Computational Geometry 45 (2012) 458–473

Contents lists available at SciVerse ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Classification of direct kinematics to planar generalized Stewart platforms ☆

Gui-Fang Zhang

College of Sciences, Beijing Forestry University, Beijing 100083, China

ARTICLE INFO

Article history:
Received 30 October 2009
Accepted 1 June 2010
Available online 27 January 2012
Communicated by D. Michelucci

Keywords:
Geometric constraint solving
Parallel robotics
Planar generalized Stewart platform
Direct kinematics
Closed-form solution

ABSTRACT

This paper presents the classification of direct kinematics for the planar generalized Stewart platform (GSP) which consists of two rigid bodies connected by three constraints between three pairs of points or lines in the base and the moving platforms. For each of the sixteen forms of planar GSPs, we give the explicit conditions on the parameters for the GSPs to have a given number of real solutions.

1. Introduction

The Stewart platform is a spatial parallel manipulator consisting of two rigid bodies: a moving platform (simply a platform), and a base whose pose (position and orientation) is fixed. The base and the platform are connected by six extensible legs. The Stewart platform is originated from the mechanism designed by Stewart for flight simulation [17] and the mechanism designed by Gough for tire test [8]. For a set of given lengths of the six legs, the pose of the platform could generally be determined. The Stewart platform has been studied extensively and has many applications. Comparing to serial mechanisms, the main advantages of the Stewart platform are its inherent stiffness and high load/weight ratio. The Stewart platform has been studied extensively and has many applications. More information on the Stewart platform can be found in [1,3,9,10,13,15]. A large portion of the work on Stewart platform is focused on the *direct kinematics* [9,10,13,15] which can be considered as a geometric constraint problem.

Although a majority of the work on Stewart platform focuses on the spatial case, several people also considered the planar Stewart platform which consists of a moving platform and a base connected by three extensible legs. The planar parallel manipulators shown in Figs. 1 and 2 are two typical planar Stewart platforms [13]. Gosselin and Merlet developed robust solving schemes and established sharper bounds for special planar Stewart platforms [7]. In [16], Pennock and Kanssner proved that the upper bound of the number of solutions for the direct kinematics of the planar Stewart platform is six. Other interesting work on the planar Stewart platform could be found in [2,11,12].

In [5], to find new and more practical parallel mechanisms for various purposes, the spatial *generalized Stewart platform* (abbr. GSP) consisting of two rigid bodies connected by six distance and/or angular constraints between six pairs of points, lines and/or planes in the base and moving platform respectively is introduced, which could be considered as the most general form of parallel manipulators with six DOFs in certain sense and a special class of geometric constraint problems.

☆ This research was partially supported by an NSFC grant (No. 60803099) and the Fundamental Research Funds for the Central Universities.
E-mail address: gfzhang@bjfu.edu.cn.

0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2010.06.007

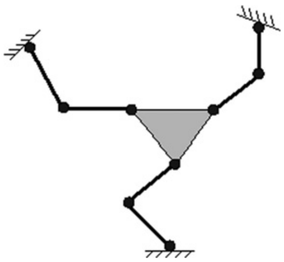

Fig. 1. 3-RPR planar parallel robot.

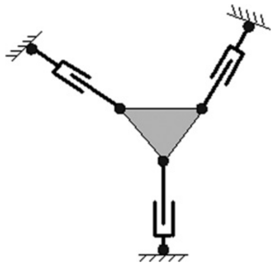

Fig. 2. 3-RPR planar parallel robot.

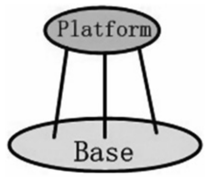

Fig. 3. Planar generalized Stewart platform.

In [20], the planar GSP shown in Fig. 3 is introduced, which could be considered as the most general form of planar parallel manipulators with three DOFs at some extent. A planar GSP consists of a fixed rigid body (called *base*) and a movable rigid body (called *platform*) connected by three distance or/and angular constraints between three pairs of points and/or lines on the base and platform. The pose of the platform is determined by the values of the three constraints.

