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GEORGE PEACOCK AND THE BRITISH ORIGINS OF SYMBOLICAL ALGEBRA

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SUMMARIES

This paper studies the background to and content of George Peacock's work on symbolical algebra. It argues that, in response to the problem of the negative numbers, Peacock, an inveterate reformer, elaborated a system of algebra which admitted essentially "arbitrary" symbols, signs, and laws. Although he recognized that the symbolical algebraist was free to assign somewhat arbitrarily the laws of symbolical algebra, Peacock himself did not exercise the freedom of algebra which he proclaimed. The paper ends with a discussion of Sir William Rowan Hamilton's criticism of symbolical algebra.

Cet article traite du fond et du contenu de l'oeuvre de George Peacock sur l'algèbre symbolique. On y affirme que pour rêpondre au problème des nombres négatifs, Peacock, un réformateur invétéré, élabora un système algébrique qui admettait des symboles, des signes, et des lois essentiellement "arbitraires." Tout en reconnaissant que l'algébriste peut énoncer quelque peu arbitrairement les lois d'une algèbre symbolique, Peacock ne fit pourtant pas lui-même de la liberté qu'il réclamait pour l'algèbre. L'article se termine par une discussion de la critique que fit William Rowan Hamilton de l'algèbre symbolique.

In diesem Aufsatz wird der Hintergrund und der Inhalt von George Peacocks Arbeiten zur symbolischen Algebra untersucht. Es wird dargelebt, wie Peacock--ein unermüdlicher Reformer--als Antwort auf das Problem der negativen Zhlen ein algebraisches System entwickelte, das im wesentlichen "wilkürliche" Symbole, Zeichen und Gesetze zuliess. Obgleich er erkannte, dass der symbolisch arbeitende Algebraiker frei ist, in gewissem Umfang die Gesetze der symbolischen Algebra willkürlich festzulegen, nutze Peacock selbst die von ihm proklamierte Freiheit der Algebra nicht aus. Schliesslich wird noch Sir William Rowan Hamiltons Kritik der symbolischen Algebra besprochen.

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George Peacock (1791-1858) is universally recognized as one of the earliest proponents of the symbolical approach to algebra. Yet there exists no single article or book to which historians of mathematics can turn for a satisfactory interpretation of the background to and content of Peacock's major algebraic work. Most summaries and evaluations of his algebraic contributions are scattered throughout textbooks on the history of mathematics and collections of biographies of great mathematicians, which for too long have served as the mainstays of the history of mathematics. Some insightful discussions of various aspects of his work are, of course, found in a few specialized sources on the history of 19th-century algebra [Nagel 1935, 448-455; Nový 1973, 190-194]. But these discussions are somewhat narrow since they are segments of more comprehensive studies in which Peacock figures as but one of many mathematicians involved in the development of modern algebra.

As contemporary historians begin to mine ever more deeply the rich published and unpublished source materials available for the study of the development of modern algebra in 19th-century England and Ireland [Hankins 1976, Dubbey 1978], the time is ripe for a reexamination of Peacock's contributions thereto. An appreciation of the algebraic work of Charles Babbage and Sir William Rowan Hamilton, for example, requires an understanding of Peacock's major ideas. This paper explores the roots and content of Peacock's work on the formulation of the symbolical approach to algebra. Through a careful study of Peacock's *Treatise on Algebra* of 1830, his "Report on the Recent Progress and Present State of Certain Branches of Analysis" of 1833, and relevant secondary materials as well, the author hopes to present a coherent interpretation of Peacock's major algebraic work [1].

We begin with a few brief remarks on Peacock's life, and suggest that he was first of all a reformer, committed not only to mathematical, but to religious, academic, and social reform as well. Evidence is presented to support the thesis already proposed by Ernest Nagel [1935], that Peacock formulated the symbolical approach to algebra in order to resolve the problem of the negative and imaginary numbers. In view of Peacock's standing as one of the earliest British advocates of symbolical algebra, the roots of his algebraic work are, to a limited extent, the very roots of British modern algebra. Following a discussion of the problem of the negative and imaginary numbers in late-18th- and early-19th-century England, the paper explains that Peacock conceived of his task as nothing less than the construction of a deductive science of symbolical algebra, with first principles sufficient to justify inclusion of the negatives and imaginaries. The principal conclusions of the section on Peacock's major algebraic ideas are: first, while he clung to quantity as the ultimate subject matter of algebra, Peacock adopted a basically symbolical approach to algebra, according to which he viewed it as a science of "arbitrary" or undefined symbols and signs governed by specified laws; and second, he recognized that the algebraist was free to assign somewhat arbitrarily the laws of algebra. Peacock usually receives credit for formulating the symbolical approach to algebra. However, credit for recognizing the freedom of algebra is usually assigned to later algebraists, especially to Hamilton, since Peacock, motivated by concern for the applicability of symbolical algebra, chose to adopt the laws of arithmetic as the laws of symbolical algebra, rather than to create new laws. Ending with a few brief remarks about Hamilton's early reaction to Peacock's work, the paper indicates the difficulty experienced by Hamilton, clearly one of the leading British algebraists of the early 19th century, in accepting what he (and probably many of his contemporaries) saw as the meaninglessness of symbolical algebra.

