Linear topological classification of certain function spaces on manifolds and CW complexes

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Abstract

For a Tychonoff space $X$, $C_p(X)$ denotes the space of all continuous real-valued functions on $X$ with the pointwise convergent topology. Two Tychonoff spaces $X$ and $Y$ are said to be $\ell$-equivalent if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. We prove that any compact topological $n$-manifold is $\ell$-equivalent to the $n$-disk. This complements a result of V. Valov on non-compact manifolds. Further the classification of CW complexes up to $\ell$-equivalence, originated by D. Pavlovskii, is completed by classifying noncompact CW complexes up to the above equivalence.

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1. Introduction and preliminaries

For a Tychonoff space $X$, $C_p(X)$ denotes the space of all real-valued continuous functions defined on $X$ with the pointwise convergent topology. It is a locally convex linear topological space. Two Tychonoff spaces $X$ and $Y$ are said to be $\ell$-equivalent if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. We prove that any compact topological $n$-manifold is $\ell$-equivalent to the $n$-disk. This complements a result of V. Valov on non-compact manifolds. Further the classification of CW complexes up to $\ell$-equivalence, originated by D. Pavlovskii, is completed by classifying noncompact CW complexes up to the above equivalence.

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**Theorem 1.1.** (1) [12, Theorem 2; 2, Theorem 1]. If $X$ is an $n$-dimensional compact CW complex, then $X$ is $\ell$-equivalent to the $n$-disk $D^n$.

(2) [8, Theorem 9; 2, Theorem 4]. If $X$ is an $n$-dimensional non-compact polyhedron with infinitely many $n$-simplexes, then $X$ is $\ell$-equivalent to the topological sum of countably many $n$-disks $\bigoplus D^n$.

**Theorem 1.2** [14]. Let $E = \mathbb{R}^n$, $I^\omega$ (= the Hilbert cube), $\mu^n$ (= the $n$-dimensional universal Menger compactum) or $R^\omega$ and suppose that $X$ is an $E$-manifold.

(1) If $X$ is compact and $E = I^\omega$ or $\mu^n$, then $X$ is $\ell$-equivalent to $E$.

(2) If $X$ is not compact, then $X$ is $\ell$-equivalent to $\bigoplus E$.

In the present paper, we proceed further in this direction and complete the classification for the class of CW complexes and the class of manifolds modeled on the above spaces and the Nöbeling space $N^{2n+1}_n$ (see Section 2 for the definition). The proofs rely on techniques of [1,2,8,12,14]. Throughout this paper, all spaces are assumed to be Tychonoff. A compact metric space is called a compactum.

**Definitions 1.3.** (1) For topological linear spaces $E$ and $F$, $E \sim F$ means that $E$ and $F$ are linearly homeomorphic.

(2) Suppose that topological linear spaces $E$ and $F$ have some norms $\| \|_E$ and $\| \|_F$ respectively, which are not necessarily compatible with the topologies. A linear continuous map $f : E \to F$ is said to be bounded if there exists a constant $K > 0$ such that $\|f(x)\|_F \leq K\|x\|_E$ for each $x \in E$. The spaces $E$ and $F$ are said to be norm-equivalent if there exists a linear homeomorphism $f : E \to F$ such that $f$ and $f^{-1}$ are bounded with respect to $\| \|_E$ and $\| \|_F$. The homeomorphism $f$ is called a norm equivalence.

(3) For topological linear spaces $E$ and $F$ with norms, the product space $E \times F$ is always assumed to be endowed with the norm defined by $\|(x, y)\|_{E \times F} = \max(\|x\|_E, \|y\|_F)$. Also $E_0^\omega$ denotes the linear subspace of $E^\omega$ defined by

$$E_0^\omega = \{(x_i) \in E^\omega \mid \lim_{i \to \infty} \|x_i\| = 0\}.$$

(4) For a Tychonoff space $X$, $C_p(X)$ denotes the space of all continuous real-valued functions with the pointwise convergent topology. For a closed subset $Y$ of $X$, $C_p(X|Y)$ denotes the subspace of $C_p(X)$ consisting of all functions vanishing on $Y$. When $X$ is compact, $C_p(X)$ is always assumed to be endowed with the sup norm. In general, for compact subset $K$ of $X$ and for $f \in C_p(X)$, let

$$\|f\|_K = \sup \{|f(x)| \mid x \in K\}.$$

(5) For a Tychonoff space $X$, let

$$(C_p(X))^\omega_C = \{(f_i) \in C_p(X)^\omega \mid \lim \|f_i\|_K = 0 \text{ for each compact subset } K \text{ of } X\}.$$

Notice that for a compact space $X$, $C_p(X)^\omega_C = C_p(X)^\omega_0$. 


(6) For a space $X$, $L_p(X)$ denotes the dual linear space of $C_p(X)$ with the weak topology. It is known that

$$L_p(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} \mid a_i \in \mathbb{R} \text{ and } x_i \in X \text{ for each } i, k \in \mathbb{N} \right\}.$$ 

Where $\delta_x$ denotes the Dirac measure at a point $x$. For $\eta = \sum_{i=1}^{k} a_i \delta_{x_i} \in L_p(X)$, the set $\{x_i \mid i = 1, \ldots, k\}$ is called the support of $\eta$ and denoted by $\text{supp } \eta$.

(7) Let $f : X \to Y$ be a continuous map between spaces. The map $f$ induces a linear continuous map $f^# : C_p(Y) \to C_p(X)$ defined by the formula $f^#(\varphi) = \varphi \circ f$ for each $\varphi \in C_p(Y)$.

(8) The $n$-disk is denoted by $D^n$. The convergent sequence $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ is denoted by $S$.

(9) A compactum $X$ is said to be $S$-stable if $X \times S$ is $\ell$-equivalent to $X$.

Some known results

The following definitions and results are key ingredients for the proofs in the present paper.

Definitions and Theorems 1.4. (1) Let $(X, A)$ be a pair of a space $X$ and its closed set $A$. A linear continuous map $u : C_p(A) \to C_p(X)$ is called an extension operator if $u(f)|A = f$ for each $f \in C_p(A)$. It is essentially proved by Dugundji [9] that, if $X$ is a metrizable space, such an operator $u$ exists and moreover if $X$ is a compactum, then $u$ can be chosen as a bounded operator.

(2) Suppose that $X$ is a metrizable space and $A$ and $B$ are closed sets of $X$ such that $A \subset B$. Then there exists a linear homeomorphism

$$\varphi : C_p(X|A) \to C_p(B|A) \times C_p(X|B).$$

When $X$ is a compactum, the homeomorphism $\varphi$ can be chosen to be a norm equivalence.

