The Conjugate Classes of Chevalley Groups of Type $(G_2)^*$

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Communicated by Nathan Jacobson

Received July 25, 1967; revised August 30, 1967

1. Preliminary

In this paper we determine the conjugate classes of Chevalley groups of type $(G_2)$ over finite fields of characteristic $\neq 2, 3$. The definition and properties of such groups are given in Chevalley [1] and Ree [5]. A matrix representation of these groups can be found in [4]. The main tools of our investigation are the properties of Chevalley groups in general ([1] III), those of groups of type $(G_2)$ ([5] Sections 2 and 3), a theorem of Lang [3], and Sylow's theorem.

Let $G$ be a Chevalley group over a finite field $K$. Denote by $K^*$ the multiplicative group of $K$. We shall use the notation $x_r(t)$, $\varphi_r$, $x_r$, $U$, $S$ $W$, $U_w^r$ as defined in [1]. In determining the conjugate classes of $G$, one has to solve various equations of type $g^{-1}xg = y$ in $G$. The main tools for solving such equations are the fundamental properties (1.1) and (1.2) of $G$, according to which elements of $G$ are written and compared, and the conjugation rules (1.3)–(1.6), (1.7)–(1.9) are also useful. These are consequences of Theorem 2 and related lemmas in [1].

(1.1) Every element $x$ of $G$ is written uniquely in the form $x = uh\omega(w)v$, where $u \in U$, $h \in S$, and $v \in U_w^r$. (We shall call $uh\omega(w)v$ the Bruhat factorization of $x$.)

(1.2) Every element $u$ of $U$ can be written uniquely in the form $u = \prod x_r(t)$, where the product is taken over the positive roots in increasing (or in any fixed) order.

(1.3) $x_r(t)x_s(u)x_r(t)^{-1} = x_r(u) \prod_{i,j} x_{ir+js}(C_{i,j}; r, s t u^{i}),$

$(i, j > 0, r, s > 0).$

* A part of this work was done while the author was a fellow of the Summer Research Institute of the Canadian Mathematical Congress in 1966.
(1.4) \[ h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(t)). \]

(1.5) \[ \omega_r x_r(t) \omega_r^{-1} = x_{w_r(s)}(\eta_{r,s,t}). \]

(1.6) \[ \omega(w)h(\chi)\omega(w)^{-1} = h(\chi'), \quad (\chi'(s) = \chi(w^{-1}(s))). \]

(1.7) Let \( u = \prod x_r(t_r) \) be an element of \( \Pi \), where the product is taken in a fixed order of the positive roots \( r \). If \( \omega(w)^{-1}u\omega(w) \in \Pi \), then \( t_r = 0 \) for all \( r \) such that \( w(r) < 0 \).

(1.8) Let

\[ x = x_r(t_r) \prod_{s > r} x_s(t_s) \in \Pi, \quad t_r \neq 0, \]

and

\[ x' = x_r(u_r') \prod_{s' > r'} x_s(u_s') \in \Pi, \quad u_r' \neq 0. \]

If \( \omega(w)^{-1}x\omega(w) = x' \), then \( w(r) \geq r' \) and \( w^{-1}(r') \geq r \).

(1.9) If \( g^{-1}h_1g = h_2 \) holds for \( h_1, h_2 \in \mathfrak{H} \) and for some \( g \in G \), then there exists \( w \in W \) such that \( \omega(w)^{-1}h_1\omega(w) = h_2 \).

Let \( G_i \) be the Chevalley group obtained from \( G \) by extending the ground field \( K \) to its extension field \( K_i \) of degree \( i \) in which \( G \) is imbedded in natural way ([1], pp. 45, 46). Denote the Cartan subgroup of \( G_i \) by \( \mathfrak{H}_i \). (Notations that are independent of the ground field, such as \( x_r(\ ) \), \( h(\ ) \), \( \varphi_r \), will be used without modification when \( K \) is extended to \( K_i \).) For each \( x \in G_i \) denote by \( x^{(q)} \) the image of \( x \) under the field automorphism of \( G_i \) induced by the automorphism: \( t \to t^q \) of \( K_i \). As a consequence of a theorem of Lang (Corollary of Theorem 1 in [2]), we have the following theorem due to Ree:

**Theorem (1.10).** Let \( \Gamma(\omega) = \{ x \in G_i \mid x = \omega^{-1}x^{(q)}(\omega) \} \), where \( \omega = \omega(w) \) for some \( w \in W \) and \( i \) is the order of \( \omega \). Then \( \Gamma(\omega) \cong G \).

Indeed, the theorem of Lang implies that \( \omega^{-1} = \zeta^{-1} x^{(q)}(\omega) \) for some \( \zeta \in G_i \). Then the inner automorphism \( y \to \zeta^{-1} y \zeta \) of \( G_i \) gives an isomorphism of \( G \) onto \( \Gamma(\omega) \).

Now, let \( G \) be the group of type \( (G_2) \) over \( K \), and assume that \( p \neq 2, 3 \). Then

\[ | G | = q^8(q^2 - 1)(q^6 - 1) = q^6(q^2 - 1)^2(q^2 + q + 1)(q^2 - q + 1), \]

\[ | \Pi | = q^6, \quad | \mathfrak{H} | = (q - 1)^6. \]
(We use the symbol $|X|$ to denote the order of a group or a group element $X$).

If $p_0$ is a prime divisor of $G$ and $p_0 \neq p$, $2$, $3$, then $p_0 \mid |S_6|$ but $p_0 \not\mid |G_6/||S_6|$, and hence $S_6$ contains a Sylow $p_0$-group of $G_6$. Therefore a $p_0$-element $x$ of $G$ is conjugate, in $G_6$, to an element $h \in S_6$. If $x = g^{-1}h_g, g \in S_6$, then $x = x^{(a)} = g^{-1}h_ag^{(a)}$, and $h$ is conjugate to $h^a$ in $G_6$. Then, from (1.9), $h = \omega^{-1}h^a\omega$ for some $\omega \in \langle \omega(w) \rangle$ and $h \in \Gamma(\omega)$. Let $S_6(\omega) = \langle h \in S_6 \mid h = \omega^{-1}h^a\omega \rangle$, then $S_6(\omega) \subseteq \Gamma(\omega)$ and (1.10) show that $G$ contains a subgroup isomorphic to $S_6(\omega)$. We shall see in Section 2 the existence of $\omega_i \in \langle \omega(w) \rangle$, $i = 2$, $3$, $6$, such that $|S_6(\omega_i)| = (q + 1)^2$, $|S_6(\omega_3)| = q^2 + q + 1$, and $|S_6(\omega_6)| = q^2 - q + 1$. Let $S_6(\omega_i), i = 2$, $3$, $6$, be fixed subgroups of $G$ isomorphic to $S_6(\omega_i)$. If $p_0 \mid |G|, p_0 \neq p, 2, 3$, then $p_0$ divides one and only one of $|S_6|$, $|S(\omega_2)|$, $|S(\omega_3)|$, or $|S(\omega_6)|$. Hence $S_6$ or one of $S(\omega_i)$, contains a Sylow $p_0$-group. Therefore every $p_0$-element of $G$ is conjugate to an element $x \in S_6$ or $S(\omega_i)$, and every $p_0$-singular element (i.e., an element whose order is divisible by $p_0$) is conjugate to an element in the centralizer $C(x)$ of $x \in S_6$ or $S(\omega_i)$.

Thus our problem breaks down naturally into two parts: (i) to determine the conjugate classes of $p$-elements by investigating $U$; (ii) to determine the conjugate classes of $p_0$-singular elements ($p_0 \neq p$) by investigating the centralizers of elements of $S_6$.

The following notation will be used throughout. $[x, y] = x^{-1}y^{-1}xy, x \sim y$: $x$ is conjugate to $y$, $C(x)$ — the centralizer of $x$ in $G$, $C_X(x)$ is the centralizer of $x$ in $X, \{x \mid \cdots\}$ is the set of $x$, ..., $\langle \cdots \rangle$ is the group generated by ..., and $K(x)$ is the conjugate class of $G$ containing $x$.

2. PROPERTIES OF GROUPS OF TYPE $(G_2)$

As in [5], let $\Sigma = \{\pm \xi_i, \xi_i - \xi_j \mid 1 \leq i, j \leq 3, i \neq j\}$ (where $\xi_1 + \xi_2 + \xi_3 = 0$) be the root system of type $(G_2)$, and choose $a = \xi_2$, $b = \xi_1 - \xi_2$ for a fundamental system of roots. Thus the set $\Sigma^+$ of positive roots arranged in increasing order are: $a = \xi_2, b = \xi_1 - \xi_2, a + b = \xi_1, 2a + b = -\xi_3, 3a + b = \xi_2 - \xi_3, \text{ and } 3a + 2b = \xi_1 - \xi_3$. We shall use both notations for roots. From [5] we quote the following commutator relations:

$$
[x_t(a), x_u(b)] = x_{a+b}(-t^2u) x_{3a+2b}(3tu^2),
[x_t(a), x_{a+b}(u)] = x_{2a+b}(3tu),
[x_t(a), x_{3a+b}(u)] = x_{3a+2b}(2tu),
[x_t(a+b), x_{a+b}(u)] = x_{3a+b}(3tu),
[x_{a+b}(t), x_{a+b}(u)] = 1,
\text{for all other pairs of } r, s \in \Sigma^+.
$$

(2.1)
Let $\chi$ be the homomorphism of the additive group generated by the roots into the multiplicative group $K^*$ such that $\chi(\xi_i) = x_i$, $i = 1, 2, 3$. The element $h(\chi)$ of the Cartan subgroup $H$ corresponding to $\chi$ will be denoted by $h(x_1, x_2, x_3)$. Then $x_1x_2x_3 = 1$, and

$$h(x_1, x_2, x_3)^{-1}x_{\xi_i-\xi_i}(t)h(x_1, x_2, x_3) = x_{\xi_i-\xi_i}(x_i^{-1}x_it),$$

(2.2)

$$h(x_1, x_2, x_3)^{-1}x_{\xi_3}(t)h(x_1, x_2, x_3) = x_{\xi_3}(x_3^{-1}t).$$

We have

$$\varphi_{\xi_1-\xi_2} \begin{pmatrix} 0 & 0 \\ z & z^{-1} \end{pmatrix} = h(z, z^{-1}, 1), \quad \varphi_{\xi_2} \begin{pmatrix} 0 & 0 \\ z & z^{-1} \end{pmatrix} = h(z^{-1}, z^{-1}, z^0),$$

etc.