PEACOCK AS A REFORMER

Peacock was born in 1791 in Denton, in northern England. In 1808 his father, a minister, sent him to a school run by James Tate, a former fellow of Sidney-Sussex College, Cambridge, to whom Peacock later dedicated his *Treatise on Algebra* [1830]. In 1809 he entered Trinity College, Cambridge, where, four years later, placing ahead of all the men of his year who competed for honors in mathematics, with the exception of the future astronomer John Frederick William Herschel, he was named second wrangler and recipient of the second Smith's prize. Elected a fellow of Trinity in 1814, he served successively as college lecturer and assistant tutor, full tutor, and sole tutor at Trinity. Because of availability of the likes of Babbage and George Airy for the few professorships in the mathematical and physical sciences at early-19th-century Cambridge, Peacock was not elected to a Cambridge professorship until 1837, when he assumed the Lowndean professorship of astronomy and geometry. Although he became dean of Ely and moved from Cambridge in 1839, he continued to hold the Lowndean professorship as a sinecure until his death in 1858, despite the protests of some of his academic colleagues [Anon. 1858, 744].

If any generalization assists the historian in the task of making sense out of Peacock's diverse activities it is that Peacock was above all a reformer--a man who struggled variously to improve British calculus and algebra, Cambridge University, and sanitation and education in the city of Ely. In 1812, as is well known, in his earliest role as a mathematical reformer, Peacock collaborated with Babbage and Herschel, then fellow undergraduates at Cambridge, in the formation of the Analytical Society. Promoting British adoption of the continental notation and methods through infiltration of such into the Cambridge mathematics program, the Society produced in 1816 an English translation of Lacroix's *Traité du calcul différentiel et du calcul intégral* and in 1820 *A Collection of Examples of the Applications of the Differential and Integral Calculus*, designed for use by tutors in the preparation of students for the mathematical tripos, the examination which determined mathematical honors at Cambridge [2]. Meanwhile, as moderator of the tripos of 1817, Peacock posed examination problems in the continental notation. When this first official employment of the continental notation at Cambridge was criticized, he responded:

> . . . *I shall never cease to exert myself to the utmost in the cause of reform, and . . . I will never decline any office which may increase my power to effect it. I am nearly certain of being nominated to the office of Moderator in the year 1818-19, and as I am an examiner in virtue of my office, for the next year I shall pursue a course even more decided than hitherto, ... It is by silent perseverance only that we can hope to reduce the many-headed monster of prejudice, and make the University answer her character as the loving mother of good learning and science.* [In Royal Society of London 1859, 538-539].

Thus, as early as 1817, Peacock was already a mathematical reformer, and verbalized what was to be the theme of his entire career--commitment to reform.

Coupled with his earlier contributions to the improvement of the calculus, Peacock's participation in the popular 19th-century struggle to reform Cambridge earned him a reputation as "a zealous advocate for progress and reform in the university" [Clark, 583]. Obituaries stressed this theme of his career, according to one, he was "all the time . . . an ardent and active politician,--a 'Reformer' by temperament and conviction,--one of the most consistent adherents of the Whig Party in the University" [Anon. 1858, 743]. Peacock merited this reputation through efforts at change within Cambridge which he exerted individually and as a member of two commissions appointed in the 1850s for such a purpose. His assault on the nonmathematical traditions of Cambridge began at least as early as 1834 when he collaborated with Joseph Romilly, Thomas Musgrave, and others in the writing of a petition to eliminate the religious examinations which barred dissenters from the university [Romilly 1967, 54]. Undaunted by the rejection of this petition by the House of Lords, he later published his *Observations on the Statutes of the University of Cambridge*, in which he explained the content of and suggested revisions in the university's laws [Peacock 1841]. In the 1850s Peacock participated officialy in the reform of Cambrige, as a member, first, of a royal commission of inquiry into the university and, later, of a parliamentary commission charged with writing new statutes for the university and its colleges. Partially as a result of the latter commission's work, Cambridge abandoned religious tests as prerequisites for admission and for the taking of all bachelors' degrees with the exception of those in divinity; the university's colleges also opened to general competition some previously "closed" scholarships and fellowships, which had been restricted to "founder's kin" and the like [Barnard 1963, 123]. Condemning Peacock's work with this commission, some of whose proposals were not well received at Cambridge, one of his contemporaries accused him of "mere love of change" rather the "zeal for improvement" [Anon. 1858, 745].

Peacock's efforts on behalf of the continental calculus, revision of Cambridge's statutes, and sanitary measures in the city of Ely show that he was basically a practical reformer. No mere visionary, he would do whatever was necessary to further a cause in which he believed, no matter how time consuming or unpopular the task. For example, as noted above, he assumed the exhausting position of moderator of the mathematical tripos and served on university committees with controversial charges; as dean of Ely, he pushed for a sewerage system in that city despite the opposition of some of its leading citizens [Royal Society of London 1859, 541].

Thus the reforming tendency was a unifying theme which ran through Peacock's entire academic and ecclesiastical career. Even in his algebraic work (the main topic of this paper), Peacock was basically a practical reformer. His major contribution to modern algebra was a textbook--his *Treatise on Algebra* of 1830--in which he developed and explained the symbolical approach for the benefit of students of mathematics [Peacock 1830, xxii]. It is possible, as John Dubbey has suggested, that some of Peacock's algebraic ideas were derived from a series of mostly unpublished essays written by Babbage [Dubbey 1977, 1978, 93-130] [3]. Yet, just as Babbage did not serve as moderator of the tripos in order to popularize the continental notation at Cambridge, so Babbage did not devote his time to writing a textbook on symbolical algebra. Rather Peacock--the inveterate practical reformmer--did.

