $|K/L| = |F/L|$ and the rest is obvious. ▮

THEOREM 7. *Let $\mathcal{G}$ be a $\mathcal{G}$-prehereditary class of modules closed under extensions. Assume further that the class $\mathcal{G}$ is inductive and that for each $F \in \mathcal{G}$ the set of all $\mathcal{G}$-pure submodules of F is also inductive. If for each $F \in \mathcal{G}$ every submodule K of F with $|K| \geq \mu_0$ is contained in a $\mathcal{G}$-pure submodule of F of the same size and every module has a $\mathcal{G}$-precover, then $\mathcal{G}$ satisfies the condition (P).*

Proof. Let λ be an arbitrary infinite cardinal. Obviously, we may assume that $\lambda \geq \mu$ and we can take any complete co-abstract set $\mathcal{K}$ of modules of cardinalities at most λ; to each member of this set we fix a $\mathcal{G}$-precover and we take κ to be the first cardinal greater than λ and all the cardinalities of all $\mathcal{G}$-precovers just mentioned.

Now let $F \in \mathcal{G}$ and $K \leq F$ be such that $|F| \geq \kappa$ and $|F/K| \leq \lambda$. If $|F/K| \leq \mu_0$ then Lemma 6 yields the existence of a submodule L of K such that $\mu_0 < |F/L| < \mu \leq \lambda$ and consequently we may without loss of generality assume that $\mu_0 < |F/K| < \lambda$. There is an $M \in \mathcal{K}$, an isomorphism $\psi : M \to F/K$, and a $\mathcal{G}$-precover $\varphi_0 : G \to M$ fixed above. Then $|G| < \kappa$, $\varphi : G \to F/K$ with $\varphi = \psi\varphi_0$, is obviously a $\mathcal{G}$-precover of F/K and we can consider the following commutative diagram

$$\begin{array}{ccc} F & = & F \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi} \\ G & \xrightarrow{\varphi} & F/K \end{array}$$

where π is the natural projection and g a homomorphism making the diagram commutative, the existence of which follows from the definition of a $\mathcal{G}$-precover. Obviously, $\operatorname{Ker} g \leq K$ and the inequalities $|F| \geq \kappa$ and $|F/\operatorname{Ker} g| = |\operatorname{Im} g| \leq |G| < \kappa$ yield that $\operatorname{Ker} g \neq 0$. Thus $\mu_0 < |F/K| \leq |\operatorname{Im} g|$ and since for an arbitrary element $0 \neq u \in \operatorname{Im} g$ we have $|Ru| \leq \mu_0$, there is a submodule B_0 of $\operatorname{Im} g$ such that $Ru \leq B_0$, $B_0 \in \mathcal{G}$, and $|B_0| < |\operatorname{Im} g|$, the class $\mathcal{G}$ being $\mathcal{G}$-prehereditary by the hypothesis.

For each $u \in \operatorname{Im} g$ we select an element $x_u \in F$ with $g(x_u) = u$; we denote by $D^- = \{x_u \mid u \in \operatorname{Im} g\}$ and by D the submodule of F generated by D^-. Obviously, $|D| = |\operatorname{Im} g| \leq |G| < \kappa$. Denoting $C_0^- = D^- \cap g^{-1}(B_0)$

and taking $0 \neq v \in \operatorname{Ker} g$ arbitrarily, the hypothesis yields the existence of a $\mathscr{G}$-pure submodule C_0 of F containing $C_0^- \cup \{v\}$ and of the same size as B_0. Since the class $\mathscr{G}$ is $\mathscr{G}$-prehereditary, there is a submodule A_0 of F such that $C_0 \leq A_0 \leq D + C_0$, $A_0 \in \mathscr{G}$, and $|A_0| < |D + C_0| = |\operatorname{Im} g|$. Continuing by induction, let us suppose that for some $k \geq 0$ the submodules $C_0 \leq A_0 \leq \cdots \leq C_k \leq A_k$ of F and the submodules $B_0 \leq B_1 \leq \cdots \leq B_k$ of $\operatorname{Im} g$ have been already constructed in such a way that $B_i \subseteq g(C_i)$ and $|B_i| = |C_i| < |\operatorname{Im} g|$, $|A_i| < |\operatorname{Im} g|$ for every $i = 0, 1, \ldots, k$, and $g(A_i) \subseteq B_{i+1}$ for every $i = 0, 1, \ldots, k-1$. Moreover, the modules $B_0, B_1, \ldots, B_k$ and $A_0, A_1, \ldots, A_k$ belong to the class $\mathscr{G}$ and the modules $C_0, C_1, \ldots, C_k$ are $\mathscr{G}$-pure in F. Since $\mathscr{G}$ is $\mathscr{G}$-prehereditary and $|g(A_k)| < |\operatorname{Im} g|$, there exists a submodule B_{k+1} of $\operatorname{Im} g$ belonging to $\mathscr{G}$, containing $g(A_k)$ and such that $|A_k| \leq |B_{k+1}| < |\operatorname{Im} g|$. Setting $C_{k+1}^- = D^- \cap g^{-1}(B_{k+1})$, the hypothesis yields the existence of a $\mathscr{G}$-pure submodule C_{k+1} of F containing $C_{k+1}^- \cup A_k$ and of the same size as B_{k+1}. Since $\mathscr{G}$ is $\mathscr{G}$-prehereditary, there is a submodule A_{k+1} of F belonging to $\mathscr{G}$ and such that $C_{k+1} \leq A_{k+1} \leq D + C_{k+1}$, $A_{k+1} \in \mathscr{G}$, $|A_{k+1}| < |D + C_{k+1}| = |\operatorname{Im} g|$. By hypotheses, $A = \bigcup_{k=0}^{\infty} A_k = \bigcup_{k=0}^{\infty} C_k$ lies in $\mathscr{G}$, it is $\mathscr{G}$-pure in F, and $B = \bigcup_{k=0}^{\infty} B_k \subseteq \operatorname{Im} g$ lies in the class $\mathscr{G}$. Now $\frac{A}{A \cap \operatorname{Ker} g} \cong g(A) = B \in \mathscr{G}$, hence $A \cap \operatorname{Ker} g$ is $\mathscr{G}$-pure in A and consequently in F, the class $\mathscr{G}$ being closed under extensions. Finally, $A \cap \operatorname{Ker} g \leq \operatorname{Ker} g \leq K$, $0 \neq v \in A \cap \operatorname{Ker} g$, and the proof is complete. ∎