Definitions and Theorems 1.5. (1) Let $f : X \to Y$ be a map between spaces $X$ and $Y$. A continuous linear operator $u : C_p(X) \to C_p(Y)$ is called a regular averaging operator of $f$ if $u(1) = 1$ and $u \circ f^# = \text{id}$. It is well known (see, for example, [12, p. 591]) that $f$ admits a regular extension operator if and only if there exists a map

$$r : Y \to P_\infty(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} \in L_p(X) \mid 0 \leq a_i \leq 1 \text{ and } \sum a_i = 1 \right\}$$

such that $\text{supp } r(y) \subset f^{-1}(y)$ for any $y \in Y$. When $f$ admits a regular averaging operator, then $C_p(X) \sim C_p(Y) \times E$, where $E$ is the topological linear subspace of $C_p(X)$ defined by

$$E = \{ g - f^# \circ u(g) \mid g \in C_p(X) \}.$$
Suppose that $U$ is a locally finite open cover of a metrizable space $X$ and let $Z = \bigoplus\{\text{cl}(U) \mid U \in \mathcal{U}\}$ and $p: Z \to X$ be the canonical projection. Then $p$ admits a regular averaging operator and hence there exists a linear homeomorphism $\psi: C_p(Z) \to C_p(X) \times E$, where $E$ is a subspace of $C_p(Z)$. When $X$ and $Z$ are compact, the homeomorphism $\psi$ can be chosen to be a norm equivalence.

The proofs of both of these results can be found in [14, p. 585 and Proposition 2.13] and are omitted (the norm condition easily follows from the proofs).

**Proposition 1.6** [12, Statement (1), p. 37]. Let $X$ be a metrizable space which is the union of two closed sets $X_1$ and $X_2$. Let $X_0 = X_1 \cap X_2$. If $C_p(X_0) \sim C_p(X_0) \times C_p(X_0)$, then $C_p(X) \sim C_p(X_1) \times C_p(X_2)$.

The following observation, which follows immediately from Theorem 1.1(1), will be used repeatedly throughout the paper.

**Observation 1.7.** If $m \leq n$ are positive integers and if $\alpha \leq \beta$ are cardinals, then $C_p(D^m)^{\alpha} \times C_p(D^n)^{\beta} \sim C_p(D^n)^{\beta}$.

### 2. Function spaces on certain manifolds

Suppose that a metric space $X$ admits a locally finite countable open cover $\mathcal{U}$ each of whose member is $\ell$-equivalent to a fixed space $Y$. By [14, Proposition 2.13], $C_p(X) \sim C_p(Y)^{\omega}$ if $X$ contains a closed copy of the topological sum $\bigoplus Y$ of countably many $Y$'s. The following theorem can be regarded as an analogue of the above result for compacta.

**Theorem 2.1.** Let $X$ be a compactum and suppose that there exists a finite open cover $\mathcal{U} = \{U_i \mid i = 1, \ldots, n\}$ such that $\text{cl}(U_i)$ is $\ell$-equivalent to an $S$-stable compactum $E$. Then $X$ is $\ell$-equivalent to $E$.

Since any compact topological manifold can be covered by finitely many open disks $\text{int} D^n$-s and $D^n$ is easily seen to be $S$-stable, we have that

**Corollary 2.2.** Any compact topological $n$-manifold is $\ell$-equivalent to $D^n$.

**Remark.** The proof of Theorem 1.1(1) due to Pavlovskii applies if a topological manifold admits a handlebody decomposition, that is, unless the manifold is a 4-dimensional nonsmoothable manifold [10, p. 136]. The proof of Theorem 2.1 presented here does not rely on such a structure theorem.

Theorem 2.1 easily generalizes to locally compact metric spaces as follows.

**Corollary 2.3.** Suppose that $X$ is a locally compact metrizable space and $E$ is an $S$-stable compactum. If each point $x$ of $X$ has an open neighborhood $U_x$ such that $\text{cl}(U_x)$ is $\ell$-equivalent to $E$, then
Proof. Decompose $X$ into the topological sum of locally compact separable metric spaces $\bigoplus X_\alpha$, where $\alpha < w(X)$. If $X_\alpha$ is compact, $X_\alpha$ is $\ell$-equivalent to $E$ by Theorem 2.1, whereas if $X_\alpha$ is not compact, $X_\alpha$ is $\ell$-equivalent to the topological sum of countably many $E$'s by [14, Proposition 2.13]. Then the conclusion easily follows. □

The scheme of the proof is similar to the one of [14, Proposition 2.13]. The compactness of $X$ prevents the countable topological sum of $E$ from being embedded into $X$ as a closed set, so it is not clear whether $C_p(X)$ contains $C_p(E)^\omega$ as a linear subspace. We avoid this difficulty by replacing $C_p(E)^\omega$ with $C_p(E)$; and apply the Bessaga–Pełczyński scheme with an examination of the norm conditions. The following lemmas are used for this purpose.

**Lemma 2.4** [1, Theorem 1]. Let $Y$ and $Z$ be compacta and suppose that there exists a linear isomorphism $\varphi: C_p(Y) \to C_p(Z)$. Then $\varphi$ is a norm equivalence.

**Lemma 2.5.** For topological linear spaces $E$ and $F$ with norms $\| \cdot \|_E$ and $\| \cdot \|_F$, we have the following.

1. If there exists a norm equivalence $\varphi: E \to F$ with respect to $\| \cdot \|_E$ and $\| \cdot \|_F$, then $E_0^\omega$ and $F_0^\omega$ are linearly homeomorphic.

2. Let $E \times F$ be endowed with the norm $\| \cdot \|_{E \times F} = \max\{\| \cdot \|_E, \| \cdot \|_F\}$. Then $E_0^\omega \times F_0^\omega$ is linearly homeomorphic to $(E \times F)_0^\omega$.

The proofs are easy and are omitted.

**Lemma 2.6.** For compacta $Y$, $Z$ and $E$, we have the following.

1. $C_p(Y \times S) \sim C_p(Y)_n^\omega$.

2. [2, Proposition 19]. If $Y$ is $\ell$-equivalent to $Z$, then $Y \times S$ is $\ell$-equivalent to $Z \times S$.

3. [2, Proposition 20]. $Y \times S$ is $S$-stable.

4. If $E$ is $S$-stable, then $C_p(E) \sim C_p(E)^n$ for each $n \in \mathbb{N}$.

**Comments on proofs.** The statement (1) is essentially proved in [14, Lemma 2.5]. The statement (2) is an easy consequence of (1), Lemma 2.4 and Lemma 2.5(1). The equivalence of (4) follows from below and an easy induction.

$$C_p(E) \times C_p(E) \sim C_p(E \times S) \times C_p(E) \sim C_p(E \times S \oplus E) \sim C_p(E \times S) \sim C_p(E).$$

**Lemma 2.7** [2, Proposition 23]. Suppose that $Y$ is a compactum which contains a closed set $A$ which is $\ell$-equivalent to $Y \times S$. Then $Y$ is $S$-stable.