THE PROBLEM OF NEGATIVE NUMBERS

The roots of Peacock's second major mathematical reform--that of algebra--can be traced back to the attack on negative numbers, waged during the late 18th and early 19th centuries by two English mathematicians, Francis Maseres and William Frend. Maseres, a fourth wrangler at Clare College, Cambridge, and Frend, a second wrangler at Christ's College and later a fellow of Jesus College, objected to the use of negative numbers; they argued that the want of an adequate definition of such numbers rendered algebraic results involving them meaningless and brought into question the very legitimacy of algebra's standing as a science. In the 18th century mathematicians defined mathematics as the science of quantity, and negative numbers variously as "quantities less than nothing" and "quantities obtained by the subtraction of a greater quantity from a lesser." Neither definition was logically satisfying. Furthermore, the latter definition raised another problem--that of the definition of the subtraction operation. As the opponents of the negatives noted, algebraists routinely defined subtraction as the taking of a lesser from a greater, and so were not free to conduct the operation upon which the second definition of a negative depended. In lieu of an adequate definition, some mathematicians had tried to justify the negatives through analogy to debts, lines drawn in certain directions, and the like [Newton 1728,3]. Dismissing this approach to justifying the negatives, Frend retorted. "when a person cannot explain the principles of a science without reference to metaphor, the probability is, that he has never thought accurately upon the subject" [Frend 1796 1, x]. Because of the lack of a satisfactory definition of negative numbers, Maseres and Frend looked upon all work involving these numbers as nonsense and maintained that by inclusion of the negatives, "the Science of Algebra, or Universal Arithmetick, has been disgraced and rendered obscure and difficult, and disgusting to men of a just taste for accurate reasoning" [Maseres 1800, lv].

In order to remedy such defects of late-18th-century algebra, Maseres and Frend called for nothing less than the total abandonment of negative numbers and the restriction, in practice as well as definition, of the subtraction operation to cases where the minuend was greater than or equal to the subtrahend. These restrictions would reduce algebra to universal arithmetic in the strictest sense, that is, to a science where symbols stood only for nonnegative numbers and signs denoted strictly arithmetical operations. Both Maseres and Frend wrote books developing such a truncated version of algebra: among them, Maseres' *Dissertation on the Use of the Negative Sign in Algebra,* published in 1758, and Frend's *Principles of Algebra*, published in two volumes in 1796 and 1799. The very first sentence of the former work announced:

> *The design of the following Dissertation is to remove from some of the less abstruse parts of Algebra, the difficulties that have arisen therein from the too extensive use of the Negative Sign, and to explain them, without considering the Negative Sign in any other light than as the mark of the subtraction of a lesser quantity from a greater.* [Maseres 1758, i]

Rejecting negative numbers entailed rejecting imaginary numbers and, of course, negative and imaginary roots of equations. Thus in his *Dissertation,* Maseres renounced the theory of equations, which had blossomed during the 17th and 18th centuries and, in particular, the fundamental theorem of algebra, which asserts the existence of (at least) one complex root of the polynomial $a_n x^n + \cdots + a_1 x + a_0 = 0$, where $n \geqslant 1$, $a_n \neq 0$, and the a_i are complex numbers. Pursuing abandonment of the negatives to its logical conclusion, Maseres argued, for example, that an equation of the form $x^2 + px = r$ has but one root. He considered the following specific case: $x^2 + 2x = 15$. Maseres stated that $x = 3$, a positive root of the equation, was acceptable, but then tried to demonstrate the absurdity of admitting the root $x = -5$. He substituted $x = -5$ into the given equation and, with the help of the law of signs, which he also criticized, derived the relationship $25 - 10 = 15$. The latter equation, he claimed, was actually an equation of the form $x^2 - 2x = 15$; in his opinion, then, the admission of the negative root, $x = -5$, resulted in the confounding of two distinct equations. Seemingly grasping at straws, he summarized:

> *This seems to be considering the two equations $xx + 2x = 15$, and $xx - 2x = 15$, or $xx + px = r$, and $xx - px = r$ (which are so many assertions quite distinct from each other, and cannot be derived from the conditions of the same problem), as if they were one and the same equation:. . . . This method of uniting together two different equations may perhaps have its uses; but, I must confess, I cannot see them: on the contrary, it should seem that perspicuity and accuracy require, that two equations, or propositions, that are in their nature different from each other, and are the results of different conditions and suppositions, should be carefully distinguished from each other, and treated of separately, each by itself, as it comes under consideration.* [Maseres 1758, 29]

In short, the crusade of Maseres and Frend against negative numbers was a crusade against all algebra beyond universal arithmetic, including the valuable theory of equations.

Although a study of British algebra textbooks and articles published in the leading British scientific journals of the late 18th and early 19th centuries demonstrates that the problem of negative numbers was a major concern of British thinkers of the period, it also shows that most British mathematicians were reluctant to renounce the riches of algebra because of foundational problems [4]. Although admitting the absence of a satisfactory definition of the negatives, most argued in favor of their retention, if only because of practical considerations. Thinkers who concurred with Maseres and Frend in recognition of the problem but not in their solution included William Greenfield, an amateur mathematician and a professor of rhetoric at the University of Edinburgh, and Robert Woodhouse, successively a fellow, Lucasian professor of mathematics, and Plumian professor of astronomy and experimental philosophy at Cambridge. In his address of 1784 before the Royal Society of Edinburgh, Greenfield admitted that

> *a complaint remains, which appears to be too well founded, that the Method of negative quantities, as has been the case with some other rules of the art, is supported, rather by induction and analogy, than by mathematical demonstration .... The very vague and unsatisfactory, and often mysterious accounts of the matter, which are given even by writers of the greatest eminence, serve only to shew, that although they are satisfied of the certainty of the method, yet they perceive that something still remains which ought to be explained, and of which no good explanation has been given.* [Greenfield 1788, 134-135]

Yet unwilling to support the reduction of algebra to universal arithmetic, Greenfield urged algebraists, such as Maseres (whom he explicitly mentioned) to "exert ... industry and ingenuity, rather to confirm than to destroy; rather to demonstrate, how far we might rely on the method of negative quantities, than to overturn at once so great a part of the labours of the modern algebraists" [Greenfield 1788, 136]. In a paper read before the Royal Society of London in 1801, Woodhouse also took a pragmatic approach to the problem. Acknowledging that "an abstract negative quantity is indeed unintelligible" [Woodhouse 1801, 96], he yet argued for retention of negative and imaginary numbers. "If operations with any characters or signs lead to just conclusions," he maintained, "such operations must be true by virtue of some principle or other" [Woodhouse 1801, 90].