Geometric constraint solving (GCS) is the key technique of parametric CAD, which allows the user to make modifications to existing designs by changing parameter values. There are four major approaches to geometric constraint solving: the numerical approach, the symbolic computation approach, the rule-based approach and the graph based approach. GCS methods may also be used in other fields like robotics, computer vision, molecular modeling, feature-based design and so on. For a review on geometric constraint solving and its applications can be found in [4] and references therein.

From the viewpoint of GCS, direct kinematics GSP is a typical geometric constraint solving problem. In [6], a general geometric constraint problem is reduced to three minimal merge patterns: (1) to compute the position of a single geometric primitive, (2) to compute the pose of a rigid body, and (3) the general merge pattern. The direct kinematics GSP is actually to merge or assemble two rigid bodies. The direct kinematics is to solve an algebraic equation system with several parameters. Using the characteristic set method [14,19], the solving of parametric equation systems is reduced to the resolution of equations in triangular form which is called closed-form solutions in [20] and hence the solving of univariate equations. In [20], it is shown that closed-form solutions to the direct kinematics of all planar GSPs could be found with the characteristic set method. With these closed-form solutions, upper bounds for the number of solutions of the direct kinematics in the general cases can also be given. For a class of GSPs involving an angular constraint, a solution to the direct kinematics based on ruler and compass constructions was provided.

The research of classification of linkages is an interesting and important problem. The reason is that we can know whether the direct kinematics exist, and obtain the number of solutions to direct kinematics directly with the given parameters furthermore, once the condition of the parameters for a planar GSP is given. In [18], Su et al. classified the movement of the RRSS spatial linkage in terms of its link dimensions with the method in [21], where the highest degree of the polynomial is four. In this paper, we give the classification of direct kinematics for sixteen planar GSPs and the explicit conditions on the parameters for the GSP to have a given number of real solutions. The rest of the paper is organized as follows. In Section 2, the basic concepts to planar GSP are given. In Section 3, we give the classification of direct kinematics for the sixteen planar GSPs. In Section 4, conclusions are given.

2. Basic concepts to planar GSP

A rigid body in the plane has three DOFs. Therefore to determine its pose, we need three geometric constraints. The planar GSP can be divided into two classes according to the three constraints added. **DDA** means there are one angular and two distance constraints to be imposed. **DDD** means there are three distance constraints to be imposed. We cannot have more than one angular constraints due to the fact that a rigid body in the plane has one rotational DOF and the rotational DOF can generally be determined by one angular constraint.

We use **LP** to represent constraint between a line $\mathbf{L}$ and a point $\mathbf{P}$ in a GSP. Thus $\mathbf{P}_1\mathbf{P}_2$ represents constraint between two points $\mathbf{P}_1$ and $\mathbf{P}_2$ in a GSP and $\mathbf{L}_1\mathbf{L}_2$ represents constraint between two lines $\mathbf{L}_1$ and $\mathbf{L}_2$ in a GSP. A GSP can be represented by the primitives involved in the three constraints. For example, **LLL-PPP** represents a GSP consisting of three lines in the platform and three points in the base, and **PPP-LLL** represents a GSP consisting of three points in the platform and three lines in the base. Thus six different sub-cases of **DDA** are **LLL-LPP**, **LLP-LPP**, **LLP-LPL**, **LPP-LLL**, **LPP-LLP** and **LPP-LPP**. Ten different sub-cases of **DDD** are **PPP-LLL**, **PPP-LLP**, **PPP-LPP**, **LLL-PPP**, **LLP-PPP**, **LPP-PPP**, **LPP-PLL**, **LLP-PPL**, **LPP-PLP** and **PPP-PPP**.

Because the primitives involved in the base and the primitives are points and lines, we can always take three points in the base and three points in the platform, respectively. For a line, we can take a point on it. Let three points in the base be $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$, and three points in the platform be $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$.

Let $\mathbf{B}_1$ be the origin of the fixed coordinate system in the base, $\mathbf{B}_1\mathbf{B}_2$ the x-axis. The coordinates of three points in the base are $\mathbf{B}_1 = (0,0), \mathbf{B}_2 = (b_1,0)$ and $\mathbf{B}_3 = (b_2,b_3)$. Because a rigid body cannot be fixed with one point, b_1,b_2,b_3 should not equal to zero simultaneously. So we could let $b_1 > 0$. And if $b_3 = 0$, three points in the base are colinear.