**Proof of Theorem 2.1.** Suppose that $X$ and $E$ are as in the hypothesis of the theorem. First we notice that the following claim implies the conclusion.
Claim. $X \times S$ is $\ell$-equivalent to $E \times S$.

For, since $\text{cl}(U_1)$ is $\ell$-equivalent to $E$ which is $S$-stable, the above claim implies that $\text{cl}(U_1)$ is $\ell$-equivalent to $X \times S$. By Lemma 2.7, $X$ is $S$-stable. Therefore, $C_p(X) \sim C_p(X \times S) \sim C_p(E \times S) \sim C_p(E)$, completing the proof.

Proof of Claim. The isomorphy between $C_p(X \times S)$ and $C_p(E \times S)$ is established by a sequence of linear homeomorphisms. The following diagram indicates the sequence. The definitions of each linear homeomorphisms will follow after the diagram.

\[
\begin{array}{c}
C_p(X \times S) \\
\downarrow \varphi_1 \\
C_p(X \times S)_0^\omega \\
\downarrow \varphi_2 \\
(C_p(E \times S) \times P)_0^\omega \\
\downarrow \varphi_3 \\
(C_p(E \times S)_0^\omega \times (C_p(E \times S) \times P)_0^\omega \\
\downarrow \varphi_4 \\
(C_p(E \times S)_0^\omega \times (C_p(E \times S) \times P)_0^\omega \\
\downarrow \varphi_5 \\
(C_p(E \times S)_0^\omega \times (C_p(E \times S) \times P)_0^\omega \\
\downarrow \varphi_6 \\
(C_p(E \times S)_0^\omega \times (C_p(E \times S) \times P)_0^\omega \\
\end{array}
\]

$\varphi_1$ and $\varphi_1$: Since $X \times S$ and $E \times S$ are $S$-stable by Lemma 2.6(3), the existence of these isomorphisms follows from Lemma 2.6(1).

$\varphi_2$ and $\varphi_2$: $X \times S$ contains $\text{cl}(U_1) \times S$ which is $\ell$-equivalent to $E \times S$ by Lemma 2.6(2). Applying Proposition 1.5(1) and Lemma 2.4, we have that $C_p(X \times S)$ is norm equivalent to $C_p(E \times S) \times C_p(X \times S)$. Let $P = C_p(X \times S) \times \text{cl}(U_1) \times S$. Then Lemma 2.5(1) implies the existence of the linear homeomorphism $\varphi_2$.

Recall that $X = \bigcup_{i=1}^k U_i$ and $\text{cl}(U_i)$ is $\ell$-equivalent to $E$. Let $Z = \bigoplus \text{cl}(U_i)$ and take the canonical projection $p : Z \to X$. Applying Proposition 1.5(2) to $p \times \text{id}_S : Z \times S \to X \times S$ and the cover $\{U_i \times S \mid i = 1, \ldots, n\}$, we see that $C_p(Z \times S)$ is norm equivalent to $C_p(Z \times S) \times Q$ for some topological linear subspace $Q$ of $C_p(Z \times S)$. Thus an application of Lemma 2.5(1) implies the existence of the isomorphism $\psi_2$.

$\varphi_3$ and $\varphi_3$: Apply Lemma 2.5(2).

$\varphi_4$ and $\varphi_4$: The isomorphism $\varphi_4$ is of the form $\varphi_4 \times \text{id}_P$, where

$\varphi_4 : C_p(E \times S)_0^\omega \to C_p(E \times S)_0^\omega \times C_p(E \times S)_0^\omega$
is defined by the formula:
\[ \varphi_4(f_1, f_2, f_3, \ldots) = ((f_1, f_3, f_5, \ldots), (f_2, f_4, \ldots)). \]
This is clearly a linear homeomorphism. The same construction yields the isomorphism \( \psi_4 \).

This completes the proof of Theorem 2.1. \( \square \)

Chigogidze [6] introduced the notion of the pseudo-interior \( \nu^n \) of the universal Menger compactum \( \mu^n \) and it is proved in [7] that \( \nu^n \) is homeomorphic to the Nöbeling space \( N_{n}^{2n+1} \) consisting of all points of \( \mathbb{R}^{2n+1} \) at most \( n \)-coordinates of which are rational. There are several reasons that we may expect the theory of Nöbeling manifolds as a finite dimensional analogue of the Hilbert space manifold theory. The following theorem is a counterpart to the results [14, Theorems 2.14 and 2.9] for the Nöbeling space. The crucial point is the following property of \( N_{n}^{2n+1} \) (see [7] for a proof), where Polish spaces refer to separable completely metrizable spaces.

Any at most \( n \)-dimensional Polish space can be embedded into \( N_{n}^{2n+1} \) as a closed set.

**Theorem 2.8.** Let \( X \) be a metrizable space. Then \( X \) is \( \ell \)-equivalent to the space \( N_{n}^{2n+1} \) if and only if \( X \) is an \( n \)-dimensional Polish space which contains a closed copy of \( N_{n}^{2n+1} \).

**Proof.** The proof is basically the same as the one of [14, Theorem 2.9]. Suppose first that \( X \) is \( \ell \)-equivalent to \( N_{n}^{2n+1} \). By [5], \( X \) is completely metrizable. Since the density of a space is preserved under \( \ell \)-equivalence (see, for example, [3, I.1.6]), \( X \) is separable. Moreover \( \dim X = n \) by [13]. Proposition 2 of [12] implies that there exists an open set of \( N_{n}^{2n+1} \) which is embedded into \( X \), but any open set of \( N_{n}^{2n+1} \) contains a closed copy of \( N_{n}^{2n+1} \). It is easy to see that we can take such copy as a closed set of \( X \).

For a proof of another implication, we show that
\[ C_p(N_{n}^{2n+1}) \cong C_p(N_{n}^{2n+1})_{\omega}, \]
and appeal to [14, Lemma 2.7]. Since \( N_{n}^{2n+1} \times S \) is an \( n \)-dimensional Polish space, it is embedded into \( N_{n}^{2n+1} \) as a closed set. The following lemma, which finishes the proof of Theorem 2.8, is essentially due to [14, Theorem 2.9], and we omit the proof.

**Lemma 2.9.** Let \( X \) be a metrizable space and suppose that \( X \times S \) is embedded into \( X \) as a closed set. Then
\[ C_p(X) \cong C_p(X)_{\omega}. \]

This completes the proof. \( \square \)

**Remark.** Any \( n \)-dimensional compactum can be embedded into the universal Menger compactum \( \mu^n \). Combining this fact with the above argument, we can avoid the use of Dranishnikov's map for the proof of [14, Theorem 2.9].
3. Function spaces on infinite CW complexes

In this section, we are concerned with $C_p(X)$ when $X$ is an infinite CW complexes. The first proposition, which is of independent interest, will be used to perform induction steps.