Greenfield and Woodhouse were joined in their attempt to salvage the negative and imaginary numbers by a few mathematicians, including John Playfair and Adrien-Quentin Buée, whose defense of these numbers offered glimpses of a symbolical approach to algebra [5]. Nagel has already pointed out that Playfair, in his paper "On the Arithmetic of Imposible Quantities," "almost guessed the secret of the nature of mathematics" [Nagel 1935, 439]. In this paper, Playfair indeed came close to recognizing algebra as a science of meaningless symbols:

> *In algebra again every magnitude being denoted by an artificial symbol, to which it has no resemblance, is*

> *liable, on some occasions, to be neglected, while the symbol may become the sole object of attention. It is not perhaps observed where the connection between them ceases to exist, and the analyst continues to reason about the characters after nothing is left which they can possibly express* . [Playfair 1778, 319]

Early in the next century, Buée, a French emigré living in London, responded to Frend's *Principles of Algebra* by suggesting that there were two kinds of algebra: (1) "*arithmétique universelle*" or universal arithmetic, in which the signs + and - stood only for addition and subtraction; and (2) "*une langue mathématique,*" or a mathematical language, in which + and - also represented what Buée called "qualities." He explained that a quality was that which was capable of a meaning or interpretation, and a negative number was actually a combination of quantity and quality:

> *whenever one has as a result of an operation a quantity preceded by the sign -, it is necessary, in order that the result has a meaning, to consider some quality. Then algebra must no longer be regarded simply as a* universal arithmetic, *but as a* mathematical language ....
>
> *Likewise, when one has said that a negative quantity is less than zero, one has once more in view this second meaning [+ and - considered as qualities rather than as signs of addition and subtraction]; because it is not the* quantity *which is less than* zero; *it is the* quality *which is inferior to* nullity [*inférieure à la* nullité]. *For example, if my debts exceed my assets I am poorer than if I had neither assets nor debts.* [Buée 1806, 25; translation is mine]

Although Buée proposed dividing algebra into two parts, universal arithmetic and some sort of mathematical language, and offered a geometrical interpretation of the complex numbers as well, his contribution to the development of symbolical algebra was otherwise minimal. As seen above, he continued to stress meaning in algebra, trying to explain the negative numbers through appeal to analogy and to the philosophical notion of quality. In his *Treatise on Algebra* Peacock noted that Buée's paper contained "some original views on the use and signification of the signs of Algebra, though presented in a very vague and unscientific form" [Peacock 1830, xxvii] [6].

PEACOCK'S SYMBOLICAL APPROACH TO ALGEBRA

Along with the work of Greenfield and the other defenders of the negatives, Peacock's formulation of the symbolical approach to algebra can be seen as a direct response to Maseres' and Frend's opposition to negative numbers. In his *Treatise on Algebra* of 1830 and "Report" of 1833, he tackled the major problems raised by the opponents of the negatives: the status of the negative numbers, the definition of the subtraction operation, and the question of algebra's standing as a science. In the "Report" he referred explicitly to the algebraic ideas of Maseres and Frend and quoted rather extensively from Frend's *Principles of Algebra*, admitting that

> *the arguments which they* [*Maseres and Frend*] *made use of were unanswerable, when advanced against the form under which the principles of algebra were exhibited in the elementary and all other works of that period, and which they have continued to retain ever since, with very trifling and unimportant alterations.* [Peacock 1833, 189-192]

A combination of factors probably aroused and fueled Peacock's interest in the problem of negative numbers and its resolution. In the early 19th century it would have been difficult for any serious English student of mathematics to have escaped contact with the problem, so often was it mentioned in the scholarly articles and algebra textbooks of the period. In his *Treatise* Peacock noted that Buée's paper of 1806 had initially directed his attention to the geometrical interpretation of the complex numbers [Peacock 1830 xxvii]. As already indicated, this paper also discussed the problem of the negatives, proposing a very vague resolution thereof. Furthermore, most major participants in the debate over negative numbers were associated with Cambridge University, where Peacock studied and then taught. Maseres and Frend were graduates of Cambridge; Frend's trial and subsequent banishment in 1793 from Cambridge for his dissenting religious and radical political views had become a *cause célèbre* at the university. Woodhouse was a fellow and professor at Cambridge during Peacock's early tenure there. Finally, conscientious as a tutor at Trinity, Peacock shared the concern of Woodhouse and others for the difficulty students manifested in following algebraic arguments. In the British universities of this period, mathematics was, after all, valued not so much for itself, but as an instrument for the training of logical minds. Possibly with this pedagogical purpose in mind, Woodhouse, in a letter of 1801 to Maseres, had agreed that "till the doctrines of negative and imaginary quantities are better taught than they are at present taught in the University of Cambridge, ... they had better *not* be taught" (in [De Morgan 1842, 462]). As already noted, Peacock himself intended his *Treatise on Algebra* of 1830 as a textbook which made algebra "perfectly accessible" to students [Peacock 1830, xxii].