The Weyl group $W$ of $\Sigma$ consists of 6 reflections $w_r : r \rightarrow -r$, $s \rightarrow s$, where $s$ is a root perpendicular to $r$, and 6 rotations generated by $w_6 : \xi_1 \rightarrow -\xi_2 \rightarrow -\xi_3 \rightarrow \cdots$. The rotations $w_6^2$ and $w_6^4$ will be denoted by $w_2$ and $w_8$ respectively. Thus $w_2 : r \rightarrow -r$ for all $r \in \Sigma$, and $w_3 : \xi_1 \rightarrow \xi_2 \rightarrow \xi_3$. Let $w_r = \varphi_{\xi_i}(-1, 0), \; r \in \Sigma$, $w_2 = w_{\xi_1-\xi_2}w_{\xi_2}, \; w_3 = w_{\xi_3}w_{\xi_1}w_{\xi_1}^{-1}$ and $w_6 = w_{66}w_{33}$. Then there exists a homomorphism of $\langle w_r \mid r \in \Sigma \rangle$ onto $W$ that sends $w_r$ into $w_r$, $r \in \Sigma$, and $w_i$ into $w_i$, $i = 2, 3, 6$, with the kernel $\langle \varphi_{\xi_1}(-1, 0) \mid r \in \Sigma \rangle$.

We shall fix a preimage $\omega(w) \in \langle \omega_r \rangle$ of each $w \in W$ as follows: For the reflections $\omega(w_r) = \omega_r$, where $r = \xi_1, -\xi_2, -\xi_3, -\xi_1, -\xi_2, -\xi_3$, and for the rotations $\omega(w_6^i) = w_6^i$. The action of $\omega(w)$ on $x_r(t)$ and on $H$ are as follows:

$$\omega(w_r)x_r(t)\omega(w_r)^{-1} = x_{\omega_r}(\eta_{r,s}t), \quad (\eta_{r,s} = \pm 1)$$

$$\omega_2^{-1}x_2(t)\omega_2 = x_{-s}(t),$$

$$\omega_3^{-1}x_3(t)\omega_3 = x_{w_3}(t).$$

(The above constants $\eta_{r,s}$ coincide with those given in [5], p. 442.)

$$\omega_{\xi_1-\xi_2}h(x_1, x_2, x_3)\omega_{\xi_1-\xi_1} = h(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}).$$

$$\omega_{\xi_3}^{-1}h(x_1, x_2, x_3)\omega_{\xi_3} = h(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}).$$

$$[[ijk]] = (123), \pi = (ij)],$$

$$\omega_2^{-1}h(x_1, x_2, x_3)\omega_2 = h(x_1^{-1}, x_3^{-1}, x_2^{-1}),$$

$$\omega_3^{-1}h(x_1, x_2, x_3)\omega_3 = h(x_3, x_1, x_2).$$

For each $r \in \Sigma$, denote by $\Phi_r$ (or $\Phi_{r,s}$) the image of the group $SL(2, K)$.
[or \(SL(2, K_2)\)] under \(\varphi_r\), and by \(\Phi_s\) the intersection of \(\tilde{\Phi}\) and \(\Phi_r\). The following relations will be used repeatedly:

\[
[\Phi_{\xi_1 - \xi_2}, \Phi_{\xi_3}] = 1,
\]

\[
\Phi_{\xi_1 - \xi_2} \cap \Phi_{\xi_3} = \langle h(-1, -1, 1) \rangle.
\]

(2.5)

The subset \(\{\xi_i - \xi_j \mid 1 \leq i, j \leq 3\}\) of \(\Sigma\) forms a root system of type \((A_3)\) and there exists an isomorphism \(\psi\) of \(SL(3, K)\) into \(G\) such that

\[
\psi \begin{pmatrix} t_1 & t_2 & 0 \\ t_3 & t_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \varphi_{\xi_1 - \xi_2} \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix},
\]

\[
\psi \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_1 & t_2 \\ 0 & t_3 & t_4 \end{pmatrix} = \varphi_{\xi_2 - \xi_3} \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}.
\]

(2.6)

The image under \(\psi\) of \(SL(3, K)\) [or \(SL(3, K_2)\)] in \(G\) (or \(G_2\)) will be denoted by \(\Psi\) (or \(\Psi_2\)). We note that \(\Psi\) is normalized by \(\omega_6\), and that

\[
\omega_2^{-1} \psi(T) \omega_2 = \psi((T^t)^{-1}),
\]

(2.7)

where \(T^t\) is the transpose of \(T\).

Let \(I(\omega) (\omega \in \{\omega(\omega)\}, \mid \omega \mid = i)\) be the subgroup of \(G_i\) as defined in (1.10) and \(\tilde{\Phi}(\omega) = I(\omega) \cap \tilde{\Phi}_i\). Then (2.4)-(2.6) yield the following:

\[
\tilde{\Phi}(\omega_2) = \{h(x_1, x_2, x_3) \mid x_1^{1+q} = 1\},
\]

\[
\tilde{\Phi}(\omega_3) = \{h(x, x^q, x^{2^q}) \mid x^{1+q+q^2} = 1\},
\]

\[
\tilde{\Phi}(\omega_6) = \{h(x, x^{-q}, x^{2^q}) \mid x^{1-q+q^2} = 1\},
\]

\[
\tilde{\Phi}(\omega_{\xi_1 - \xi_2}) = \{h(x, x^q, x^{-q-1}) \mid x^{2^{q-1}} = 1\},
\]

\[
\tilde{\Phi}(\omega_{\xi_2 - \xi_3}) = \{h(x, x^{-q}, x^{2^{q-1}}) \mid x^{2^{2q-1}} = 1\},
\]

(2.7)

\[
\Phi_{r,2} \cap I(\omega_2) = \left\{ \varphi_r \begin{pmatrix} t_1 & t_2 \\ -t_2 & t_1 \end{pmatrix} \right\} \left| t_1^{1+q} + t_2^{1+q} = 1 \right\} \simeq U(2, K_2),
\]

\[
\Psi_2 \cap I(\omega_2) \simeq U(3, K_2).
\]

Here \(U(i, K_2)\) denote the unitary group of degree \(i\) over \(K_2\).
THEOREM (2.8). The group $G$ contains a subgroup of order $(q + 1)^2$ isomorphic to the direct product of two cyclic groups of order $q + 1$, and cyclic subgroups of orders $q^2 - 1$, $q^2 + q + 1$, and $q^2 - q + 1$.

3. CONJUGATE CLASSES OF $p$-ELEMENTS

We divide the set of elements of $U$ into the following three subsets:

$$S_1 = \{x_a(x_1) x_b(x_2) u_2 \mid x_1, x_2 \in K^*, u_2 \in [U, U]\}$$
$$S_2 = \{x_a(z) u_2 \mid z \in K^*, u_2 \in [U, U]\},$$
$$S_3 = \{x_b(t) u_2 \mid t \in K, u_2 \in [U, U]\}.$$

THEOREM (3.1). (i) Every element of $S_1$ is conjugate to $x_a(1) x_b(1)$.

(ii) $g^{-1} x g = y, x \in S_1, y \in U, g \in G$, implies that $g \in U\delta$ and $y \in S_1$.

(iii) $|C(x_a(1) x_b(1))| = q^2$.

Proof. (i) We show that every element $x = x_a(x_1) x_b(x_2) u_2$ of $S_1$ can be transformed into $x_a(1) x_b(1) u'$ by an element of $U\delta$. First, by using (2.2), $x$ can be transformed into $x' = x_a(1) x_b(1) u', u' \in [U, U]$. Let

$$x_1 = x_a(1) x_b(1) x_r(t_r) \prod_{s > r} x_s(t_s),$$

where $r \geq a + b, t_r \neq 0$. Then (2.1) shows that there exists an element $x_0$ in $X_{r-a}$ (if $r \neq 3a + 2b$) or in $X_{r-b}$ (if $r = 3a + 2b$), such that

$$x_0^{-1} x_1 x_0 = x_a(1) x_b(1) \prod_{s > r} x_s(t'_s).$$

Then by successive applications of this reduction, we can obtain $x' \sim x_a(1) x_b(1)$.

(ii) Let $g = uh\omega(w)v$ be the Bruhat factorization of $g$, then we have

$$\omega(w)^{-1}(uh)^{-1}xuh\omega(w) = v^{-1}yv.$$  

From (2.1) and (2.2), we have $(uh)^{-1}xuh \in S_1$; then (1.7) implies that $\omega(a) > 0$ and $\omega(b) > 0$, and hence $\omega = 1$ and $g \in U\delta$. Consequently, $y \in S_1$.

(iii) Put $x = y = x_a(1) x_b(1)$ in (ii), then we have

$$C(x_a(1) x_b(1)) = C_{U\delta}(x_a(1) x_b(1)).$$
On the other hand, (i) shows that \( U \mathcal{H} \) is acting, by conjugation, transitively on \( S_1 \). Thus

\[
| C_{U \mathcal{H}}(x_a(1) x_b(1)) | = | U \mathcal{H} || S_1 | = q^2.
\]

**Theorem (3.2).** Every element of \( S_2 \) is conjugate to an element in \( S_3 \).

**Proof.** A reduction similar to the one used in the proof of (3.1) (i) yields that \( x_a(z) u_2 \sim x_a(1) x_{3a+2b}(t) \) for any \( z \in K^* \), \( u_2 \in [U, U] \), and for some \( t \in K \). Then

\[
\omega_0^{-1} x_a(1) x_{3a+2b}(t) \omega_0 = x_{2u+b}(-1)x_0(-t) \in S_3.
\]

We use the following properties of \( S_3 \) to classify the elements of \( S_3 \) into conjugate classes:

(a) The quotient group \( S = S_3 / \mathfrak{X}_{3a+2b} \) is Abelian and all the elements in a coset \( y x_{3a+2b} \) are conjugate, if \( y \notin \mathfrak{X}_{3a+2b} \).