Assuming that point $\mathbf{D}$ is the foot of perpendicular of point $\mathbf{P}_3$ to $\mathbf{P}_1\mathbf{P}_2$, let point $\mathbf{D}$ be the origin of the moving coordinate system in the platform. The coordinate of point $\mathbf{D}$ in the fixed coordinate system is $\mathbf{D} = (x_3,x_4)$. Let $\angle(\mathbf{B}_1\mathbf{B}_2,\mathbf{P}_1\mathbf{P}_2) = \theta$, $x_1 = \cos\theta, x_2 = \sin\theta$. The moving coordinates of three points in the platform are $\mathbf{P}_1 = (-h_1,0)$, $\mathbf{P}_2 = (h_2,0)$, $\mathbf{P}_3 = (0,h_3)$, where h_1,h_2 are two nonnegative parameters [12]. Because a rigid body cannot be fixed with one point, h_1,h_2,h_3 should not equal to zero simultaneously, we could let $h_1 + h_2 > 0$. $\mathbf{P}_1\mathbf{P}_2$ is the x-axis of the moving coordinate system. Their coordinates in the fixed coordinate system are

$$\begin{cases} \mathbf{P}'_1 = (-h_1x_1 + x_3, -h_1x_2 + x_4), \\ \mathbf{P}'_2 = (h_2x_1 + x_3, h_2x_2 + x_4), \\ \mathbf{P}'_3 = (-h_3x_2 + x_3, h_3x_1 + x_4). \end{cases}$$

There exist at most three lines in the base which satisfy the three distance constraints. Let the parametric equations of these lines be

$$\begin{cases} \mathbf{L}_1\colon\ \mathbf{P} = \mathbf{B}_3 + u_1\mathbf{s}_1 \quad \big(\mathbf{s}_1 = (l_1,m_1), |\mathbf{s}_1| = 1\big), \\ \mathbf{L}_2\colon\ \mathbf{P} = \mathbf{B}_2 + u_2\mathbf{s}_2 \quad \big(\mathbf{s}_2 = (l_2,m_2), |\mathbf{s}_2| = 1\big), \\ \mathbf{L}_3\colon\ \mathbf{P} = \mathbf{B}_1 + u_3\mathbf{s}_3 \quad \big(\mathbf{s}_3 = (l_3,m_3), |\mathbf{s}_3| = 1\big). \end{cases}$$

There exist at most three lines in the platform which satisfy the three distance constraints. Let the parametric equations of these lines be

$$\begin{cases} \mathbf{L}_{01}\colon\ \mathbf{P} = \mathbf{P}_3 + u_1\mathbf{s}_1 \quad \big(\mathbf{s}_1 = (l_1,m_1), |\mathbf{s}_1| = 1\big), \\ \mathbf{L}_{02}\colon\ \mathbf{P} = \mathbf{P}_2 + u_2\mathbf{s}_2 \quad \big(\mathbf{s}_2 = (l_2,m_2), |\mathbf{s}_2| = 1\big), \\ \mathbf{L}_{03}\colon\ \mathbf{P} = \mathbf{P}_1 + u_3\mathbf{s}_3 \quad \big(\mathbf{s}_3 = (l_3,m_3), |\mathbf{s}_3| = 1\big). \end{cases}$$

Although we use the same $\mathbf{s}_i$ in $\mathbf{L}_i$ and $\mathbf{L}_{0i}$ $(i = 1,2,3)$, there will cause no confusion. The reason is that lines $\mathbf{s}_i$ in $\mathbf{L}_i$ and $\mathbf{L}_{0i}$ $(i = 1,2,3)$ will not appear in the same cases when the three constraints between the base and the platform are three distance constraint simultaneity. After the three distance constraint are imposed, the corresponding parametric equations of three lines in the platform are