Influenced by Maseres and Frend, Peacock began his *Treatise on Algebra* of 1830 and "Report" of 1833 with the claim that there were within algebra no adequate foundations for the negative numbers and the unrestricted subtraction operation and, therefore, that algebra's standing as a deductive science was suspect. Together with the opponents of the negatives, he dismissed the popular idea that the principles of arithmetic were also the principles of algebra. He pointed out, for example, that it was erroneous to maintain that the subtraction operation of algebra could be justified by an appeal to that of arithmetic. He noted that, although the subtraction operation of algebra had been borrowed from arithmetic with no statement of any extension in its meaning or application, the arithmetical operation was clearly limited to those cases in which the minuend was greater than or equal to the subtrahend, while the algebraic operation was unrestricted. As he generalized,

> *in the exposition of the principles of algebra, arithmetic has always been taken for its foundation, and the names of the fundamental operations in one science have been transferred to the other without any immediate change of their meaning, yet it has generally been found necessary subsequently to enlarge this very narrow basis of so very general a science, though the reason of the necessity of doing so, and the precise point at which, or the extent to which, it was done, has usually been passed over without notice. The science which was thus formed was perfectly abstract, in whatever manner we arrived at its fundamental conclusions; and those conclusions were the same whatever view was taken of their origin, or in whatever manner they were deduced; but a serious error was committed in considering it as a science which admitted of strict and rigorous demonstration, when it certainly possessed no adequate principles of its own, whether assumed or demonstrated, which could properly justify the character which was thus given to it.* [Peacock 1833, 188-189]

While he concurred with Maseres and Frend in diagnosing the problems of algebra, Peacock proposed a remedy very different from theirs. In order to establish algebra as a science, Maseres and Frend had scuttled all questionable algebraic entities and reduced algebra to universal arithmetic in the strictest sense. Peacock, on the other hand, shared the reluctance of Woodhouse and other defender of the negatives to lose those parts of mathematics in which the negatives figured, and so sought to construct a deductive science of algebra in which the negatives, imaginaries, and an unrestricted subtraction operation were preserved. As he explained in his "Report," the universal arithmetic of Maseres and Frend could not be accepted in place of algebra because

> *there were a great multitude of algebraical results and propositions, of unquestionable value and of unquestionable consistency with each other, which were irreconcilable with such a system, or, at all events, not deducible from it; and amongst them, the theory of the composition of equations, which Harriot had left in so complete a form, and which made it necessary to consider negative and even impossible quantities as having a real existence in algebra, however vain might be the attempt to interpret their meaning.* [Peacock 1833, 190-191]

Because of these pragmatic considerations, Peacock wrote the *Treatise* "with a view of conferring upon Algebra the character of a demonstrative science, by making its first principles coextensive with the conclusions which were founded upon them" [Peacock 1830, v]. In short, he sought to justify all the conclusions of traditional algebra by rewriting the first principles of algebra.

Basic to Peacock's work on the foundations of algebra was his well-known distinction between arithmetical algebra and symbolical algebra, which he described as two independent sciences, distinguished from one another by the assumption in the latter of the existence of the negatives and an unrestricted subtraction operation. Arithmetical algebra was universal arithmetic in the strictest sense which, as he noted in the "Report," had been clearly and logically developed by Frend [Peacock 1833, 191]; symbolical algebra, defined variously throughout the *Treatise* and "Report," was above all traditional algebra, that is, algebra including the negatives, imaginaries, and unrestricted subtraction operation. According to Peacock, the mathematician moved from arithmetical to symbolical algebra as follows:

> *The assumption . . . of the independent existence of the signs + and - . . . renders the performance of the operation denoted by - equally possible in all cases: and it is this assumption which effects the separation of arithmetical and symbolical Algebra, and which renders it necessary to establish the principles of this science upon a basis of their own: for the assumption in question can result from no process of reasoning from the principles or operations of Arithmetic, and . . . it must be considered therefore as*

> *an independent principle, which is suggested as a means of evading a difficulty which results from the application of arithmetical operations to general symbols.* [Peacock 1830, viii-ix]

Alternately, later in the same work, Peacock wrote:

> *If, however, we* generalize *the operation denoted by -, so that it may admit of application in all cases, we shall then find the independent existence of this sign will follow as a necessary consequence, and we shall thus introduce a class of quantities, whose existence was never contemplated in Arithmetic or Arithmetical Algebra. . . .*
>
> *This* generalization *of the operation denoted by -, is in reality an* assumption, *inasmuch as it is not a consequence deducible from the operation of subtraction as defined and used in Arithmetic and Arithmetical Algebra.* [Peacock 1830, 70-71]

Although somewhat fuzzy about the relative primacy of the above-mentioned assumptions--that of the existence of the independent signs + and - and that of the unrestricted subtraction operation--Peacock thus revealed the crux of his solution to the problem of negative numbers. The negatives and unrestricted subtraction operation arose when arithmetic was treated in the most general manner. But there were no arithmetical foundations for either. In lieu of such, he now introduced them into algebra by assumption, without definition or arithmetical justification, as "a means of evading a difficulty" [7].