(b) Both \( \mathfrak{X}_a \) and \( \mathfrak{X}_{-a} \) (hence \( \Phi_a \)) normalize \( S_3 \) and centralize \( \mathfrak{X}_{3a+2b} \).

Property (a), together with the relation \( \omega_0^{-1} \mathfrak{X}_{3a+2b} \omega_0 = \mathfrak{X}_b \), implies that every element of \( S_3 \) is conjugate to an element of the form \( II_i x_{ia+b}(t_i) \). We shall regard \( S \) as a vector space over \( K \), and

\[
x_0(t_0) x_{a+b}(t_1) x_{2a+b}(t_2) x_{3a+2b}(t_3) \mathfrak{X}_{3a+2b}
\]

will be denoted by \((t_0, t_1, t_2, t_3)\). The conjugation by an element \( x \in \mathfrak{X} \Phi_a \) induces a linear transformation in \( S \). The image of \((t_0, t_1, t_2, t_3)\) under this transformation will be denoted by \((t_0, t_1, t_2, t_3)x\). We shall also use the the symbol \( s_1 \sim s_2 \) in case \( s_1, s_2 \in S \), and \( s_2 = s_1 x \) for some \( x \in \mathfrak{X} \Phi_a \). We have

\[
(t_0, t_1, t_2, t_3) h(x_1, x_2, x_3^{-1}x_2^{-1}) = (x_1^{-1}x_2^{-1} t_0, x_1^{-1}t_1, x_1^{-1}x_2^{-1}t_2, x_1^{-1}x_2^{-2}t_3), \quad \text{(3.3)}
\]

\[
(t_0, t_1, t_2, t_3) \omega_a = (-t_0, -t_2, t_1, t_0). \quad \text{(3.4)}
\]

By combining (2.1), (3.3) and (3.4), we obtain

\[
(t_0, t_1, t_2, t_3) \Phi_a \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = (t_0, t_1, t_2, t_3) \begin{pmatrix} \alpha^a & \alpha^a \beta & \alpha \beta^a & -\beta^a \\ 3 \alpha^a \gamma & \alpha(\alpha \delta + 2 \beta \gamma) & \beta(2 \alpha \delta + \beta \gamma) & -3 \beta \delta \gamma \\ 3 \alpha \beta \gamma & \gamma(2 \alpha \delta + \beta \gamma) & \delta(\alpha \delta + 2 \beta \gamma) & -3 \beta \delta \gamma \\ -\gamma^a & -\gamma^a \delta & -\gamma \delta \gamma & \delta^a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \quad \text{(3.5)}
\]

We determine the orbits of \( \mathfrak{X} \Phi_a \) in \( S - \{0\} \). By a suitable element of \( \mathfrak{X}_a \), then by \( \omega_a \), if necessary, every element of \( S - \{0\} \) can be transformed into
an element of the forms \((z_0, 0, 0, 0), (z_0, z_2, 0), (z_0, 0, z_3), (z_0, 0, z_2, z_3), \)
or \((0, z_1, 0, 0)\), where \(z_i \in K^*\) for \(0 \leq i \leq 3\). Next, by transforming these elements by an element of \(\mathfrak{H}\), we have

\[
\begin{align*}
(x_0, 0, 0, 0) &\sim (1, 0, 0, 0), \\
(0, z_1, 0, 0) &\sim (0, 1, 0, 0), \\
(z_0, 0, z_2, 0) &\sim (1, 0, 1, 0) \quad \text{or} \quad (1, 0, \lambda, 0)
\end{align*}
\]

where \(\lambda\) is a fixed nonsquare in \(K\). In case \(q \equiv -1 \pmod{3}\),

\[
(z_0, 0, 0, z_3) \sim (1, 0, 0, 1),
\]

and in case \(q \equiv 1 \pmod{3}\),

\[
(z_0, 0, 0, z_3) \sim (1, 0, 0, 1), (1, 0, 0, \mu) \quad \text{or} \quad (1, 0, 0, \mu^{-1}),
\]

where \(\mu\) is a fixed noncube in \(K\). Furthermore, we have

\[
(1, 0, 0, \mu^{-1}) \omega \eta h(\mu^{-1}, -1, -\mu) = (1, 0, 0, \mu),
\]

\[
(1, 0, 0, 1) \varphi_a \left(1 \begin{array}{c}
\frac{1}{2} \\
-1
\end{array} \right) h(1, \frac{1}{2}, 2) = (1, 0, 1, 0).
\]

Unlike the above classifications, a rather involved computation is needed for elements of type \((z_0, 0, z_2, z_3)\). We first transform these elements into elements of the form \((1, 0, -1, z)\) by \(h(-z_2 z_3^{-1}, z_2^{-1}z_3, -z_2^{-1}) \omega \eta \zeta(-1)\), then use the following identities derived from (3.3) and (3.4):

\[
(1, 0, -1, t + t^{-1}) \varphi_a \left(1 \begin{array}{c}
\frac{1}{2} \\
-t
\end{array} \right) h(1, 1 - t^2, (1 - t^2)^{-1}) = (1, 0, 0, t^{-1})
\]

\[
(1, 0, -1, s^3 + s^{-3}) F(s) = (1, 0, 1, 0),
\]

where

\[
F(s) = \varphi_a \left(\begin{array}{c}
\frac{s^2 + s^{-2}}{2(s^2 + s^{-2} + 1)} \\
\frac{s + s^{-1}}{2(s^2 + s^{-2} + 1)}
\end{array} \right)
\]

\[
\times h \left(\begin{array}{c}
\frac{s - s^{-1}}{2} \\
\frac{2(s^2 + s^{-2} + 1)}{s + s^{-1}}, \frac{1}{s^2 + s^{-2} + 1}
\end{array} \right).
\]

Put \((1, 0, -1, z) = (1, 0, -1, t + t^{-1})\), where \(t\) is either an element of \(K^*\).
or an element of \( K_2 \) such that \( t^{1+q} = 1 \). In case \( t \in K^* \), \((1, 0, -1, t + t^{-1})\) can be transformed into \((1, 0, 0, t^{-1})\) by (3.6). In case \( t \in K_2 \) (and cubic roots of \( t \) exist in \( K_2 \)), select \( s \) such that \( s^3 = t \) and \( s^{1+q} = 1 \); then, by using (3.7), we have

\[
(1, 0, -1, t + t^{-1}) F(s) h((s - s^{-1})^{-1}, s - s^{-1}, 1) = ((s - s^{-1})^3, 0, 1, 0) \\
\sim (1, 0, \lambda, 0).
\]

Note that \( F(s) h((s - s^{-1})^{-1}, s - s^{-1}, 1) \) is contained in \( S_3 \); i.e., it can be written with the coefficients belonging to \( K \). Finally, suppose that \( t \in K_2 \), \( t^{1+q} = 1 \) and \( t \) is a noncube in \( K_2 \). This occurs only when \( q = -1 \) (mod 3).

Let \( \nu \) be a fixed element of \( K_2 \) of exponent \( q + 1 \) so that every element \( t \) with the above property can be written as \( \nu^{3j+1} \) for some integer \( j \). We show that

\[
(1, 0, -1, \nu^{3j+1} + \nu^{-3j-1}) \sim (1, 0, -1, \nu + \nu^{-1}) \tag{3.8}
\]

for every integer \( j \). Let \( \theta \) be a cubic root of \( \nu \). Since (3.7) yields

\[
(1, 0, -1, \nu^{3j+1} + \nu^{-3j-1}) F(\theta^{3j+1}) = (1, 0, 1, 0)
\]

over the extension \( K_\theta \) of \( K \), it suffices to show that all the coefficients of \( F(\theta^{3j+1}) F(\theta)^{-1} \) may be taken from \( K \). For the sake of simplicity, put \( \theta^k + \theta^{-k} = c(k\theta) \), \( \theta^k - \theta^{-k} = s(k\theta) \). Then we have

\[
F(\theta^{3j+1}) F(\theta)^{-1} = \phi_n \begin{pmatrix}
\frac{s((9j+3)\theta)}{s((9j+3)\theta)} & \frac{s(6j\theta)}{s(3j\theta)} \\
\frac{s((9j+3)\theta)}{s(3j\theta)} & \frac{s((9j+3)\theta)}{s(3j\theta)}
\end{pmatrix}
\times h \left( 1, \frac{s((9j+3)\theta)}{s(3\theta)}, \frac{s(3\theta)}{s((9j+3)\theta)} \right).
\]

Since \( s(k\theta) s(\theta) = c((k + 1)\theta) - c((k - 1)\theta) \), every coefficient on the right can be written in terms of \( c(3k\theta) = \nu^k + \nu^{-k} \in K \). This proves (3.8).

Summarizing the above results, we have:

**Theorem** (3.9). Every element of \( S_3 - \{1\} \) is conjugate to one of the elements \( x_0(1), x_{a+b}(1), x_0(1) x_{2a+b}(1), x_b(1) x_{2a+b}(\lambda) \), and either \( x_0(1) x_{2a+b}(\mu) \) or \( x_0(1) x_{2a+b}(-1) x_{2a+b}(\zeta) \) according as \( q = 1 \) or \( -1 \) (mod 3). Here \( \lambda \) is a fixed nonsquare in \( K \), \( \mu \) is a fixed noncube in \( K \), and \( \zeta \) is a fixed element of \( K \) such that the polynomial \( x^3 - 3x - \zeta \) is irreducible over \( K \).