$$\begin{cases} \mathbf{L}_{11}\colon\ \mathbf{P} = \mathbf{P}'_3 + u_1\mathbf{s}_{11}, \quad |\mathbf{s}_{11}| = 1,\ \mathbf{s}_{11} = (l_1x_1 - m_1x_2, l_1x_2 + m_1x_1), \\ \mathbf{L}_{22}\colon\ \mathbf{P} = \mathbf{P}'_2 + u_2\mathbf{s}_{22}, \quad |\mathbf{s}_{22}| = 1,\ \mathbf{s}_{22} = (l_2x_1 - m_2x_2, l_2x_2 + m_2x_1), \\ \mathbf{L}_{33}\colon\ \mathbf{P} = \mathbf{P}'_1 + u_3\mathbf{s}_{33}, \quad |\mathbf{s}_{33}| = 1,\ \mathbf{s}_{33} = (l_3x_1 - m_3x_2, l_3x_2 + m_3x_1). \end{cases}$$

In the following sections, we use $|\mathbf{PL}|$ to denote the distance between point $\mathbf{P}$ and line $\mathbf{L}$, and $|\mathbf{P}_1\mathbf{P}_2|$ to denote the distance between two points $\mathbf{P}_1$ and $\mathbf{P}_2$, where the distance between two points is more than zero.

3. Classification of direct kinematics to planar generalized Stewart platform

3.1. Case DDA

For **DDA**, we will impose angular constraint firstly. Because the expressions of angular constraint only involves unit vectors parallel to the corresponding line on the platform or the base. So we need only to consider angular constraints between two unit vectors. Let $\mathbf{s}_1$ be unit vector on the base which is parallel to line $\mathbf{B}_1\mathbf{B}_2$ and $\mathbf{s}_2$ unit vector on the platform which is parallel to line $\mathbf{P}_1\mathbf{P}_2$, and $\angle(\mathbf{s}_1, \mathbf{R}\mathbf{s}_2) = \theta$. Assuming that the rotational matrix is $\mathbf{R} = (r_{ij})_{2\times 2}$ and the angular constraint is $\cos\theta = x_1 (x_2 = \sin\theta)$. Let $\mathbf{s}_1 = (1, 0)$ and $\mathbf{s}_2 = (1, 0)$, we can obtain the following equation system.

$$\begin{cases} \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}, \\ det(\mathbf{R}) = 1, \\ \mathbf{s}_1 \cdot \mathbf{R}\mathbf{s}_2 = x_1. \end{cases} \tag{1}$$

Equation system (1) can be reduced to the following triangular form with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \mathbf{r}_{12}^2 - 1 + x_1^2 = 0, \\ \mathbf{r}_{21} + r_{12} = 0, \\ \mathbf{r}_{22} - x_1 = 0, \\ \mathbf{r}_{11} - x_1 = 0. \end{cases} \tag{2}$$

It is obvious that equation system (2) has two real solutions if $x_1 \neq 1$. If $x_1 = 1$, equation system (2) has one real solution.

After the angular constraint is imposed, we will impose the two remaining distance constraints simultaneously. It is clear that imposing distance constraints will not break the angular constraint imposed previously. Thus we only need to solve an equation system consisting of two distance constraints.

3.1.1. Case LL-PP

In this case, each of the two distance constraints is between a line in the platform and a point in the base. Let the distance constraints be $|L_{11}B_3| = d_{13}$ and $|L_{22}B_2| = d_{22}$. The equation system is as follows, where $d_1 = \pm d_{13}$ and $d_2 = \pm d_{22}$.

$$\begin{cases} (l_1x_2 + m_1x_1)(-h_3x_2 + x_3 - b_2) - (l_1x_1 - m_1x_2)(h_3x_1 + x_4 - b_3) - d_1 = 0, \\ (l_2x_2 + m_2x_1)(h_2x_1 + x_3 - b_1) - (l_2x_1 - m_2x_2)(h_2x_2 + x_4) - d_2 = 0. \end{cases} \tag{3}$$