Incorporating the assumptions of the existence of the undefined negatives and of an undefined and unrestricted subtraction operation into symbolical algebra required a radical revision of the standard notion of algebra as some sort of vague generalization of arithmetic in which symbols were substituted for numbers. In place of this notion Peacock defined symbolical algebra as "the science which treats of the combinations of arbitrary signs and symbols by means of defined though arbitrary laws" [Peacock 1830, 71]. As he explained, algebraic symbols were "arbitrary" in the sense that they were "not merely . . . the general representatives of numbers, but of every species of quantity" [Peacock 1830, ix]. Algebraic signs, such as the independent signs + and -, were "arbitrary" since they were also introduced into symbolical algebra without definition. If, however, the signs of symbolical algebra were arbitrary, how then were algebraic operations to be determined? In his *Treatise*, Peacock mandated: "the definitions of those operations must regard the laws of their combination *only*" [Peacock 1830, ix]; in the "Report," he explained: "In arithmetical algebra, the definitions of the operations determine the rules; in symbolical algebra, the rules determine the meaning of the operations, or more properly speaking, they furnish the means of interpreting them" [Peacock 1833, 200]. In short, in lieu of the definitions of symbols and signs demanded by his predecessors, Peacock used laws of combination to determine the content of symbolical algebra, thus shifting the emphasis in algebra from the meaning of symbols and signs to the laws of operation. It was this aspect of his work, similar in spirit and effect to the introduction into physics during the Scientific Revolution of concern for the "how" rather than the "why", upon which Augustus De Morgan, Duncan F. Gregory, George Boole, and other British pioneers of mathematics constructed modern abstract algebra during the second third of the 19th century.

Although symbolical algebra was basically a science of arbitrary or undefined symbols and signs, Peacock explained that interpretation of algebraic symbols and signs could follow, even if it did not precede, algebraic manipulation [Peacock 1833, 195]. The negative numbers, for example, were initially meaningless, but could admit of many possible, although no necessary, interpretations. Peacock cautioned, however, that "it is only by their [the symbols'] ceasing to be abstract numbers that we shall be enabled to interpret the affections which the signs + and - (or any other signs) essentially attached to them may be supposed to express" [Peacock 1830, ix]. If an algebraic symbol, say a, stood only for an abstract number, then Peacock admitted that $-a$ made no sense. (This was, after all, the argument of Maseres and Frend.) When, however, such a symbol represented every species of quantity, an interpretation for $-a$ could easily be found; if, for example, $+a$ was a line drawn in a certain direction, the $-a$ was a line drawn in the opposite direction.

In his discussions of the laws governing algebraic operations Peacock enunciated a frequently overlooked restricted version of the principle of the freedom of mathematics--that the algebraist rather arbitrarily assigns the laws of symbolical algebra. In his *Treatise* and "Report" he had described these laws as "defined though arbitrary." In the former work, he also maintained that "we may *assume* any laws for the combination and incorporation of such symbols, so long as our assumptions are independent, and therefore not inconsistent with each other" [Peacock 1830, 71]. Reiterating this idea, the "Report" declared:

> *in symbolical algebra, the rules determine the meaning of the operations .... we might call them* arbitrary as*sumptions, in as much as they are* arbitrarily *imposed upon a science of symbols and their combinations, which might be adapted to any other assumed system of consistent rules.* [Peacock 1833, 200-201]

Since Peacock described the symbols and signs of symbolical algebra as arbitrary, he would have been hard-pressed to argue that there existed a single necessary set of rules, governing such symbols and signs. Instead, as illustrated above, he defended the arbitrariness of the laws of symbolical algebra and thus the freedom of the algebraist.

Peacock's declaration that the laws of symbolical algebra were assigned rather arbitrarily by mathematicians was almost totally ignored, first by his contemporaries and later by historians of mathematics, who have usually ascribed recognition of the freedom of mathematics to Hamilton, the creator of the quaternions, and to the inventors of non-Euclidean geometry [8]. Peacock's loss of credit for this idea is partially due to the fact that he did not exercise the freedom of algebra which he proclaimed. Rather, because of his concern for the applicability of symbolical algebra, he simply adopted the laws of arithmetic as the laws of algebra. In both the *Treatise* and "Report" declarations of the freedom of algebra were accompanied by statements of what Peacock saw as the major problem associated with the exercise of that freedom--that of the applicability or usefulness of an algebra built upon arbitrary symbols, signs, and laws. Thus in the earlier work he pointed out that such an algebra of arbitrary symbols and signs "must terminate in the consequences of their laws of combination" [Peacock 1830, xi] and expressed the fear that it "may. . . be one of useless and barren speculations"[Peacock 1830, 71]. In the "Report," he explained this problem of applicability as follows:

> *If . . . in the construction of such a system, we looked to the assumption of such rules of operation or of combination only, as would be sufficient, and not more than sufficient, for deducing equivalent forms, without any reference to any subordinate science, we should be altogether without any means of interpreting either our operations or their results, and the science thus formed would be one of symbols only, admitting of no applications whatever.* [Peacock 1833, 200]

Concern for applicability caused Peacock to concentrate less on exploration of the essential arbitrariness of the laws of symbolical algebra than on the selection and defense of such laws as would ensure the usefulness of his new algebra.

The principle of selection of the laws of symbolical algebra adopted by Peacock was, of course, the now famous *principle of the permanence of equivalent forms*, according to which the laws of arithmetical algebra were adopted as the laws of symbolical algebra, thus guaranteeing that the latter would be useful at least as a generalization of arithmetic. To the enunciation, demonstration, and application of this principle Peacock devoted considerably more space than to the treatment of the arbitrariness of algebraic laws. As he explained in the *Treatise* in order to avoid creating a science of "useless and barren speculations," he decided to "choose some subordinate science as the guide merely, and not as the foundation of our assumptions, and frame them in such a manner that Algebra may become that most general form of that science, when the symbols denote the same quantities which are the objects of its demonstration" [Peacock 1830, 71]. The decision to use what Peacock termed a "subordinate science" as the "science of suggestion," or that science the laws of which suggested those of symbolical algebra, led to his formal statement of the principle of the permanence of equivalent forms:

> *Whatever form is Algebraically equivalent to another, when expressed in general symbols, must be true, whatever those symbols denote.*
>
> *Conversely, if we discover an equivalent form in Arithmetical Algebra or any other subordinate science, when the symbols are general in form though specific in their nature, the same must be an equivalent form, when the symbols are general in their nature as well as in their form.* [Peacock 1830, 104]

As already indicated, Peacock chose arithmetical algebra as the science of suggestion for symbolical algebra [1830, 74]. Thus his adoption of the permanence principle can be seen not only as a resolution of the problem of the applicability of symbolical algebra, but as a retreat to the standard earlier practice of the transference of the laws of arithmetic to algebra.