Next, we determine the orders of the centralizers of the above elements.
By so doing we shall see that no pair of these elements are conjugate in \( G \). Let

\[
\begin{align*}
y_1 &= x_{3a+b}(1) \\
y_2 &= x_{2a+b}(1) \\
y_3 &= x_{a+b}(1) x_{3a+b}(-1) \\
y_4 &= x_{a+b}(1) x_{3a+b}(-1) \\
y_5 &= x_{b}(1) x_{3a+b}(u) \\
y_6 &= x_{b}(1) x_{2a+b}(-1) x_{3a+b}(\xi).
\end{align*}
\]

We first determine \( C_U(y_i) \) and \( K^*_+ = \{ u^{-1}y_iu \mid u \in U \} \) by conjugating \( y_i \) with all the elements in \( U \). Using the fact that \([X_r, X_s] = 1\) if \( r + s \) is not a root, we have \( C_U(y_i) \supseteq \langle X_r \mid r \in R_i \rangle \), where

\[
\begin{align*}
R_1 &= \Sigma^+, \\
R_2 &= \{ b, 2a + b, 3a + b, 3a + 2b \}, \\
R_3 &= \{ a + b, 3a + b, 3a + 2b \}, \\
R_4 &= \{ a + b, 2a + b, 3a + 2b \}, \\
R_5 &= \{ 2a + b, 3a + b, 3a + 2b \}.
\end{align*}
\]

For each \( i \), write elements \( u \in U \) in the form

\[
u = \prod_{r \in R_i} x_r(t_r) \prod_{s \in \Sigma^+ - R_i} x_s(t_s),
\]

where the product \( \prod x_s(t_s) \) is taken in increasing order of \( s \in \Sigma^+ - R_i \); then compute \( u^{-1}y_iu \) by using (2.1). We have

\[
u^{-1}y_1u = y_1,
\]

\[
u^{-1}y_2u = y_2x_{3a+b}(-3t_a) x_{3a+2b}(-3t_a+b),
\]

\[
u^{-1}y_3u = y_3x_{2a+b}(2\lambda t_a) x_{3a+b}(-3\lambda t_a b) x_{3a+2b}(t_a t_b + 3\lambda t_a t_b),
\]

where \( i = 3, 4 \), \( \lambda_3 = 1 \), \( \lambda_4 = \lambda \), and \( t_i = -1 - 3\lambda_i + 3\lambda_i^2t_a \),

\[
u^{-1}y_4u = x_b(1) x_{a+b}(t_a) x_{2a+b}(t_a^2) x_{3a+b}(\mu - t_a^3) x_{3a+2b}(-t_a^3 + t_a t_b - \mu t_b + t_a + t_b),
\]

\[
u^{-1}y_5u = x_b(1) x_{a+b}(t_a) x_{2a+b}(t_a^2 - 1) x_{3a+b}(-t_a + 3t_a + \xi) x_{3a+2b}(-t_a^2 - \xi t_b + t_a + t_b).
\]

The above identities give us \( K^*_+ = \{ u^{-1}y_iu \} \), and enable us to determine \( C_U(y_i) \).

\[
\begin{align*}
C_U(y_1) &= U, \\
C_U(y_2) &= \langle X_b, X_{2a+b}, X_{3a+b}, X_{3a+2b} \rangle, \\
C_U(y_3) &= \langle X_{a+b}, X_{3a+b}, X_{3a+2b}, X_i \rangle, \\
C_U(y_4) &= \langle X_{a+b}, X_{2a+b}, X_{3a+2b}, X_i \rangle, \\
C_U(y_5) &= \langle X_{2a+b}, X_{3a+b}, X_{3a+2b}, X_i \rangle, \\
C_U(y_6) &= \langle X_{2a+b}, X_{3a+b}, X_{3a+2b}, X_i \rangle.
\end{align*}
\]

(3.11)
where
\[ X_a = \langle x_a(-3t) x_{2a+b}(t) | t \in K \rangle, \]
\[ X_3 = \langle x_3(-3t) x_{2a+b}(t) | t \in K \rangle, \]
\[ X_5 = \langle x_5(t) x_{3a+1b}(t) x_{3a+2b}(t) | t \in K \rangle, \]
\[ X_6 = \langle x_5(3t) x_{a+b}(t) | t \in K \rangle. \]

The groups \( X_i \) in (3.11) are one-parameter subgroups in the sense that
\[ X_i = \{ x_i(t) (t \in K) \text{ and } q(t, x_i(t_1) x_i(t_2) = q(t, + t), t_1, t_2 \in K \}. \]

We know from (3.10) that \( |K^+| = q^2, i \geq 2 \). Hence
\[ |C_{i}(y_i)| = q^4 \quad (i \geq 2). \quad (3.12) \]

We note here that if \( u \notin C_{\U}(y_i) \), then \( uh \notin C(y_i) \) for any \( h \in \F \). Hence
\[ C(y_i) \cap \U \F = C_{\U}(y_i) C_{\F}(y_i). \]

By using (1.7) and (1.8), we next show that
\[ C(y_i) \subseteq \U \F \cup \U \F \omega(w_a)U_{w_a}^{\omega} \quad (i \neq 2) \quad (3.13) \]

and (1.8) implies that
\[ \omega(w)^{-1}(uh)^{-1}y_i(uh) \omega(w) = vy_i v^{-1}, \quad (3.14) \]

and (1.8) implies that
\[ w(r_i) \geq r_i, \quad w^{-1}(r_i) \geq r_i. \]

By checking the action of each \( w \in W \) on \( \Sigma \), we can see that the above conditions are satisfied only by the following elements of \( W \) besides 1: \( w_a \) in case \( r_i = r_1 \); \( w_b \) in case \( r_i = r_2 \); \( w_a \) and \( w_{3a+b} \) in case \( r_i = r_3 = r_4 \); \( w_a \), \( w_{2a+b} \) and \( w_{3a+b} \) in case \( r_i = r_5 = r_6 \). Assume that \( w = w_{3a+b} \) satisfies (3.14) for \( y_i = y_3 \) or \( y_4 \) and for any \( u, v \in \U, h \in \F \). Then, by substituting the explicit form of \( u^{-1}y_i u \) in (3.10), we have
\[ \omega(w_{3a+b})^{-1}h^{-1}_{x_{a+b}(\lambda_i)} x_{2a+b}(2\lambda_i t_a) x_{3a+b}(-1 - 3\lambda_i t_a^2) \]
\[ x_{3a+2b}(...) h \omega(w_{3a+b}) \in \U. \]

On the other hand, \( w_{3a+b}(2a + b) < 0 \) and \( w_{3a+b}(3a + b) < 0 \), and hence
2\lambda t_a = 0 \text{ and } -1 - 3\lambda t_a^2 = 0 \text{ by (1.7). This is not possible. In a similar manner, we can delete } \omega_{xa+b} \text{ and } \omega_{3xa+b} \text{ from the case } r_t - r_a - r_b. \text{ These restrictions on } w \text{ and (1.1) yield (3.13).}

The centralizers of \( y_1 \) and \( y_2 \) are now readily determined. We have \( C_5(y_1) = S_a, C_5(y_2) = S_b \text{ from (2.2), } \omega(w_a) \in C(y_1), \omega(w_b) \in C(y_2) \text{ from } [\Phi_a, \Phi_{3a+2b}] = 1, [\Phi_b, \Phi_{2a+b}] = 1, \text{ and } U_{w_a}^* = x_a \subseteq C(y_1), U_{w_b}^* = x_b \subseteq C(y_2). \text{ Then from (3.12) we can derive,}

\[
C(y_1) = U_S \cup U_S \omega(w_a) x_a = U_{\Phi_a}
\]

\[
|C(y_1)| = q^2(q^2 - 1), (3.15)
\]

\[
C(y_2) = C_u(y_2) S_b \cup C_u(y_2) \omega(w_b) x_b
\]

\[
= \langle X_r \mid r = b, 2a + b, 3a + b, 3a + 2b, -b \rangle, (3.16)
\]

\[
|C(y_2)| = q^4(q^2 - 1).
\]

For \( y_i, i \geq 3 \), we reduce the problem as follows: We know from (3.13) that \( C(y_i) \) [from now on \( y_i \) stands for \( y_3, \ldots, y_6 \) only] is contained in \( U_S \cup U_S \omega(w_a)x_a = U_{\Phi_a}S = S_3\Phi_aS \), where \( S_3 = \langle X_r \mid r \geq b \rangle \) as defined at the beginning of this section. Thus every element of \( C(y_i) \) can be written as \( sx, s \in S_3 \), \( x \in \Phi_aS \). Then \( sx \in C(y_i) \) implies that \( y_i = x^{-1}s^{-1}y_ixs \equiv x^{-1}y_i, x \text{ (mod } x_{3a+2b}) \). In other words, \( x \) must fix the vector \( \tilde{y}_i \in S \) that represents \( y_i x_{3a+2b} \), or (using the notations introduced with \( S \)) \( \tilde{x}_i x = \tilde{y}_i \). We shall first determine the group

\[ M_i = \langle x \mid x \in \Phi_aS, \tilde{y}_i x = \tilde{y}_i \rangle, \]

for each \( i \). It will turn out that every element of \( M_i \) actually centralizes \( y_i. \)

Suppose we have shown that \( M_i \subseteq C(y_i) \). Then \( sx \in C(y_i), s \in S_3, x \in \Phi_aS \), if and only if \( x \in M_i \) and \( s \in C(y_i) \cap S_3 = C_u(y_i) \). Moreover, \( S_3 \cap M_i \subseteq S_3 \cap \Phi_aS = 1. \text{ Thus we will have}

\[
C(y_i) = C_u(y_i)M_i, (3.17)
\]

\[
|C(y_i)| = |C_u(y_i)||M_i|. \]

The problem is, therefore, to determine \( M_i \) and to prove that \( M_i \subseteq C(y_i) \).