Recall that a hereditary torsion theory $\sigma = (\mathscr{T}, \mathscr{F})$ for the category R-mod consists of two abstract classes $\mathscr{T}$ and $\mathscr{F}$, the σ-torsion class and the σ-torsion-free class, respectively, such that $\operatorname{Hom}(T, F) = 0$ whenever $T \in \mathscr{T}$ and $F \in \mathscr{F}$, the class $\mathscr{T}$ is closed under submodules, factor modules, extensions, and direct sums, the class $\mathscr{F}$ is closed under submodules, extensions, and direct products, and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathscr{T}$ and $F \in \mathscr{F}$. To each hereditary torsion theory it is associated a radical filter of left ideals $\mathscr{L} = \{I \leq R \mid R/I \in \mathscr{T}\}$ and the torsion part $\sigma(M) = T$ of the module M consists of all elements $a \in M$ with $(0 : a) \in \mathscr{L}$. The torsion theory σ is said to be of *finite type* if the filter $\mathscr{L}$ contains a cofinal subset of finitely generated left ideals.

COROLLARY 8. *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory of finite type for the category* R-mod, *then the class $\mathscr{F}$ satisfies the condition* (P).

Proof. The class $\mathscr{F}$ is hereditary, inductive, and closed under extensions. Since σ is of finite type, the $\mathscr{F}$-pure submodules of members of $\mathscr{F}$ are inductive and every module has an $\mathscr{F}$-precover by the theorem of Teply [T]. To finish the proof it remains to show that every submodule K of a σ-torsion-free module F with $|K| \geq \mu_0$ is contained in an $\mathscr{F}$-pure submodule L of F of the same size as K. Denoting $\sigma(F/K) = L/K$, the submodule L is obviously $\mathscr{F}$-pure in F and it remains to show that $|L| = |K|$. However, for each $x \in L \setminus K$ there exists a finitely generated left ideal $I = Ra_1 + Ra_2 + \cdots + Ra_n \in \mathscr{L}$ with $Ix \subseteq K$. If we associate to the element x the finite subset $\{a_1 x, \ldots, a_n x\} \subseteq K$, then whenever $a_1 x = a_1 y, \ldots, a_n x = a_n y$ for some $x, y \in L \setminus K$ we have $I(x - y) = 0$ and consequently $x - y \in \sigma(F) = 0$ yields $x = y$. From this we immediately see that $|L| = |K|$ and, as remarked above, an application of Theorem 7 finishes the proof. ∎

COROLLARY 9. *The class $\mathscr{F}$ of all torsion-free abelian groups satisfies the condition (P).*

Proof. This is by Corollary 8. ∎

Remark 10. The preceding corollary says that if λ is a given infinite cardinal, then there is a cardinal $\kappa > \lambda$ such that for any torsion-free abelian group F with $|F| \geq \kappa$ and any of its subgroup K with $|F/K| \leq \lambda$ the subgroup K contains a non-zero subgroup pure in F. This fact follows from the above theory of precovers and as far as we know, the first direct proof has been done by the first author in [Bi1] in the case when F/K is a p-group and in [Bi2] in the general case.

Note added in proof. The Flat Cover Conjecture has been solved independently by E. Enochs, R. El Bashir, and L. Bican; cf. "All modules have flat covers," preprint.

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