While Peacock clearly deserves some credit for recognizing the freedom of the symbolical algebraist to assign laws of combination, the description of Peacock as one of the original formulators of the principle of the freedom of mathematics must be qualified by admission of the following points: (1) he refused to exercise the freedom of algebra beyond stating that, as an algebraist, he had chosen to assign the laws of arithmetic to symbolical algebra; (2) in a few passages in the "Report" and the revised edition of the *Treatise* which appeared in 1842-1845, he seemed to abandon the principle of the freedom of algebra and to argue that the laws of arithmetic were necessarily the laws of algebra; and (3) he clearly believed that an element of freedom was involved in symbolical algebra only, and not in arithmetic or geometry. The first point was discussed above. The second, I believe, is the main reason why Peacock's recognition of the freedom of algebra has been missed by most historians of mathematics.

Put simply, he sometimes wrote as if that freedom did not exist. Although his "Report" included the explicit statement of algebraic freedom quoted above, it also referred to a necessary connection between the laws of arithmetic and those of symbolical algebra:

> *But though the science of arithmetic, or of arithmetical algebra, does not furnish an adequate foundation for the science of symbolical algebra, it necessarily* suggests *its principles, or rather its laws of combination; for in as much as symbolical algebra, though arbitrary in the authority of its principles, is not arbitrary in their application, being required to include arithmetical algebra as well as other sciences, it is evident that their rules must be identical with each other, as far as those sciences proceed together in common.* [Peacock 1833, 195]

Furthermore, in the second and substantially revised edition of the *Treatise on Algebra*, Peacock stated: "I believe that no views of the nature of Symbolical Algebra can be correct or philosophical which made the selection of its rules of combination arbitrary and independent of arithmetic" [Peacock 1845 2, 453]. These two passages were quoted by Elaine Koppelman as evidence that Peacock "insisted that the laws of arithmetic must always be valid" in symbolical algebra [Koppelman 1971, 216]. Koppelman's statement is, of course, correct as far as it goes, but it should be qualified by recognizing that in the first edition of the *Treatise* and in the "Report," Peacock argued that the necessity of adopting arithmetical laws as the laws of symbolical algebra rested upon the somewhat arbitrary acceptance of arithmetical algebra as the science of suggestion for symbolical algebra [9].

Not only did Peacock stop short of creating arbitrary symbolical laws--sometimes writing as if he had renounced any idea of mathematical freedom--he also restricted this freedom to symbolical algebra. Arithmetic and geometry, he argued, were based upon axioms which were "necessary and self-evident consequences of the definitions"; in symbolical algebra, however, there were "properly speaking, no axioms, since the propositions, *immediately* deducible from the definitions and assumptions, must be considered rather as the necessary and immediate consequences of defined operations, than the necessary and self-evident results of reasoning" [Peacock 1830, 112-113]. In the "Report" he explained: "We are supposed to be in possession of a science of arithmetical algebra . . . whose laws of combination are capable of strict demonstration, without the aid of any principle which is not furnished by our knowledge of common arithmetic" [Peacock 1833, 206]. Thus, according to Peacock, arithmetic and geometry were based upon self-evident, necessary axioms from which were derived the laws of these two sciences. Symbolical algebra, on the other hand, was not based on self-evident axioms, but on defined operations. The freedom of mathematics, then, was exercised only in symbolical algebra, in the definition of its operations through assignment of its law of combination.

NEGATIVE REACTION TO PEACOCK'S SYMBOLICAL APPROACH

In order to place a scientific development in historical perspective, the historian of science must frequently supplement accounts of the acceptance of the development with accounts of its rejection. Thus, in a study of early reaction to Peacock's major algebraic work, it is not sufficient to note that De Morgan, Gregory, and Boole accepted, with some variations, Peacock's symbolical approach to algebra. The rejection of the approach by Sir William Rowan Hamilton--a brilliant mathematician whose contributions to the development of abstract algebra at least equaled those of the aforementioned mathematicians--is as interesting and perhaps as significant as its adoption by these others. The story of Hamilton's opposition underscores the revolutionary nature of Peacock's work, as no account of its acceptance can.

In a letter of 1835 to John T. Graves, a close friend and mathematical correspondent, Hamilton tried to explain his reluctance to join Peacock's school:

> *we* [*Hamilton and Graves*] *belong to opposite poles in Algebra; since you, like Peacock, seem to consider Algebra as a "System of Signs and of their combinations," somewhat analogous to syllogisms expressed in letters; while I am never satisfied unless I think that I can look beyond or through the signs to the things signified. I habitually desire to find or make in Algebra a system of demonstrations resting at last on intuitions, analogous in some way or other to Geometry as presented by Euclid.* [Graves 1885 2, 143]

Again, about ten years later, Hamilton offered the following description of his early negative reaction to Peacock's *Treatise* of 1830:

> *When I first read that work . . . and indeed for a long time afterwards, it seemed to me, I own . . . that the author designed to reduce algebra to a mere system of symbols, and* nothing *more; an affair of pothooks and hangers, of black strokes upon white paper, to be made according to a fixed but arbitrary set of rules: and I refused, in my own mind, to give the high name of*

> Science *to the results of such a system; as I should, even now, think it a stretch of courtesy, however it may be allowed by custom, to speak of chess as a "science," though it may well be called a "scientific game."* [Graves 1885 2, 528].