\( C(y_3), C(y_4). \text{ By using (3.5) and (3.3) we can show that } (0, \lambda_t, 0, -1)x = (0, \lambda_t, 0, -1) \text{ holds only if } x \text{ is } h(1, \epsilon_1, \epsilon_1) \text{ or}

\[
q_i a \left( \frac{-1/2}{-3/2 \lambda_t}, \frac{\epsilon_a(\epsilon_a(-3\lambda_t)^{1/2})}{1/2} \right) h(1, \epsilon_1, \epsilon_1),
\]

where \( \epsilon_1 = \pm 1, \epsilon_a = \pm 1. \)
Thus we have,

\[ M_3 = \langle \varphi_a \left( \frac{-1}{2}, \frac{(-3)^{1/2}}{2}, h(1, -1, -1) \right) \rangle, \]

\[ M_4 = \langle h(1, -1, -1) \rangle, \]

if \( q \equiv 1 \pmod{3} \), and

\[ M_2 = \langle h(1, -1, -1) \rangle, \]

\[ M_4 = \langle \varphi_a \left( \frac{-1}{2}, \frac{(-3\lambda)^{1/2}}{2}, h(1, -1, -1) \right) \rangle, \]

if \( q \equiv -1 \pmod{3} \). (2.2) shows that \( h(1, -1, -1) \in C(y_i), i = 3, 4 \). Put

\[ \varphi_a \left( \frac{-1}{2}, \frac{(-3\lambda)^{1/2}}{2}, \frac{-1}{2} \right) = f_i. \]

Then \( y_i f_i = y_i \) implies that \( f_i, y_i, f_i = y_i x_{3a+2b}(t) \) for some \( t \in K \). Since \( [\Phi_a, \Phi_{3a+2b}] = 1 \), we have \( f_i, y_i f_i = y_i x_{3a+2b}(kt) \). On the other hand, \( f_i^3 = 1 \). Hence \( 3t = 0 \) and \( f_i \in C(y_i) \). Thus \( M_i \subseteq C(y_i) \) is proved and we have,

\[
| C(y_3) | = \begin{cases} 6q^4 & \text{if } q \equiv 1 \pmod{3} \\ 2q^4 & \text{if } q \equiv -1 \pmod{3} \end{cases} \quad \text{(3.18)}
\]

\[
| C(y_4) | = \begin{cases} 2q^4 & \text{if } q \equiv 1 \pmod{3} \\ 6q^4 & \text{if } q \equiv -1 \pmod{3} \end{cases} \]

\( C(y_5) \). It is assumed that \( q \equiv 1 \pmod{3} \). Again using (3.3) and (3.5), we can see that an element of \( M_5 \) must be in \( \langle h(\omega, \omega, \omega) \rangle \) or of the form \( \omega(\omega, \omega) h(\mu_0, -\mu_0^2, -\mu_0) \), where \( \omega = \frac{1}{3}(-1 + (-3)^{1/3}) \) and \( \mu_0 \) is a cubic root of \( \mu \). Since \( \omega \in K \), but \( \mu_0 \notin K \), we have

\[ M_5 = \langle h(\omega, \omega, \omega) \rangle. \]

Then (2.2) shows that \( M_5 \subseteq C(y_5) \).

\[ | C(y_5) | = 3q^4. \quad \text{(3.19)} \]

\( C(y_6) \). In this case the elements of \( M_6 \) are not easily obtained from (3.3) and (3.5). We recall that \( \zeta = \lambda + \lambda^{-1} \), where \( \lambda \) is a noncube in \( K_2 \). Identity (3.6) shows that \( y_6 \sim y_5 \) in \( G_2 \). We know from the above argument on \( M_5 \) that the number of elements \( x \in \Phi_{6过去的2} \), such that \( y_5 x = y_5 \), is still 3. Hence \( | M_6 | \) is 3 or 1. On the other hand, (3.9) shows that

\[ (1, 0, -1, \zeta) F(\theta) = (1, 0, -1, \zeta) F(\omega \theta), \]

where \( \theta \in K_6 \) is a cubic root of \( \lambda \).
THE CONJUGATE CLASSES OF CHEVALLEY GROUPS OF TYPE $(G_2)$

Moreover,

$$F(\omega \theta)F(\theta)^{-1} = q_0 \begin{pmatrix} (\theta^3 - \theta^{-3})^{-1} \begin{pmatrix} \theta^3 \omega^2 - \theta^{-3} \omega^{-2} & -\omega + \omega^{-1} \\ \omega - \omega^{-1} & \theta^3 \omega - \theta^{-3} \omega^{-1} \end{pmatrix} 
$$

is defined over $K$. Hence

$$M_6 = \langle F(\omega \theta)F(\theta)^{-1} \rangle.$$

An argument similar to the one used in the proof of $f_i \in C(y_i), i = 3, 4$, shows that $M_i \subseteq C(y_i)$.

$$|C(y_0)| = 3q^4. \quad (3.21)$$

This completes the determination of the conjugate classes of $p$-elements. Including the identity class, the group $G$ contains 7 conjugate classes of $p$-elements.

4. CONJUGATE CLASSES OF $p_0$-ELEMENTS, $p_0 \mid q^6 - 1$

We shall first determine the centralizers of elements of $\mathfrak{H}$ and $\mathfrak{H}(\omega(\omega))$.

THEOREM (4.1). $C(h(\chi)) = \langle \mathfrak{H}_r, \Phi_r \mid \chi(r) = 1 \rangle$.

**Proof.** If $\chi(r) = 1$, then $x_r(t)$ and $x_{-r}(t)$ for any $t \in K$ commute with $h(x)$ by (1.4). Hence $\Phi_r = \langle \mathfrak{X}_r, \mathfrak{X}_{-r} \rangle \subseteq C(h(\chi))$, and $\langle \mathfrak{H}_r, \Phi_r \mid \chi(r) = 1 \rangle \subseteq C(h(\chi))$.

Let $h = h(x)$ and suppose that $x \in C(h)$. If $x = u \omega(\omega) v$ is the Bruhat factorization of $x$, then $(h^{-1}u h') h'^{-1} \omega(\omega)(h^{-1}v h')$, where $h'^{-1} \omega(\omega) = h^{-1} \omega(\omega) h$, is also the Bruhat factorization of $x = h^{-1}x h = x$. Hence $h, \omega(\omega) \in C(h)$. Again the uniqueness of the factorization $u = \prod x_r(r)$, $v = \prod x_r(\chi(r)r)$, $h u h^{-1}$ shows that $t_r = 0$ for all $r$, such that $\chi(r) \neq 1$. Hence $u \in \langle \mathfrak{X}_r \mid \chi(r) = 1 \rangle$. Similarly, $v \in \langle \mathfrak{X}_r \mid \chi(r) = 1 \rangle$. Now we show that if $\omega(\omega) \in C(h)$, then $\omega(\omega)$ is a product of $\omega_r, r \in \Sigma, \chi(r) = 1$, which will imply that $\omega(\omega) \in \langle \Phi_r \mid \chi(r) = 1 \rangle$. Let $h = h(x_1, x_2, x_3)$. If $\omega(\omega) = \omega_{r'}$, $r \in \Sigma$, then (2.4) shows that $\chi(r) = 1$. If $\omega(\omega) = \omega_{\xi_1}$, then $z_i = \pm 1$ for all $i$, but $z_2 = 1$ for some $k$. Then $\chi(\xi_k) = \chi(\xi_k - \xi_3) = 1$, where $(ijk) = (23)$. On the other hand, $\omega_3 = \omega_{\xi_3 - \xi_2} \omega_{\xi_2}$. If $\omega(\omega) = \omega_3$, then $z_1 = z_2 = z_3$, and hence $\chi(\xi_1 - \xi_3) = \chi(\xi_2 - \xi_3) = 1$; but, then $\omega_3 = \omega_{\xi_3 - \xi_2} \omega_{\xi_1 - \xi_2}$. Finally, (2.6) and (2.7) show that $\omega_3 \in C(h(\chi))$ only if $h(\chi) = 1$. This completes the proof of (4.1).
As an immediate consequence of (4.1) we obtain (4.2) below. For the sake of simplicity, we shall denote \(\xi_0\) by \(c\) (and \(\xi_1 - \xi_8\) by \(b\)).

\[
C(h(-1, -1, 1)) = \langle \Phi_c, \Phi_e, s \rangle,
C(h(\omega, \omega, \omega)) = \Psi \quad \text{if } \omega \in K
C(h(z, z^{-1}, 1)) = \langle \Phi_c, s \rangle \quad (z^2 \neq 1), \tag{4.2}
C(h(z^{-1}, z^{-1}, z^2)) = \langle \Phi_c, s \rangle \quad (z^2 \neq 1, z^3 \neq 1),
C(h(z_1, z_2, z_3)) = s \quad (z_i z_j^{-1} \neq 1, 1 \leq i, j \leq 3).
\]

(4.1) and (4.2) remain valid when the ground field \(K\) is replaced by its extension \(K_i\). Let \(\Phi_{s} = \Phi_{r, 2} \cap \Gamma(\omega_2)\). We have

\[
C_{\Gamma(\omega_2)}(h(-1, -1, 1)) = \langle \Phi_h, \Phi_c, s \rangle(\omega_2),
C_{\Gamma(\omega_2)}(h(\omega, \omega, \omega)) = \Psi_2 \cap \Gamma(\omega_2) \simeq U(3, K_2),
C_{\Gamma(\omega_2)}(h(z, z^{-1}, 1)) = \langle \Phi_c, s \rangle(\omega_2) \quad (z^{2+1} = 1, z^2 \neq 1),
C_{\Gamma(\omega_2)}(h(z^{-1}, z^{-1}, z^2)) = \langle \Phi_h, s \rangle(\omega_2) \quad (z^{2+1} = 1, z^2 \neq 1, z^3 \neq 1),
C_{\Gamma(\omega_2)}(h(z_1, z_2, z_3)) = s(\omega_2) \quad (z_i z_j^{-1} \neq 1, 1 \leq i, j \leq 3), \tag{4.3}
C_{\Gamma(\omega_2)}(h(z, z^q, z^{-q-1})) = \delta(\omega_2) \quad (z^{2q-1} = 1, z^{q+1} \neq 1),
C_{\Gamma(\omega_2)}(h(z, z^{-q}, z^{q-1})) = \delta(\omega_2) \quad (z^{2q-1} = 1, z^{q+1} \neq 1),
C_{\Gamma(\omega_2)}(h(z, z^q, z^{q-1})) = \delta(\omega_2) \quad (z^{2q+q+1} = 1, z^3 \neq 1),
C_{\Gamma(\omega_2)}(h(z, z^{-q}, z^{q+1})) = \delta(\omega_2) \quad (z^{2q-q+1} = 1, z^3 \neq 1).
\]

We shall now determine the conjugate classes \(K(x)\) for \(|x| = 2\) and \(|x| = 3\). It is known that \(G\) has one class of involutions ([3], [6]). The proof of (4.4) is given here for the sake of completeness.