A proper map $f : X \to Y$ is a closed map such that $f^{-1}(K)$ is compact for each compact subset $K$ of $Y$. For a proper map $f : X \to Y$ between locally compact metrizable spaces, the mapping cylinder $M_f$ is defined by

$$M_f = X \times [0, 1] \amalg Y / (x, 0) \sim f(x), \quad x \in X.$$ 

It is easy to see that $M_f$ is metrizable.

**Proposition 3.1.** For any proper map $f : X \to Y$ between locally compact metrizable spaces, we have that

$$C_p(M_f) \approx C_p(X \times [0, 1]) \times C_p(Y).$$

**Proof.** Since $Y$ is closed in $M_f$, we have that

$$C_p(M_f) \approx C_p(Y) \times C_p(M_f|Y)$$

by Proposition 1.5(1). It is easy to see that

$$C_p(M_f|Y) \approx C_p(X \times [0, 1]|X \times 1) \approx C_p(X \times [0, 1]|X \times 0) \quad (2)$$

and by Proposition 1.5(1) again, we have that

$$C_p(X \times [0, 1]|X \times 0) \approx C_p(X \times [0, 1]|X \times S) \approx C_p(X \times S|X \times 0). \quad (3)$$

The proof of [14, Theorem 2.5] is applied to prove that

$$C_p(X \times [0, 1]|X \times S) \approx C_p(X \times [0, 1]|X \times \{0, 1\})^\omega, \quad (4)$$

and

$$C_p(X \times S|X \times 0) \approx C_p(X)^\omega. \quad (5)$$

By [14, Lemma 2.3], we see that

$$C_p(X \times [0, 1])^\omega \approx C_p(X \times [0, 1]|X \times \{0, 1\})^\omega \approx C_p(X)^\omega. \quad (6)$$

Furthermore,

$$C_p(X \times [0, 1]) \approx C_p(X \times [0, 1])^\omega \approx C_p(X)^\omega \quad (7)$$

by [14, Theorem 2.5].

The conclusion now easily follows by using (1)–(7) in this order. □

Our first goal is to prove Theorem 3.3 which is a generalization of Theorem 1.1 to noncompact CW complexes.

**Definition 3.2.** A cell $e$ of a CW complex $X$ is called a principal cell if $e$ is not contained in the closure of any other cell of $X$. For a CW complex $X$, $\mathcal{P}_i(X)$ denotes the set of all principal $i$-cells. Let $\tau_i(X)$ be

$$\tau_i(X) = \begin{cases} |\mathcal{P}_i(X)| & \text{if } |\mathcal{P}_i(X)| \geq \omega, \\ 1 & \text{if } |\mathcal{P}_i(X)| < \omega. \end{cases}$$
Theorem 3.3. Let $X$ be an $n$-dimensional CW complex ($n \geq 1$). Then

$$C_p(X) \sim \prod_{i=1}^{n} C_p(D^i)^{\tau_i(X)}.$$  

The proof proceeds by an induction on $n$. Theorem 1.1(1) deals with the compact case. First we prove the theorem when $n = 1$. Notice that a noncompact 1-dimensional CW complex is an infinite graph.

Lemma 3.4. Let $\tau \geq \omega$ be a cardinal and let $I_\tau = \bigvee_{\alpha < \tau} [0, 1]_\alpha$ be the quotient space obtained from $\bigoplus_{\alpha < \tau} [0, 1]_\alpha$ by identifying all 0's into one point. Then

$$C_p(I_\tau) \sim C_p(D^1)^{\tau}.$$  

Proof. Let 0 be the point of identification of $I_\tau$ and let $D(\tau)$ be the discrete space with the cardinality $\tau$. Clearly,

$$C_p(I_\tau) \sim C_p(0) \times C_p(I_\tau | 0),$$  

and it is easy to see that

$$C_p(I_\tau | 0) \sim C_p([0, 1] \times D(\tau)) \sim C_p((0, 1])_0^{\tau}. \hspace{1cm} (2)$$

Let $I = [0, 1]$ and $J = [1/2, 1]$, and let $i: J \to I$ be the inclusion. Clearly there exists an extension operator $u: C_p(J) \to C_p(I|0)$ and this implies by Definitions and Theorems 1.4 that

$$C_p(I|0) \sim C_p(J) \times E \text{ for some topological linear subspace } E \text{ of } C_p(I|0). \hspace{1cm} (3)$$

Notice also that

$$C_p(J) \sim C_p(J \oplus \text{a point}) \hspace{1cm} (\text{by Theorem 1.1(1)})$$

$$\sim C_p(J) \times C_p(0). \hspace{1cm} (4)$$

Therefore we have the following sequence of isomorphisms.

$$C_p(I_\tau) \sim C_p(0) \times C_p(I_\tau | 0) \hspace{1cm} (1)$$

$$\sim C_p(0) \times C_p(J)^{\tau} \times E^{\tau} \hspace{1cm} (2) \text{ and (3)}$$

$$\sim C_p(J)^{\tau} \times C_p(0)^{\tau} \times E^{\tau} \hspace{1cm} (4)$$

$$\sim C_p(I|0)^{\tau} \times C_p(0)^{\tau} \hspace{1cm} (3)$$

$$\sim C_p(I)^{\tau}. \hspace{1cm} (5)$$

This completes the proof. $\Box$

Lemma 3.5. If $\dim X = 1$, then Theorem 3.3 holds.

Proof. The scheme of the proof is similar to the one of [14, Proposition 2.13]. Since $\dim X = 1$, it is easy to get a triangulation $K$ of $X$ so that the closure of each 1-cell
of $X$ contains at most three 1-simplexes of $K$. For a vertex $v$ of $K$, let $\text{st}(v, K)$ be the open star of $v$. The key for the proof is the fact that the collection $\{\text{st}(v, K) \mid v \in K^{(0)}\}$ forms a locally finite open cover of $X$ (see, for example, [11, Corollary 3, p. 293]). Let $S_v = \text{cl} \text{st}(v, K)$ and let $\tau(v)$ be the cardinality of the set of all 1-simplexes containing $v$. Then $S_v$ is homeomorphic to $I_{\tau(v)}$. Let $Z = \bigoplus S_v$ be the topological sum of $S_v$'s and take the canonical projection $p: Z \to X$. By Proposition 1.5(2), we have that

(1) $C_p(Z) \sim C_p(X) \times E$ for some topological linear space $E$.

Let $\{\sigma_\alpha \mid \alpha < \tau\}$ be the collection of all 1-cells of $X$, all of which are principal. “Shrink” each $\sigma_\alpha$ into $s_\alpha$ to form a closed discrete collection $\{s_\alpha \mid \alpha < \tau\}$. It is easy to see that there exists an extension operator $u: C_p(\bigcup_{\alpha < \tau} s_\alpha) \to C_p(X)$, which implies by Definitions and Theorems 1.4 that

(2) $C_p(X) \sim C_p(\bigoplus_{\alpha < \tau} s_\alpha) \times C_p(X|\bigcup_{\alpha < \tau} s_\alpha)$.