Thus, at least in the 1830s and early 1840s, Hamilton could not accept Peacock's modern idea of symbolical algebra as a science of basically arbitrary symbols and laws. He clearly desired to retreat to the traditional, and what he regarded as the more philosophically satisfying, view of algebra as a science of meaningful symbols governed by necessary principles stemming from intuition.

Although Hamilton's solution to the problem of the negative and imaginary numbers rested on his definition of algebra as the science of pure time [Hamilton 1837], his reaction to Peacock's symbolical approach should not be dismissed as the eccentric view of a mathematician obsessed with metaphysics. Rather it should be taken as the thoughtful response of an early-19th-century mathematician who was well steeped in traditional mathematics and so expected all mathematics to be meaningful and necessary in more than a strictly logical way. In short, his rejection of the symbolical approach underscores the novelty of Peacock's algebraic work, in which the centuries-old notion of the necessity of mathematical principles was abandoned and the notion of the arbitrariness of the symbols and laws of algebra introduced. In his expression of concern for the direction in which Peacock was taking algebra, Hamilton probably spoke for the majority of his mathematical contemporaries and certainly for those modern mathematicians and philosophers of mathematics who still object to an approach to algebra which makes it akin to a game, albeit an exceptionally fruitful game [10].

On this issue of meaning in mathematics, Hamilton was on the losing side; Peacock, on the winning. In the middle third of the 19th century De Morgan, Gregory, and Boole adopted and refined Peacock's symbolical approach to algebra, and in 1843, with the invention of the quaternions, Hamilton, somewhat ironically, became the first to exercise the freedom of mathematics which Peacock had merely proposed. By 1846 even Hamilton was willing "to admit that there is a sort of symbolical science, or *science of language*, which well deserves to be studied, abstraction being made for a while of *meaning*, or interpretation" [Graves 1885 2, 521]. In short, Peacock's reforming tendencies, directed at an algebra marred by the problem of negative numbers, resulted in the formulation of the symbolical approach to algebra which, capped by Hamilton's work on the quaternions, emerged as the modern abstract approach to algebra, clearly the dominant approach from the late 19th century through the present.

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NOTES

A shorter version of this paper was presented in March 1979 at the annual meeting of the Wisconsin Section of the Mathematical Association of America at the University of Wisconsin-Eau Claire.

1. Peacock also published a revised version of his *Treatise on Algebra* [1842-1845]. However, the present paper is based on an analysis of the first edition of the *Treatise* and the "Report" of 1833, since Peacock's reputation as one of the earliest British proponents of the symbolical approach to algebra rests upon these two earlier works.

2. For a description of the origins and activities of the Analytical Society, see Dubbey [1963; 1978, 31-50] and Koppelman [1971, 179-187].

3. This paper does not explore the question of the possible influence of Babbage's unpublished "Philosophy of Analysis" on Peacock's algebraic ideas, since at the time of the writing of the paper the present author had not seen the relevant unpublished manuscript.

4. The problem of negative numbers was discussed in the *Transactions of the Royal Society of Edinburgh* by William Greenfield [1788]; in the *Philosophical Transactions of the Royal Society of London* by John Playfair [1778], Robert Woodhouse [1801], Adrien-Quentin Buée [1806], and Davies Gilbert [1831]; in Charles Hutton's *Mathematical and Philosophical Dictionary* [1795 2, 147]; and in algebra textbooks by John Walker [1812, 71] and John Bonnycastle [1820 2, 3-4]. Unlike Maseres and Frend, these authors advocated retention of the negatives.

5. Nagel has suggested that Woodhouse also deserves some credit for laying "the basis for the subsequent development of algebra" because of "his penetrating remarks on the nature of symbolic reasoning" (in [Woodhouse 1801]) [Nagel 1935, 446].

6. In an interesting study of the development of British symbolical algebra, Koppelman [1971] has argued that the change in algebra from emphasizing the meaning or nature of symbols to stressing the laws of their combination resulted from the early-19th-century work on the calculus of operations by Peacock and other leading British mathematicians of the period. Although the calculus of operations certainly contributed to the emergence of the symbolical approach to algebra, the preceding comments suggest that there were indications of this approach in late-18th-century British algebra.

7. The imaginary numbers were introduced into symbolical algebra through the assumption of the existence of the sign $\sqrt{-1}$. As Peacock declared: "if therefore the primitive and *recognized* signs of affection be + and - or +1 and -1, considered as symbolical multipliers of the symbols or quantities they precede, I assume $1^{1/n}$ or $(-1)^{1/n}$ or any other expressions which are symbolically equivalent to the, to represent the appropriate signs of affection of the corresponding arithmetical roots [Peacock 1830, xxviii]. In the "Report," however, Peacock referred not only to the independent signs + and -, but also to the "quantities ... $-c$ and $+c$" [Peacock 1833, 195].

8. Peacock's statement of the arbitrariness of the laws of symbolical algebra was mentioned, but not explored, by Nagel [1935, 454-455] and Novy [1973, 192-194].

9. As far as I have been able to determine, the revised version of Peacock's *Treatise on Algebra* [1842-1845] does not include any statement of the arbitrarines of symbolical laws equivalent to those found in the *Treatise* of 1830 or the "Report" of 1833.

10. For a fairly recent general discussion of the problem of meaning in mathematics, see [Birkhoff 1975, esp. 507-521, 529-533].

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