**Theorem (4.4).** Every involution of \(G\) is conjugate to \(h(-1, -1, 1)\).

**Proof.** If \(q = 1 \pmod{4}\), then the subgroup \(<s, \omega_{\xi_1-\xi_8}, \omega_{\xi_8}> = s \cup s \omega_{\xi_1-\xi_8} \cup s \omega_{\xi_8} \cup s \omega_{\xi_8}\) contains a Sylow 2-group of \(G\), and hence every involution of \(G\) is conjugate to an involution of this subgroup. The subgroup \(s\) contains the 3 involutions \(h(-1, -1, 1) = h_0, h(1, -1, -1) = \omega_3 h_0 \omega_3, h(-1, 1, -1) = \omega_3 h_0 \omega_3^{-1}\). The involutions in \(s \omega_{\xi_1-\xi_8}\) are of the form

\[
h(z, -z^{-1}, -1) \omega_{\xi_1-\xi_8} = \varphi_{\xi_1-\xi_8} \begin{pmatrix} 0 & z_i \\ z_i^{-1} & 0 \end{pmatrix} \varphi_{\xi_8} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]
THE CONJUGATE CLASSES OF CHEVALLEY GROUPS OF TYPE \((G_3)\)

Since

\[
\begin{pmatrix}
0 & z_1 \\
z^{-1} & 0
\end{pmatrix}
\sim
\begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix}
\]

in \(SL(2, K)\), the above element is conjugate to

\[
\varphi_{\xi_1 - \xi_2}
\begin{pmatrix}
i & 0 \\
0 & i
\end{pmatrix}
\varphi_{\xi_3}
\begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix} = h(1, -1, -1).
\]

We can show similarly that every involution in \(\mathfrak{H}_3\) is conjugate to \(h(1, -1, -1)\). Every element \(x = h(z_1, z_2, z_3)\) of \(\mathfrak{H}_3\) is an involution. Since at least one \(z_i\) is a square in \(K\), an element of \(\langle \omega_3 \rangle\) will transform \(x\) into an element of the form

\[
h(z, z^{-1}, 1)h(z'^{-1}, z'^{-1}, z'^3) = \varphi_{\xi_1 - \xi_2}
\begin{pmatrix}
0 & z_1 \\
-z^{-1} & 0
\end{pmatrix}
\varphi_{\xi_3}
\begin{pmatrix}
0 & z' \\
-z'^{-1} & 0
\end{pmatrix},
\]

\(z, z' \in K\).

This element is also conjugate to \(h(1, -1, -1)\). If \(q \equiv -1 \pmod{4}\), then we can show that every involution of \(\Gamma(\omega_3)\) is conjugate to \(h(-1, -1, 1)\) in \(\Gamma(\omega_3)\) by replacing \(G, \mathfrak{H}_3 \) and \(SL(2, K)\) in the foregoing proof by \(\Gamma(\omega_3), \mathfrak{H}(\omega_2)\) and \(U(2, K_2)\). Note that \(\omega(w) \in \Gamma(\omega_2)\) for all \(w \in W\).

**Theorem (4.5).** The group \(G\) has 2 conjugate classes of elements of order 3. These classes may be represented by \(h(\omega, \omega^{-1}, 1)\) and \(h(\omega, \omega, \omega)\), if \(q \equiv 1 \pmod{3}\).

**Proof.** The subgroup \(\langle \mathfrak{H}_3, \omega_2 \rangle = \mathfrak{H}_3 \cup \mathfrak{H}_3 \cup \mathfrak{H}_3^{-1}\) contains a Sylow 3-subgroup of \(G\) in the case \(q \equiv 1 \pmod{3}\). Every element of order 3 in \(\mathfrak{H}_3\) is of the form \(h(z_1, z_2, z_3), z_3^3 = 1\), and a suitable element of \(\langle \omega(\omega) \rangle\) will transform it into \(h(\omega, \omega^{-1}, 1)\) or \(h(\omega, \omega, \omega)\). Every element in \(\mathfrak{H}_3 \omega_3\) or \(\mathfrak{H}_3 \omega_3^{-1}\) is of order 3. (Therefore every 3-element of \(G\) of order \(> 3\) is conjugate to an element in \(\mathfrak{H}_3\) if \(q \equiv 1 \pmod{3}\). This fact will be quoted in the proof of (4.6).) Take an arbitrary element

\[
n = h(z_1, z_2, z_3)\omega_3 = \psi
\begin{pmatrix}
0 & z_1 \\
0 & 0 \\
z_3 & 0
\end{pmatrix}
\]

of \(\mathfrak{H}_3 \omega_3\), and let

\[
S = \frac{1}{\omega - \omega^3}
\begin{pmatrix}
1 & z_1 & z_1 z_3 \\
z_1^{-1} & \omega & \omega^{-1} z_2 \\
z_1^{-2} z_2^{-1} & \omega^{-1} z_2^{-1} & \omega
\end{pmatrix}
\]

Then \(\psi(S)^{-1}x\psi(S) = h(1, \omega, \omega^{-1})\). We can show similarly that every element of \(\mathfrak{H}_3 \omega_3^{-1}\) is conjugate to \(h(1, \omega, \omega^{-1})\).
In the case $q \equiv -1 \pmod{3}$, we can show that every element of order 3 in $\Gamma(\omega_3)$ is conjugate to $h(\omega, \omega^{-1}, 1)$ or $h(\omega, \omega, \omega)$ simply by replacing $S$ by $S(\omega_3)$ in the above proof. Again note that $\psi(s)$ is in $\Gamma(\omega_3)$ if $h(z_1, z_2, z_3) \in S(\omega_3)$.

The conjugate classes $K(x), (|x|, q^3 \pm q + 1) \neq 1, 3$, are readily determined. If $g \in \Gamma(\omega_3)$ and $(|g|, q^3 \pm q + 1) \neq 1, 3$, then $g$ is conjugate to an element in $C(h)$ for some $h \in S(\omega_3), h^3 \neq 1$. But $C(h) = S(\omega_3)$ from (4.3), and $g$ itself is conjugate to an element in $S(\omega_3)$. Now two elements $h_1$ and $h_2$ of $S(\omega_3)$ are conjugate in $G_3$ if and only if $h_2 = \omega_3^{-1}h_1\omega_3^i$ from (1.9), (2.4), and (2.7). On the other hand, $\omega_6 \in \Gamma(\omega_3)$. Furthermore, $h \neq \omega_6^{-1}h\omega_6^i$ if $h \in S(\omega_3), h^3 \neq 1$, and $\omega_6 \neq \omega_3^i$. Therefore the elements of the set $\{h \in S(\omega_3), h^3 \neq 1\}$ are divided into $\frac{1}{3}(q^3 + q - 2)$ conjugate classes of $\Gamma(\omega_3)$ if $q \equiv -1 \pmod{3}$. A similar result is obtained for the elements $g$ of $\Gamma(\omega_3)$ such that $(|g|, q^3 \pm q + 1) \neq 1, 3$.

Next we determine the conjugate classes $K(x), (|x|, q^3 \pm q + 1) \neq 1, 3$. For this purpose we introduce some notation. Let $T_0$ be a fixed element of $SL(2, K_2)$ such that $T_0^{-1}T_0^\phi = (0, 1)\phi$, and $\xi_2 = \phi_0(T_0) \phi_2(T_0)$; then $\xi_2 \in G_2$ and $\xi_2^{-1}\phi_0^{-1} = \omega_2^{-1}$. Denote by $\gamma$ the inner automorphism $y \rightarrow \gamma y\gamma^{-1}$ of $G_2$. Then $\gamma(\Gamma(\omega_3)) = G$, $\gamma(\phi_0) = \phi_0$, and $\gamma(\phi_4) = \phi_4$. Let $\gamma(\mathcal{F}) = \langle \phi_0, \phi_4, \mathcal{F} \rangle$, $h_0 = h(-1, -1, 1)$, $\mathcal{R} = \gamma(\mathcal{F}(\omega_3))$, and $\mathcal{R}_r = \gamma(\mathcal{F}(\omega_3) \cap \mathcal{F}_r)$, $r = b, c$. Let

$$h_1 = \phi_r \begin{pmatrix} \alpha \lambda_1 \\ 0 \lambda_1^{-1} \end{pmatrix}, \quad k_1 = \gamma \left( \phi_r \begin{pmatrix} \alpha \lambda_1 \\ 0 \lambda_2 \end{pmatrix} \right),$$

where $\lambda_1$ and $\lambda_2$ are elements of $K_2$ of exponents $2(q - 1)$ and $2(q + 1)$ respectively. Then $h_1k_1, h_2k_2, h_3k_3, h_4k_4 \in \mathcal{F} - \phi_0\mathcal{F}$ and

$$\mathcal{H} = \mathcal{S}_b\mathcal{S}_0 + \mathcal{S}_b\mathcal{S}_c(h_0k_0),$$

$$\mathcal{R} = \mathcal{R}_b\mathcal{R}_r + \mathcal{R}_b\mathcal{R}_c(h_0k_c),$$

$$\langle h_1\gamma^{-1}(h_0) \rangle = \langle \phi_s(T_0)^{-1}(h_0k_0) \phi_s(T_0) \rangle = \mathcal{S}(\omega_3),$$

where $(r, s) = (b, c)$ or $(c, b)$. Let $\mathcal{R}(\omega_3) = \langle (h_0k_c) \rangle$ and $\mathcal{R}(\omega_0) = \langle h_0k_c \rangle$.