We show next that

(3) $C_p(Z) \sim C_p(\bigoplus_{\alpha < \tau} s_\alpha)$.

**Proof of (3).** By Lemma 3.4, $S_\alpha$ is $\ell$-equivalent to $\bigoplus_{\tau(v)} I$ and hence $Z$ is $\ell$-equivalent to $\bigoplus_v \bigoplus_{\tau(v)} I$. In this sum, each $I$ corresponds to an 1-simplex of $K$ and each 1-simplex of $K$ “appears” at most twice under this correspondence. Since there are infinitely many 1-simplexes, we see that $\bigoplus_v \bigoplus_{\tau(v)} I$ is homeomorphic to the topological sum of $|K^{(1)}|$-many $I$'s. As each 1-cell of the original CW complex structure of $X$ contains at most three 1-simplexes of $K$, we see that $Z$ is $\ell$-equivalent to $\bigoplus_{\alpha < \tau} s_\alpha$. 

Let $F = C_p(Z)$. Since $\tau_1(X) \geq \omega$, we have that $F^\omega \sim F$. We are now in the situation to which the Bessage–Pelczyński method is applied as follows.

$$C_p(X) \sim C_p\left(\bigoplus_{\alpha} s_\alpha\right) \times C_p\left(X | \bigcup_{\alpha < \tau} s_\alpha\right)$$

$$\sim F \times C_p\left(X | \bigcup_{\alpha < \tau} s_\alpha\right)$$

$$\sim F^\omega \times F \times C_p\left(X | \bigcup_{\alpha < \tau} s_\alpha\right)$$

$$\sim F^\omega \times C_p(X)$$

$$\sim C_p(X)^\omega \times E^\omega \times C_p(X)$$

$$\sim C_p(X)^\omega \times E^\omega$$

$$\sim F^\omega \sim F.$$

This completes the proof. 

**Proof of Theorem 3.3.** The proof proceeds by an induction. We may assume that $X$ is not compact. The $n = 1$ case is proved in Lemma 3.5. We assume that theorem holds for all CW complexes of dimension $\leq n - 1$.

Suppose that $X$ is an $n$-dimensional infinite CW complex and let $\{\sigma_\alpha \mid \alpha < \tau_n(X)\}$ be the collection of all $n$-cells of $X$. As in Lemma 3.5, shrink each $\sigma_\alpha$ into $s_\alpha$ so that
{s_\alpha \mid \alpha < \tau_n(X)} forms a closed discrete collection. Let \( S^{n-1}_\alpha \) be the \((n-1)\) sphere indexed by \( \alpha < \tau_n(X) \) and fix a homeomorphism \( f_\alpha : S^{n-1}_\alpha \rightarrow \partial s_\alpha \). The collection \( \{f_\alpha\} \) defines a map \( f : \bigoplus S^{n-1}_\alpha \rightarrow X \). Define \( Y \) and \( Z \) by \( Y = X^{(n-1)} \) and \( Z = X - \bigcup s_\alpha \), and observe that \( Z \) is the mapping cylinder of \( f \). Applying Proposition 3.1, we have that

\[
C_p(Z) \sim C_p\left( \bigoplus S^{n-1}_\alpha \times I \right) \times C_p\left( X^{(n-1)} \right).
\]

The subset \( Y \) and \( Z \) are closed in \( X \) and \( Y \cap Z = \bigoplus \partial s_\alpha \). By Theorem 1.1(1),

\[
C_p\left( \bigoplus \partial s_\alpha \right) \times C_p\left( \bigoplus \partial s_\alpha \right) \sim \prod_{\alpha < \tau} C_p(\partial s_\alpha) \times C_p(\partial s_\alpha)
\]

\[
\sim \prod_{\alpha < \tau} C_p(\partial s_\alpha) \sim C_p\left( \bigoplus \partial s_\alpha \right).
\]

Now apply Proposition 1.6 and we have that

\[
C_p(X) \sim C_p'(Y) \times C_p(Z).
\]

Let \( \sigma_i \) be the cardinality of the set of all principal \( i \)-cells of \( X^{(n-1)} \) which is contained in the closure of an \( n \)-cell of \( X \). It is easy to see that, for each \( i \leq n-1 \), we have that

\[
\tau_i(X) + \sigma_i = \tau_i\left( X^{(n-1)} \right) \text{ and } \sigma_i = \tau_n(X).
\]

(Use the closure finite condition for the last equality.) The desired equivalence is obtained as follows.

\[
C_p(X) \sim C_p\left( \bigoplus s_\alpha \right) \times C_p'(Z)
\]

\[
\sim C_p(D^n)^{\tau_n(X)} \times C_p(D^n)^{\tau_n(X)} \times C_p\left( X^{(n-1)} \right)
\]

\[
\sim C_p(D^n)^{\tau_n(X)} \times \prod_{i=1}^{n-1} C_p(D^i)^{\sigma_i + \tau_i(X)}
\]

\[
\sim \prod_{i=1}^{n} C_p(D^i)^{\tau_i(X)} \quad \text{(apply Observation 1.7 here)}.
\]

This completes the proof. \( \square \)

The expression of \( C_p(X) \) in Theorem 3.3 may not be “irreducible” in the sense that Observation 1.7 can be used to “cancel” some terms \( C_p(D^i)^{\tau_i(X)} \). The irreducible expression is defined in the following way.

Let \( \alpha_1 = \max\{\tau_i(X) \mid i = 1, \ldots, n\} \) and let \( i_1 = \max\{s \mid \tau_s(X) = \alpha_1\} \). Recursively define \( \alpha_{t+1} = \max\{\tau_i(X) \mid i > i_t\} \) and \( i_{t+1} = \max\{s \mid s > i_t \text{ and } \tau_s(X) = \alpha_{t+1}\} \).

Then repeated applications of Observation 1.7 show that

\[
\prod_{i=1}^{n} C_p(D^i)^{\tau_i(X)} \sim \prod_{s=1}^{k} C_p(D^{i_s})^{\alpha_s}, \quad (#)
\]

where \( i_k = n \) and \( \alpha_s > \omega \) for each \( s < k - 1 \). The sequence \( ((\alpha_s); (i_s)) \) is called the irreducible sequence of \( X \) and the expression (\#) is called the irreducible expression of
$C_p(X)$. Notice that $i_1 < i_2 < \cdots < i_k$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_k$. The next theorem completes the classification of function spaces on finite dimensional CW complexes.

**Theorem 3.6.** Let $((i_s); (\alpha_s))$ and $((j_s); (\beta_s))$ be two irreducible sequences. Then

$$
\prod_{s=1}^{k} C_p(D^i_s)^{\alpha_s} \sim \prod_{t=1}^{\ell} C_p(D^j_t)^{\beta_t}
$$

if and only if $k = \ell$, $i_s = j_s$ and $\alpha_s = \beta_s$ for each $s = 1, \ldots, k$.