If $m_2l_1 - l_2m_1 \neq 0$, equation system (3) can be reduced to triangular form (4) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} (m_2l_1 - l_2m_1)\mathbf{x}_3 + \big((l_2m_1 + m_2l_1)(b_1 - b_2) + (l_2l_1 - m_2m_1)b_3\big)x_2^2 \\ \quad + \Big(\big((m_2m_1 - l_2l_1)(b_1 - b_2) + (l_2m_1 + m_2l_1)b_3\big)x_1 - m_1m_2h_2 - d_1m_2 - l_1h_3m_2 + m_1d_2\Big)x_2 \\ \quad + (l_2l_1h_3 - d_2l_1 + l_2d_1 + m_2h_2l_1)x_1 - b_1l_1m_2 - l_2l_1b_3 + l_2m_1b_2 = 0, \\ (m_2l_1 - l_2m_1)\mathbf{x}_4 + \big((m_2m_1 - l_2l_1)(b_1 - b_2) + (l_2m_1 + m_2l_1)b_3\big)x_2^2 \\ \quad + \Big(\big((l_2m_1 + m_2l_1)(b_2 - b_1) + (m_2m_1 - l_2l_1)b_3\big)x_1 + l_2l_1h_3 - d_2l_1 + l_2d_1 + m_2h_2l_1\Big)x_2 \\ \quad + (m_1m_2h_2 + d_1m_2 - m_1d_2 + l_1h_3m_2)x_1 + m_1m_2(b_2 - b_1) - l_1b_3m_2 = 0. \end{cases} \tag{4}$$

If $m_2l_1 - l_2m_1 = 0$, two lines in the platform are parallel. There is no solution and the pose of the platform cannot be fixed.

3.1.2. Case LP-PL

In this case, one distance constraint is between a point in the platform and a line in the base, the other is between a line in the platform and a point in the base. Let the distance constraints be $|L_{11}B_3| = d_{13}$ and $|P_2'L_2| = d_{22}$. The equation system is as follows, where $d_1 = \pm d_{13}$ and $d_2 = \pm d_{22}$ respectively.

$$\begin{cases} (l_1x_2 + m_1x_1)(-h_3x_2 + x_3 - b_2) - (l_1x_1 - m_1x_2)(h_3x_1 + x_4 - b_3) - d_1 = 0, \\ m_2(h_2x_1 + x_3 - b_1) - l_2(h_2x_2 + x_4) - d_2 = 0. \end{cases} \tag{5}$$

If $(m_2l_1 - l_2m_1)x_1 - (l_2l_1 + m_2m_1)x_2 \neq 0$, equation system (5) can be reduced to triangular form (6) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \big((m_2l_1 - l_2m_1)x_1 - (l_2l_1 + m_2m_1)x_2\big)\mathbf{x}_3 + (l_2m_1 - m_2l_1)h_2x_2^2 + \big(-(l_2l_1 + m_2m_1)h_2x_1 + l_2l_1b_2 + l_2m_1b_3 \\ \quad + m_1d_2 + m_1m_2b_1\big)x_2 + (-l_1d_2 + l_2m_1b_2 - l_2l_1b_3 - l_1m_2b_1)x_1 + m_2h_2l_1 + l_2d_1 + l_2l_1h_3 = 0, \\ \big((m_2l_1 - l_2m_1)x_1 - (l_2l_1 + m_2m_1)x_2\big)\mathbf{x}_4 - (l_2l_1 + m_2m_1)h_2x_2^2 + \big((-l_2m_1 + m_2l_1)h_2x_1 + m_2m_1b_3 \\ \quad + m_2l_1(b_2 - b_1) - l_1d_2\big)x_2 + \big(m_2m_1(b_2 - b_1) - m_1d_2 - m_2l_1b_3\big)x_1 + m_1m_2h_2 + m_2d_1 + m_2l_1h_3 = 0. \end{cases} \tag{6}$$

If $(m_2l_1 - l_2m_1)x_1 - (l_2l_1 + m_2m_1)x_2 = 0$, there is no finite solution and the pose of the platform cannot be determined.