**Lemma (4.6).** (i) If $x \in G, (|x|, q^3 \pm 1) \neq 1, 3$, then $x$ is conjugate to an element in $\mathcal{H}$.

(ii) If $x \in G, (|x|, q^3 \pm 1) = (|x|, q^4 \pm q^2 + 1) = 3, |x| \neq 3$, then $|x| = 3p$ and $x$ is conjugate to an element in $\mathcal{F}$ or $\mathcal{R}$ or $\mathcal{R}(\omega_2)$.

**Proof.** (i) If $|x|$ is even, then a power of $x$ is an involution and (4.4), (4.2) yield the desired result. If $|x|$ is divisible by $3^p$, then a power $y$ of $x$ is of order $3^p$ and $y$ is conjugate to an element $h$ in $S$ or $R$, as we have noted in the proof of (4.5). Hence $x$ is conjugate to an element in $C(h), h^3 \neq 1, h \in S$,
or $\mathfrak{R}$. If $|x|$ is divisible by a prime $p_0 \neq 2, 3$, then either $\mathfrak{S}$ or $\mathfrak{R}$ contains a Sylow $p_0$-group of $G$, and again $x$ is conjugate to an element in $C(h), h^3 \neq 1, h \in \mathfrak{S}$, or $\mathfrak{R}$. In the case $h \in \mathfrak{S}$, (4.2) yields the desired result. In the case $h \in \mathfrak{R}$, (4.4) shows that $\eta^{-1}(x)$ is conjugate to an element in $\eta(\langle \Phi^+_h, \Phi^+_h, \mathfrak{S}(\omega_3) \rangle) = \mathfrak{R}$.

(ii) The conditions on $|x|$ imply that $|x| = 3p^e$ for some $e \geq 1$. Then (4.5), (4.2), and (4.4) yield (ii).

Lemma (4.7). Every element of $\mathfrak{S}$ is conjugate to an element in one of the following subgroups:

$$\mathfrak{S}, \mathfrak{R}, \mathfrak{A}(\omega_3), \mathfrak{A}(\omega_3), \mathfrak{S}_b \mathfrak{S}_c, \mathfrak{S}_c \mathfrak{S}_h, \mathfrak{R}_h \mathfrak{A}_c, \mathfrak{A}_c \mathfrak{X}_b, \langle h_0, X_b X_c \rangle.$$

Proof. For any $x_r \in \Phi_r$, we can find $g_r \in \Phi_r$ such that $y_r = g_r x_r g_r \in \mathfrak{S}_r$, $\mathfrak{R}_r$, or $\langle h_0, X_r \rangle$. Hence, for any element $x_b x_c$ in $\Phi_b \Phi_c$, we have $(g_b g_c)^{-1} x_b x_c (g_b g_c) = y_b y_c$, as required. If $x \in \mathfrak{S} - \Phi_b \Phi_c$, then $x_2 \in \Phi_b \Phi_c$ and we have $g^{-1} x_2^g = y_b y_c$ for some $g \in \Phi_b \Phi_c$, $y_r \in \mathfrak{S}_r$, $\mathfrak{R}_r$, or $\langle h_0, X_r \rangle$. We solve $(g^{-1} x_2^g) = y_b y_c$ for $g^{-1} x_2^g$. Let $x = x_b x_c (h_b h_c), x_r \in \Phi_r$, and $y_r = g^{-1} x_r g$. Then $y_r \in \Phi_r h_r = \Phi_r h_r$ and $y_r^2 \in \Phi_r$. From $(g^{-1} x_2^g) = y_b y_c$ and $\Phi_b \cap \Phi_c = \langle h_0 \rangle$, we obtain either $y_b^2 = y_b, y_c^2 = y_c$, or $y_b^2 = h_0 y_b, y_c^2 = h_0 y_c$; in any case, $y_r^2 \in \mathfrak{S}_r, \mathfrak{R}_r$, or $\langle h_0, X_r \rangle$. Then a simple computation of $2 \times 2$ matrices shows that $y_r^4 = 1$. If $y_r^4 = 1$, then $g_r^{-1} y_r g_r \in \mathfrak{S}_r h_r$ or $\mathfrak{R}_r h_r$ for some $g_r \in \Phi_r$. In the case $q = 1 (\mod 4)$, let $y_r^* = \varphi_r(T)$; then $T$ must be of the form

$$\begin{pmatrix} \alpha \lambda_1 & \beta \lambda_1 \\ \gamma \lambda_1 & -\alpha \lambda_1 \end{pmatrix}, \quad \alpha, \beta, \gamma \in K.$$

If $\beta \neq 0$, then

$$T' = \begin{pmatrix} 1 & 0 \\ -\alpha \beta^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \beta \lambda_1 i & 1 \\ -\frac{1}{2} (\beta \lambda_1 i)^{-1} \end{pmatrix} \in SL(2, K)$$

transforms $T$ into $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Similarly, we can find an element in $SL(2, K)$ that transforms $T$ into $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $\gamma \neq 0$; if $\beta = \gamma = 0$, then $T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Thus $y_r^*$ is transformed into

$$\varphi_r \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \varphi_r \begin{pmatrix} \lambda_r^{-1} i & 0 \\ 0 & -\lambda_r i \end{pmatrix} \varphi_r \begin{pmatrix} \lambda_r & 0 \\ 0 & \lambda_r^{-1} \end{pmatrix} \in \mathfrak{S}_r h_r$$

by an element in $\Phi_r$. In the case $q = 1 (\mod 4)$, let $\eta^{-1}(y_r^*) = \varphi_r(U)$; then

$$U = \begin{pmatrix} \alpha \lambda_2 & \beta \lambda_2^{-1} \\ (\beta \lambda_2^{-1} \gamma) & -\alpha \lambda_2 \end{pmatrix}, \quad \alpha^{1+q} + \beta^{1+q} = 1.$$
Let

\[
U' = \begin{pmatrix}
\beta\lambda_2^{-1} & \alpha\lambda_2 - i \\
-\alpha\lambda_2 + i & (\beta\lambda_2^{-1})^0
\end{pmatrix}
\]

where \( \delta \) is an element of \( K_2 \) such that \( \delta^{-(1+a)} = -2(1 + 2\alpha\lambda_2i) \), the determinant of the first matrix. Then \( \varphi_r(U') \in \Phi_r^* \), and it transforms \( \varphi_r(U) \) into an element in \( \gamma^{-1}(\mathcal{H}_r \mathcal{K}_r) \). Therefore \( \gamma_r \) is transformed into an element in \( \mathcal{H}_r \mathcal{K}_r \) by an element in \( \Phi_r \). We have thus shown that every element of \( \mathcal{G} - \Phi_0 \mathcal{G}_c \) is conjugate to an element in \( \mathcal{H}, \mathcal{R}, \mathcal{R}(\omega_h) \), or \( \mathcal{R}(\omega_e) \) and the proof of (4.7) is completed.

The conjugate classes \( \mathcal{R}(x), x \in \mathcal{S}, \mathcal{R}, \mathcal{R}(\omega_h), \mathcal{R}(\omega_e) \) are readily determined. All the elements in the above subgroups are conjugate in \( G_2 \) to an element in \( \mathcal{S}_2 \). From (1.9) we know that an element in one of the subsets \( \mathcal{S} - \langle h_0 \rangle \), \( \mathcal{R} - \langle h_0 \rangle \), \( \mathcal{R}(\omega_h) - (\mathcal{R}_b \cup \mathcal{H}_c) \), \( \mathcal{R}(\omega_e) - (\mathcal{R}_b \cup \mathcal{H}_c) \), cannot be conjugate in \( G_2 \) (a fortiori, cannot be so in \( G \)) to an element in another one of these subsets. Again from (1.9) we know that two elements of \( \mathcal{S} \) (or \( \mathcal{R} \)) are conjugate to each other if and only if one can be transformed to the other by some \( \omega(w) \) [or \( \eta(\omega(w)) \)]. We can also see that two elements of \( \mathcal{R}(\omega_h) \) [or \( \mathcal{R}(\omega_e) \)] are conjugate to each other if and only if one can be transformed to the other by \( 1, \eta(\omega_h), \omega_e, \) or \( \eta(\omega_e) \omega_e \) [or by \( 1, \omega_h, \eta(\omega_e), \) or \( \omega_e \eta(\omega_e) \)].