**Proof.** Let $\Delta(i_s; \alpha_s) = \bigoplus_{\alpha_s} D^i_s$ and for simplicity, let $X_s = \Delta(i_1; \alpha_1) \oplus \cdots \oplus \Delta(i_s; \alpha_s)$ and $Y_t = \Delta(j_1; \beta_1) \oplus \cdots \oplus \Delta(j_t; \beta_t)$. Suppose that there exists a linear homeomorphism $\theta : C_p(X_k) \rightarrow C_p(Y_\ell)$.

First of all, since $\dim X_k = \dim Y_\ell$, it follows that $i_k = j_\ell$. We prove that

$$\alpha_k = \beta_\ell. \quad (1)$$

Let $\theta^* : L_p(Y_\ell) \rightarrow L_p(X_k)$ be the dual isomorphism. Define a subset $L$ of $X_k$ by

$$L = \bigcup \{ \text{supp} (\theta^*(\delta_y)) \mid y \in \Delta(j_\ell; \beta_\ell) \} \subset X_k.$$ 

For any dense set $D$ of $\Delta(j_\ell; \beta_\ell)$, the set $\bigcup \{ \text{supp} (\theta^*(\delta_y)) \mid y \in D \}$ is dense in $L$ by the continuity of $\theta^*$. Hence, $d(L) \leq d(\Delta(j_\ell; \beta_\ell))$, where $d(E)$ denotes the density of $E$. Let $P = \bigcup \{ D^j \mid D^j \cap L \neq \emptyset \}$. Since any disk is separable, $d(L) = d(P)$. This means that

$$d(P) \leq d(\Delta(j_\ell; \beta_\ell)) = \beta_\ell.$$ 

Now we proceed as in [14, Theorem 2.16]. The same argument as the one in [14, p. 593, lines 23–35], one can prove that

$$C_p(Y_\ell/\Delta(j_\ell; \beta_\ell)) \sim C_p(X_k/P) \times E$$

for some topological linear space $E$.

Notice that $Y_\ell/\Delta(j_\ell; \beta_\ell)$ and $X_k/P$ are locally compact metrizable spaces. The proof of [8, Theorem 6] works in this situation to conclude that

$$\dim X_k/P \leq \dim Y_\ell/\Delta(j_\ell; \beta_\ell) = j_\ell - 1.$$ 

The above inequality forces $P$ to contain $\Delta(i_k = j_\ell; \alpha_k)$ and hence $d(P) \geq \alpha_k$. Therefore $\alpha_k \leq \beta_\ell$ and by symmetry $\alpha_k \geq \beta_\ell$, so the equality holds.

Next we prove that

$$C_p(X_{k-1}) \sim C_p(Y_{\ell-1}). \quad (2)$$

The above proof of (1) shows that there exists a subset $P$ of $X_k$ containing $\Delta(i_k = j_\ell; \alpha_k = \beta_\ell)$ with $d(P) = \alpha_k$ such that $C_p(Y_{\ell-1}) \sim C_p(X_k/P) \times E$. Recalling that $\alpha_k > \alpha_{k-1} > \cdots > \alpha_1$, we see that $C_p(X_k/P) \sim C_p(X_{k-1})$. Thus

$$C_p(Y_{\ell-1}) \sim C_p(X_{k-1}) \times E$$

for some topological linear space $E$. Therefore

$$C_p(X_{k-1}) \sim C_p(Y_{\ell-1}).$$

The above proof of (1) shows that there exists a subset $P$ of $X_k$ containing $\Delta(i_k = j_\ell; \alpha_k = \beta_\ell)$ with $d(P) = \alpha_k$ such that $C_p(Y_{\ell-1}) \sim C_p(X_k/P) \times E$. Recalling that $\alpha_k > \alpha_{k-1} > \cdots > \alpha_1$, we see that $C_p(X_k/P) \sim C_p(X_{k-1})$. Thus
for some topological linear space $E$, and by the symmetry again, $C_p(X_{k-1}) \sim C_p(Y_{\ell-1}) \times F$ for some topological linear space $F$. Since $\alpha_{k-1} \geq \omega$ and $\beta_{\ell-1} \geq \omega$, we see that

$$C_p(Y_{\ell-1})^\omega \sim C_p(Y_{\ell-1})$$
and $C_p(X_{k-1})^\omega \sim C_p(X_{k-1})$.

Now apply the Bessaga–Pelczyński scheme as follows.

$$C_p(X_{k-1}) \sim C_p(Y_{\ell-1}) \times F$$
$$\sim C_p(Y_{\ell-1})^\omega \times C_p(Y_{\ell-1}) \times F$$
$$\sim C_p(Y_{\ell-1})^\omega \times C_p(X_{k-1})$$
$$\sim C_p(X_{k-1})^\omega \times E^\omega \times C_p(X_{k-1})$$
$$\sim C_p(X_{k-1})^\omega \times E^\omega \sim C_p(Y_{\ell-1})^\omega$$
$$\sim C_p(Y_{\ell-1}).$$

This proves (3) and a downward induction completes the proof. □

It would be worth mentioning a special case when $X$ is a locally compact finite dimensional polyhedron. Any such polyhedron $X$ is a countable complex and $\tau(X) = 1$ or $\omega$. An application of Observation 1.7 simplifies the expression of $C_p(X)$ as follows.

**Corollary 3.7.** Let $X$ be a locally compact $n$-dimensional polyhedron and let $p(X) = \max\{i \mid \text{there are infinitely many principal } i\text{-simplexes}\}$. Then

$$C_p(X) \sim C_p(D^n)^\omega \times C_p(D^n).$$

Two locally compact finite dimensional polyhedra $X$ and $Y$ are $\ell$-equivalent if and only if $\dim X = \dim Y$ and $p(X) = p(Y)$.

**Remark.** It follows from [14, Corollary 2.6] that $C_p(D^n)$ is not linearly homeomorphic, but is homeomorphic to $C_p(D^n)^\omega$. This implies that, if $k \leq n$, $C_p(D^k)^\omega \times C_p(D^n)$ is homeomorphic to

$$C_p(D^k)^\omega \times C_p(D^n)^\omega \sim (C_p(D^k) \times C_p(D^n))^\omega \sim C_p(D^n)^\omega$$

(by Observation 1.7) and the last term is homeomorphic to $C_p(D^n)$. This observation and the above corollary imply that, for any locally compact $n$-dimensional polyhedron $X$, $C_p(X)$ is homeomorphic to $C_p(D^n)$.