3.1.3. Case LP-PP

In this case, one distance constraint is between a line in the platform and a point in the base, the other is between a point in the platform and a point in the base. Let the two distance constraints be $|L_{11}B_3| = d_{13}$ and $|P'_2B_2| = t_{22}$. The equation system is as follows, where $d_1 = \pm d_{13}$ and $d_2 = t_{22}^2 > 0$.

$$\begin{cases} (l_1x_2 + m_1x_1)(-h_3x_2 + x_3 - b_2) - (l_1x_1 - m_1x_2)(h_3x_1 + x_4 - b_3) - d_1 = 0, \\ (h_2x_1 + x_3 - b_1)^2 + (h_2x_2 + x_4)^2 - d_2 = 0. \end{cases} \tag{7}$$

If $m_1x_2 - l_1x_1 \neq 0$, equation system (7) can be reduced to triangular form (8) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \mathbf{x}_3^2 + 2\big(\big(\big(m_1^2 - l_1^2\big)(b_2 - b_1) - 2m_1l_1b_3\big)x_2^2 + \big(\big(\big(l_1^2 - m_1^2\big)b_3 + 2l_1m_1(b_1 - b_2)\big)x_1 - l_1(m_1h_2 + l_1h_3 + d_1)\big)x_2 \\ \quad + \big(l_1^2h_2 - m_1d_1 - m_1l_1h_3\big)x_1 + m_1l_1b_3 - m_1^2b_2 - l_1^2b_1\big)\mathbf{x}_3 + 2\big(\big(m_1^2 - l_1^2\big)h_2b_3 + 2m_1l_1h_2(b_2 - b_1)\big)x_2^3 \\ \quad + \big(2\big(\big(m_1^2 - l_1^2\big)h_2(b_2 - b_1) - 2l_1b_3m_1h_2\big)x_1 + \big(m_1^2 - l_1^2\big)\big(b_3^2 - b_2^2 + b_1^2 + h_2^2 - d_2\big) + 2d_1m_1h_2 + 4m_1b_2l_1b_3 \\ \quad + 2l_1h_3m_1h_2\big)x_2^2 + 2\big(\big(\big(m_1^2 - l_1^2\big)b_2b_3 - l_1^2h_3h_2 + l_1m_1\big(d_2 - b_3^2 - b_1^2 - h_2^2 + b_2^2\big) - d_1l_1h_2\big)x_1 \\ \quad + (l_1b_2 + m_1b_3)d_1 + 2l_1m_1h_2b_1 + (m_1h_3 + l_1h_2)l_1b_3 + \big(l_1^2h_3 - m_1l_1h_2\big)b_2\big)x_2 + 2\big((m_1b_2 - l_1b_3)d_1 \\ \quad + m_1b_2l_1h_3 - l_1^2(h_3b_3 + h_2b_1)\big)x_1 + l_1^2\big(b_1^2 + h_2^2 - d_2\big) + (l_1h_3 + d_1)^2 + (m_1b_2 - l_1b_3)^2 = 0, \\ (m_1x_2 - l_1x_1)\mathbf{x}_4 + (l_1x_2 + m_1x_1)x_3 - (m_1b_3 + l_1b_2)x_2 - (m_1b_2 - l_1b_3)x_1 - d_1 - l_1h_3 = 0. \end{cases} \tag{8}$$

If $m_1x_2 - l_1x_1 = 0$, we can get the following equation system.

$$\begin{cases} x_1^2 + x_2^2 - 1 = 0, \\ (l_1x_2 + m_1x_1)(-h_3x_2 + x_3 - b_2) - (l_1x_1 - m_1x_2)(h_3x_1 + x_4 - b_3) - d_1 = 0, \\ (h_2x_1 + x_3 - b_1)^2 + (h_2x_2 + x_4)^2 - d_2 = 0, \\ m_1x_2 - l_1x_1 = 0. \end{cases} \tag{9}$$

If $m_1 \neq 0$, $l_1 \neq 0$, equation system (9) can be reduced to triangular form (10) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \mathbf{x}_2^2 - l_1^2 = 0, \\ l_1\mathbf{x}_1 - m_1x_2 = 0, \\ x_1\mathbf{x}_3 - b_2x_1 - m_1(d_1 + l_1h_3) = 0, \\ m_1\mathbf{x}_4^2 + 2h_2l_1x_1\mathbf{x}_4 + 2(b_2 - b_1)(d_1 + l_1h_3 + m_1h_2)x_1 \\ \quad + m_1\big((d_1 + l_1h_3)^2 + 2m_1h_2(d_1 + l_1h_3) + (b_2 - b_1)^2 - d_2 + h_2^2\big) = 0. \end{cases} \tag{10}$$

It is obviously that for the case that $m_1x_2 - l_1x_1 = 0$, only when $x_2^2 = l_1^2$, we have solution and the pose of the platform can be determined.