It remains to determine the conjugate classes of the \( p \)-singular elements. From (4.6) and (4.7) we know that if \( |x| = mp \), \( m \neq 1, e \neq 0 \), then \( e = 1 \) and \( m \) is a divisor of \( q - 1 \) or \( q + 1 \). Furthermore, if \( |x| \neq 2p, 3p \), then \( x \) is conjugate to an element in \( \mathcal{S}_0 \mathcal{X}_p, \mathcal{S}_c \mathcal{X}_p, \mathcal{R}_b \mathcal{X}_p, \) or \( \mathcal{R}_c \mathcal{X}_p \). An element \( h, \alpha_x(t) \in \mathcal{S}_r \mathcal{X}_s \), \( t \neq 0 \), can be transformed into \( h, \alpha_x(1) \) by an element of \( \mathcal{S} \), and an element \( k, \alpha_x(t) \in \mathcal{R}_r \mathcal{X}_s \) can be transformed into \( k, \alpha_x(1) \) by an element in \( \mathcal{R}(\omega_e) \). Then, by using (1.9) and the fact that \( x_1 \sim x_2 \) implies \( x_1^p \sim x_2^p \), we can see that a pair of elements \( x_1 \) and \( x_2 \), \( x_1^{2p} \neq 1, x_2^{2p} \neq 1 \), taken from the union of the subsets \( \mathcal{S}_0 x_0(1), \mathcal{S}_c x_0(1), \mathcal{R}_b x_0(1) \) and \( \mathcal{R}_c x_0(1) \), are conjugate to each other if and only if they belong to the same subset and \( x_1^p = x_2^p \). Using this fact, we can find the number of conjugate classes \( \mathcal{R}(x), |x| = mp \), \( m \neq 1, 2, 3 \). For the centralizers, we have

\[
C(h_r, \alpha_x(1)) = \mathcal{S}_r \mathcal{X}_s \quad (h_r \in \mathcal{S}_r, h_r^2 \neq 1, h_r^3 \neq 1)
\]
\[
C(k_r, \alpha_x(1)) = \mathcal{R}_r \mathcal{X}_s \quad (k_r \in \mathcal{R}_r, k_r^2 \neq 1, k_r^3 \neq 1)
\]

\( (r = b, c \quad \text{and} \quad s = c, b) \)

which can be shown as follows:

\[
C(h_r x_0(1)) \subseteq C(h_r) \cap C(x_0(1)) = (\Phi_c \mathcal{H}_r \cup \Phi_s \mathcal{H}_r(h_r \mathcal{H}_r)) \cap C(x_0(1)).
\]
Take an arbitrary element $g = g \in \Phi_s$, in the coset $\Phi_s \subseteq \ker(h, h_s)$. Then $g^{-1}x_s(1)g = g^{-1}x_s(\lambda)g_s$, where $\lambda$ is a nonsquare in $K$; thus $g^{-1}x_s(\lambda)g_s \neq x_s(1)$ for any $g_s \in \Phi_s$. Hence no element of this coset centralizes $x_s(1)$ and we have

$$C(h, x_s(1)) \subseteq \Phi_s \subseteq \ker(h, h_s)$$

Similarly,

$$C(h, x_s(1)) \subseteq (\Phi_s \cup \Phi_s(h, h_s)) \subseteq \ker(h, h_s)$$

If $x \in G$, $|x| = 2p$, then $x$ is conjugate to one of elements $h_o x_s(1)$, $h_o x_s(1)$, $h_o x_s(1)$, or $h_o x_s(1)$, where $\lambda$ is a nonsquare in $K$.

Finally, the conjugate classes $K(x)$, $|x| = 3p$. In the case $q \equiv 1 \pmod{3}$, $x^p \sim h(m, \omega^{-1}, 1)$ or $h(\omega, m, \omega)$, and accordingly $x \sim h(\omega, \omega^{-1}, 1)$ or $x \sim h(\omega, \omega, \omega)$, where $\gamma$ is a $p$-element of $\Psi$. The subgroup $\Psi$, being isomorphic to $SL(3, K)$, has 4 conjugate classes of $p$-elements which may be represented by $x_{e_1-e_2(1)}$, $x_{e_1-e_2(1)}$, $x_{e_1-e_2(1)}$, $x_{e_1-e_2(1)}$ or $x_{e_1-e_2(1)}$, where $\mu$ is a noncube in $K$. Hence $y$ is conjugate to one of these 4 elements and we can see that $G$ has 5 conjugate classes of elements of order $3p$. Similarly, in the case $q \equiv -1 \pmod{3}$, every element of order $3p$ of $\Omega(\omega_2)$ is conjugate to $h(\omega, \omega^{-1}, 1)$ or $h(\omega, \omega, \omega)$, where $y_i \sim \Phi_{\epsilon i} y_i \in \Phi_{\epsilon i} y_i$, and $y_2, \ldots, y_5$ are representatives of 4 conjugate classes of $p$-elements of $\Psi_2 \cap \Omega(\omega_2) \simeq U(3, K)$.

This completes the determination of the conjugate classes of $p$-singular elements for all $p_0 | q^3 - 1$. The results of Section 4 are summarized in the following table. Each entry $S$ of the first column denotes a set of elements in $\Phi_{\epsilon i} y_i$ for some $\omega$. Every $p_0$-singular element of $G$ is conjugate to an element in one and only one of the sets $\Phi_{\epsilon i} y_i$, where $\Phi_{\epsilon i}$ is an isomorphism of $\Omega(\omega)$ onto $G$ such that $\Phi_{\epsilon i}(S) \subseteq G$. The second column denotes the number of conjugate classes of $G$ into which the elements of the set $\Phi_{\epsilon i}(S)$ are divided. The third column denotes the order of the centralizer of an element in $\Phi_{\epsilon i}(S)$. $\epsilon = 1$ or $-1$ according as $q \equiv 1$ or $-1 \pmod{3}$.

<table>
<thead>
<tr>
<th>$h(-1, -1, 1)$</th>
<th>1</th>
<th>$q^2(q^2 - 1)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(\omega, \omega, \omega)$</td>
<td>1</td>
<td>$q^2(q^2 - 1)(q^2 + \epsilon)$</td>
</tr>
<tr>
<td>$h(\omega, \omega^{-1}, 1)$</td>
<td>$\frac{1}{2}(q - 3)$</td>
<td>$q(q - 1)^2(q + 1)$</td>
</tr>
<tr>
<td>$z^2 = 1$, $z^2 \neq 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h(\omega, \omega^{-1}, z^2)$</td>
<td>$\frac{1}{2}(q - 4 - \epsilon)$</td>
<td>$q(q - 1)^2(q + 1)$</td>
</tr>
<tr>
<td>$z^2 = 1$, $z^2 \neq 1$, $z^3 \neq 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ h(x, x, x, x) = \frac{1}{12} (q^2 - 8q + 17 + 2\epsilon) \quad (g - 1)^2 \]

\[ z_t^{q-1} = 1, \quad x_t x_j^{q-1} \neq 1 \]

\[ h(x, x, x, 1) = \frac{1}{2}(q - 1) \quad q(g - 1)(g + 1)^2 \]

\[ z_t^{q+1} = 1, \quad x^2 \neq 1 \]

\[ h(x, x, x^2, 1) = \frac{1}{2}(q - 2 + \epsilon) \quad q(g - 1)(g + 1)^2 \]

\[ x_t^{q+1} = 1, \quad x^2 \neq 1, \quad x^3 \neq 1 \]

\[ h(x, x, x, 1) = \frac{1}{12} (q^2 - 4q + 5 - 2\epsilon) \quad (g + 1)^2 \]

\[ z_t^{q+1} = 1, \quad x_t x_j^{q+1} \neq 1 \]

\[ h(x, x^2, x^{q-1}) = \frac{1}{2}(q - 1)^2 \quad q^3 - 1 \]

\[ x_t^{q+1} = 1, \quad x^2 \neq 1 \]

\[ h(x, x^{q-1}, x^{q+1}) = \frac{1}{2}(q^2 + q - 1 - \epsilon) \quad q^2 + q + 1 \]

\[ x_t^{1+q+q^2} = 1, \quad x^3 \neq 1 \]

\[ h(x, x^{q-1}, x^{q+1}) = \frac{1}{2}(q^2 - q - 1 + \epsilon) \quad q^3 - q + 1 \]

\[ x_t^{1+q+q^2} = 1, \quad x^3 \neq 1 \]

\[ h(-1, -1, 1) \equiv (1) \quad q^2(q^2 - 1) \]

\[ h(-1, -1, 1) \equiv (1) \quad q^2(q^2 - 1) \]

\[ h(-1, -1, 1) \equiv (1) \quad 2q^2 \]

\[ h(-1, -1, 1) \equiv (1) \quad 2q^2 \]

\[ h(\omega, \omega, \omega) \equiv \Psi^* \quad q^2(q - \epsilon) \]

\[ y \in \Phi_0 \quad \text{or} \quad \Psi^* \]

\[ h(\omega, \omega, \omega) \equiv \Psi^* \quad 3q^2 \]

\[ y \in \Psi \quad \text{or} \quad \Psi_2 \cap J(\omega_2) \]

\[ h(x, x, x, 1) \equiv (1) \quad \frac{1}{2}(q - 3) \quad q(g - 1) \]

\[ z_t^{q-1} = 1, \quad x^2 \neq 1 \]

\[ h(x, x, x, 1) \equiv (1) \quad \frac{1}{2}(q - 4 - \epsilon) \quad q(g - 1) \]

\[ x_t^{q-1} = 1, \quad x^2 \neq 1, \quad x^3 \neq 1 \]

\[ h(x, x, x, 1) \equiv (1) \quad \frac{1}{2}(q - 1) \quad q(g - 1) \]

\[ x_t^{q+1} = 1, \quad x^2 \neq 1 \]

\[ h(x, x, 1) \equiv (1) \quad \frac{1}{2}(q - 2 + \epsilon) \quad q(g + 1) \]

\[ x_t^{q+1} = 1, \quad x^2 \neq 1, \quad x^3 \neq 1 \]

\[ h(x, x, x, 1) \equiv (1) \quad \frac{1}{2}(q - 1) \quad q(g + 1) \]

\[ x_t^{q+1} = 1, \quad x^2 \neq 1 \]

\[ h(x, x, x, 1) \equiv (1) \quad \frac{1}{2}(q - 2 + \epsilon) \quad q(g + 1) \]

\[ x_t^{q+1} = 1, \quad x^2 \neq 1, \quad x^3 \neq 1 \]
I am indebted to Professor Rimhak Ree for bringing the result of S. Lang to my attention. Professor Ree has obtained the character table of the groups of type $(G_2)$ for the cases $q \equiv 1 \pmod{6}$.

REFERENCES