Next we deal with infinite dimensional CW complexes. Recall that for any CW complex $X$, $X = \text{dirlim}(X(1) \rightarrow X(2) \rightarrow \cdots)$, where all bonding maps are inclusions. For an infinite dimensional CW complex $X$, let $\alpha_i(X)$ be the cardinality of the set of all $i$-cells which are contained in the closures of infinitely many cells. Then $C_p(X)$ can be expressed as follows.
Theorem 3.8. For an infinite dimensional CW complex $X$,

$$C_p(X) \sim \prod_{i=1}^{\infty} C_p(D^i)^{\tau_i(X)} \times \prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i(X)}.$$  

Proof. By the above remark, we see that

$$C_p(X) \sim \varprojlim \left( C_p(X^{(1)}) \leftarrow i_k^* C_p(X^{(2)}) \leftarrow \cdots \right),$$  \hspace{1cm} (1)

where $i_k : X^{(k)} \to X^{(k+1)}$ is the inclusion.

Let $\Sigma_n$ be the set of all $n$-cells of $X$ and let $\sigma_n = |\Sigma_n|$. The set $\Sigma_n$ is decomposed into three sets as follows.

$$\Sigma_n = A_n \cup B_n \cup P_n,$$

where $A_n$ is the set of all $n$-cells which are contained in the closures of infinitely many cells, $B_n$ is the set of all $n$-cells which are contained in the closure of a principal cell of dim $\geq n + 1$, and $P_n$ is the set of all principal $n$-cells.

Let $\beta_n = |B_n|$, then $\sigma_n = \alpha_n(X) + \beta_n + \tau_n(X)$.

A careful examination of the proof of Theorem 3.3 (the induction step) reveals that there exists a linear homeomorphism

$$\varphi_n : C_p(X^{(n+1)}) \to C_p(X^{(n)}) \times C_p(D^{n+1})^{\sigma_n}$$

which makes the above diagram commutative. This implies that

$$C_p(X) \sim \prod_{i=1}^{\infty} C_p(D^i)^{\tau_i(X)} \times \prod_{i=1}^{\infty} C_p(D^i)^{\beta_i} \times \prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i(X)}.$$  

By Observation 1.7 again, the second term of the last formula can be "cancelled" as follows. Let $B_{ij}$ ($i < j$) be the set of all $i$-cells of $B_i$ which is contained in the closure of a principal $j$-cell. Clearly $B_i = \bigcup_{j > i} B_{ij}$ and let $\beta_{ij} = |B_{ij}|$. Then

$$\prod_{i=1}^{\infty} C_p(D_i)^{\tau_i(X)} \times \prod_{i=1}^{\infty} C_p(D_i)^{\beta_i} \sim \prod_{i=1}^{\infty} \left( C_p(D_i)^{\tau_i(X)} \times \prod_{k < i} C_p(D_k)^{\beta_{ki}} \right)$$

$$\sim \prod_{i=1}^{\infty} C_p(D_i)^{\tau_i(X)} \text{ by Observation 1.7.}$$

Therefore we have the desired conclusion. \qed
Let us specialize the above result to polyhedra. Let $h_i : \sigma^i \to \sigma^{i+1}$ be the standard embedding of the $i$-simplex into the $(i+1)$-simplex as one of the faces and let

$$\sigma^\infty = \text{dirlim} \left( \sigma^1 \overset{h_1}{\to} \sigma^2 \overset{h_2}{\to} \cdots \right).$$

If $X$ is an infinite dimensional polyhedron, regarded as a CW complex in the standard manner, then it is clear that $\alpha_n(X) = \text{cardinality of the set of all } n\text{-simplexes which is contained in an “infinite simplex” } \sigma^\infty$. It is then easy to see that $\alpha_n(X)$ does not depend on $n$ and denoted by $\alpha(X)$. From Theorem 3.8, we have that

**Corollary 3.9.** If $X$ is an infinite dimensional polyhedron then

$$C_p(X) \sim \prod_{i=1}^{\infty} C_p(D^i)^{\tau_i(X)} \times \prod_{i=1}^{\infty} C_p(D^i)^{\alpha(X)}.$$

**Remark.** It is easy to construct an infinite dimensional countable polyhedron $K$ such that $\tau_i(K) = 0$ for each $i$ and $K$ contains $c$-many $\sigma^\infty$'s. According to the above corollary,

$$C_p(K) \sim \prod_{i=1}^{\infty} C_p(D^i)^{\omega} \sim C_p(\sigma^\infty)$$

and the last term is not linearly homeomorphic to $C_p(\sigma^\infty)^c$. (This follows from an examination of the densities of the underlying spaces. See, for example, [3, 1.1.6].) This example indicates that in Corollary 3.9, one cannot replace the second term with $C_p(\sigma^\infty)^{\sigma(X)}$, where $\sigma(X)$ is the cardinality of the set of all $\sigma^\infty$'s.

It remains to classify the space $\prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i}$ and this is performed in the next theorem. In statements (1) and (2) below, we deal with special cases and the statement (3) claims that these two cases exhaust all the possibility. For a sequence $(\alpha_i)$ of cardinals, take a sequence of sets $(A_i)$ such that $|A_i| = \alpha_i$. Let $\mathbf{\alpha}_i^\infty = \bigoplus_{j \geq i} A_j$.

**Theorem 3.10.** (1) For two sequences $(\alpha_i)$ and $(\beta_i)$ of infinite cardinals,

$$\prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i} \sim \prod_{i=1}^{\infty} C_p(D^i)^{\beta_i} \text{ if and only if } \alpha_i^\infty = \beta_i^\infty \text{ for each } i.$$

(2) $\prod_{i=1}^{\infty} C_p(D^i) \sim \prod_{i=1}^{\infty} C_p(D^i)^{\omega}$.

(3) For any sequence $(\alpha_i)$ of cardinals, the space $\prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i}$ is linearly homeomorphic to a space of the type (1) above.

**Proof.** Let $\Delta(i, \alpha_i) = \bigoplus_{i} A_i^i$, then $C_p(\bigoplus \Delta(i, \alpha_i)) \sim \prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i}$.

(1) Suppose first that $\bigoplus \Delta(i, \alpha_i)$ and $\bigoplus \Delta(i, \beta_i)$ are $\ell$-equivalent. We show that $\alpha_i^\infty = \beta_i^\infty$ for each $i$. The argument is similar to the one of Theorem 3.6. We sketch the outline of the proof. Take a linear homeomorphism $\theta : C_p(\bigoplus \Delta(i, \alpha_i)) \to C_p(\bigoplus \Delta(i, \beta_i))$ and let $K = \bigoplus_{j=1}^{\infty} \Delta(j, \alpha_j)$ and $L = \bigcup \{ \text{supp} \theta^*(\delta_y) \mid y \in K \}$. As in Theorem 3.6, $d(L) \leq d(K)$. Let $P$ be the union of all $D^j$'s of $\bigoplus \Delta(i, \beta_i)$ which intersect $L$, then we have
that $d(P) = d(L) \leq d(K)$ and further there exists a topological linear space $E$ such that

$$C_p\left(\bigoplus_{j=1}^{i-1} \Delta(j, \alpha_j)\right) \sim C_p\left(\bigoplus_{j=1}^{\infty} \Delta(j, \alpha_j) / K\right) \sim C_p\left(\bigoplus_{j=1}^{\infty} \Delta(j, \beta_j) / P\right) \times E.$$ 

Apply (the proof of) [8, Theorem 6] again to show that

$$\dim \bigoplus_{j=1}^{\infty} \Delta(j, \beta_j) / P \leq \dim \bigoplus_{j=1}^{i-1} \Delta(j, \alpha_j) = i - 1,$$

which means that $P$ contains $\bigoplus_{j=1}^{\infty} \Delta(j, \beta_j)$. Therefore

$$\beta_i^\infty = d(K) \geq d(P) \geq \alpha_i^\infty.$$ 

By symmetry, we obtain the reverse inequality.

Conversely, suppose that $\alpha_i^\infty = \beta_i^\infty$ for each $i$. For each $i$, let $\gamma_i = \max(\alpha_i, \beta_i)$.

Under the above assumption, we prove that

$$\prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i} \sim \prod_{i=1}^{\infty} C_p(D^i)^{\gamma_i} \sim \prod_{i=1}^{\infty} C_p(D^i)^{\beta_i}.$$ 

**Proof of (4).** For each $i$, let $A_i$ and $B_i$ be sets such that $\alpha_i = |A_i|$, $\beta_i = |B_i|$, and

$A_i \supset B_i$ if $\gamma_i = \alpha_i$, and $A_i \subseteq B_i$ if $\gamma_i = \beta_i$.

Let $C_i = (A_i - B_i) \cup (B_i - A_i)$ and $\delta_i = |C_i|$. Clearly

$$\alpha_i = \beta_i + \delta_i$$

if $\gamma_i = \alpha_i$, and

$$\beta_i = \alpha_i + \delta_i$$

if $\gamma_i = \beta_i$.

Since $\delta_i \leq \alpha_i \leq \alpha_i^\infty$ if $\gamma_i = \alpha_i$ and $\delta_i \leq \beta_i \leq \beta_i^\infty$ if $\gamma_i = \beta_i$, there exists an injection $C_i \rightarrow \bigoplus_{j \geq i} A_j$ or $\bigoplus_{j \geq i} B_j$ according to either of the above cases. Decompose $C_i$ into $\bigoplus_{j \geq i} C_{ij}$ such that $C_{ij}$ is embedded into $A_j$ or $B_j$. Let $\delta_{ij} = |C_{ij}| \leq \alpha_j$ or $\beta_j$. Then

$$\prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i} \sim \prod_{\alpha_i \geq \beta_i} C_p(D^i)^{\alpha_i} \times \prod_{\beta_i > \alpha_i} C_p(D^i)^{\beta_i} \sim \prod_{\alpha_i \geq \beta_i} C_p(D^i)^{\alpha_i} \times \prod_{\beta_i > \alpha_i} C_p(D^i)^{\alpha_i + \delta_i} \sim \prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i} \times \prod_{\beta_i > \alpha_i} C_p(D^i)^{\delta_{i+s}} \sim \prod_{i=1}^{\infty} \left(C_p(D^i)^{\alpha_i} \times \prod_{t=0}^{i} C_p(D^{i-t})^{\delta_{i-t}}\right) \sim \prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i}$$

(by Observation 1.7 and $\delta_{i-t-t} \leq \alpha_i$).
By symmetry, we see that \( \prod_{i=1}^{\infty} C_p(D^i)^{\gamma_i} \sim \prod_{i=1}^{\infty} C_p(D^i)^{\beta_i} \) and this completes the proof of (4) and hence the proof of (1).

(2) \( \bigoplus \Delta(i, 1) \) is naturally embedded in \( \bigoplus \Delta(i, \omega) \) as a closed set. It is easy to construct a closed embedding of \( \bigoplus \Delta(i, \omega) \) into \( \bigoplus \Delta(i, 1) \). Then the desired isomorphy follows from [14, Lemma 2.7].

(3) Let \( (\alpha_i) \) be a sequence of cardinals. We have three cases to consider.

Case 1. All but finitely many \( \alpha_i \)'s are infinite cardinals.

Case 2. All but finitely many \( \alpha_i \)'s are finite cardinals.

Case 3. There are infinitely many infinite cardinals and infinitely many finite cardinals.

For Case 1, let \( K > 0 \) be such that \( \alpha_i \geq \omega \) for each \( i \geq K \). Observation 1.7 applies to see that

\[
\prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i} \sim \prod_{i=1}^{K} C_p(D^i)^{\omega} \times \prod_{i > K} C_p(D^i)^{\alpha_i},
\]

which is a space of the type (1). If Case 2 occurs, let \( L > 0 \) be such that \( \alpha_i < \omega \) for each \( i \geq L \). Then a similar computation to the above with the help of (2) shows that

\[
\prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i} \sim \prod_{i=1}^{L} C_p(D^i)^{\alpha_i} \times \prod_{i > L} C_p(D^i)^{\omega},
\]

which also reduces to the type (1). Finally for Case 3, decompose \( \mathbb{N} \) into countable collection of disjoint sets as \( \mathbb{N} = \bigcup_{i=1}^{\infty} A_i \cup B_i \) such that, with respect to the standard order,

\[
A_1 < B_1 < A_2 < B_2 < \cdots
\]

(that is, for each \( a_i \in A_i \) and \( b_i \in B_i \), we have that \( a_1 < b_1 < a_2 < b_2 < \cdots \), and

\[
\alpha_i < \omega \text{ if } \alpha_i \in \bigcup_j A_j \text{ and } \alpha_i \geq \omega \text{ if } \alpha_i \in \bigcup_j B_j.
\]

Then we have that

\[
\prod_{i=1}^{\infty} C_p(D^i)^{\alpha_i} \sim \prod_{\alpha_i \in A_1} C_p(D^i)^{\omega} \times \prod_{\alpha_i \in B_1} C_p(D^i)^{\alpha_i} \times \prod_{\alpha_i \in A_2} C_p(D^i)^{\omega} \times \cdots
\]

by Observation 1.7 and (2).

This completes the proof of (3) and the proof of the theorem. \( \square \)

Let us apply the above result to infinite dimensional locally compact polyhedra. If \( X \) is such a polyhedron, \( X \) contains no \( \sigma^\infty \), so \( C_p(X) \sim \prod_{i=1}^{\infty} C_p(D^i)^{\tau_i(X)} \) and \( \tau_i(X) = 1 \) or \( \omega \). From Theorem 3.10, we have that

**Corollary 3.11.** For any locally compact infinite dimensional polyhedron \( X \),

\[
C_p(X) \sim \prod_{i=1}^{\infty} C_p(D^i).
\]
References