If $m_1 = 0$ and $m_1x_2 - l_1x_1 = 0$, we can get $l_1 = \pm 1$ and $x_1 = 0$. Thus we can get the following equation system.

$$\begin{cases} x_2^2 - 1 = 0, \\ l_1x_2(-h_3x_2 + x_3 - b_2) - d_1 = 0, \\ (x_3 - b_1)^2 + (h_2x_2 + x_4)^2 - d_2 = 0. \end{cases} \tag{11}$$

Equation system (11) can be reduced to triangular form (12) and (13) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \mathbf{x}_2 - 1 = 0, \\ l_1\mathbf{x}_3 - d_1 - l_1(b_2 + h_3) = 0, \\ \mathbf{x}_4^2 + 2h_2\mathbf{x}_4 + 2(h_3 - b_1 + b_2)l_1d_1 + (h_3 - b_1 + b_2)^2 + h_2^2 - d_2 + d_1^2 = 0. \end{cases} \tag{12}$$

$$\begin{cases} \mathbf{x}_2 + 1 = 0, \\ l_1\mathbf{x}_3 + d_1 - l_1(b_2 - h_3) = 0, \\ \mathbf{x}_4^2 - 2h_2\mathbf{x}_4 + 2(h_3 + b_1 - b_2)l_1d_1 + (h_3 + b_1 - b_2)^2 + h_2^2 - d_2 + d_1^2 = 0. \end{cases} \tag{13}$$

If $l_1 = 0$ and $m_1x_2 - l_1x_1 = 0$, we can get $m_1 = \pm 1$ and $x_2 = 0$. Thus we can get the following equation system.

$$\begin{cases} x_1^2 - 1 = 0, \\ m_1x_1(x_3 - b_2) - d_1 = 0, \\ (h_2x_1 + x_3 - b_1)^2 + (h_2x_2 + x_4)^2 - d_2 = 0. \end{cases} \tag{14}$$

Equation system (14) can be reduced to triangular form (15) and (16) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \mathbf{x}_1 - 1 = 0, \\ m_1\mathbf{x}_3 - d_1 - m_1 b_2 = 0, \\ \mathbf{x}_4^2 - d_2 + d_1^2 + 2m_1(b_2 - b_1 + h_2)d_1 + b_2^2 + 2(h_2 - b_1)b_2 + b_1^2 - 2h_2 b_1 + h_2^2 = 0, \end{cases} \tag{15}$$

$$\begin{cases} \mathbf{x}_1 + 1 = 0, \\ m_1\mathbf{x}_3 + d_1 - m_1 b_2 = 0, \\ \mathbf{x}_4^2 - d_2 + d_1^2 + 2m_1(b_1 - b_2 + h_2)d_1 + b_2^2 - 2(b_1 + h_2)b_2 + b_1^2 + 2h_2 b_1 + h_2^2 = 0. \end{cases} \tag{16}$$

3.1.4. Case PP-LL

In this case, each of the two distance constraints is between a point in the platform and a line in the base. Let the two distance constraints be $|P_3'L_1| = d_{31}$ and $|P_2'L_2| = d_{22}$. The equation system is as follows, where $d_1 = \pm d_{31}$ and $d_2 = \pm d_{22}$ respectively.

$$\begin{cases} m_1(-h_3x_2 + x_3 - b_2) - l_1(h_3x_1 + x_4 - b_3) - d_1 = 0, \\ m_2(h_2x_1 + x_3 - b_1) - l_2(h_2x_2 + x_4) - d_2 = 0. \end{cases} \tag{17}$$

If $l_1m_2 - m_1l_2 \neq 0$, equation system (17) can be reduced to triangular form (18) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} (l_1m_2 - l_2m_1)\mathbf{x}_3 + l_2(m_1h_3 - l_1h_2)x_2 + l_1(m_2h_2 + l_2h_3)x_1 + l_2(d_1 + m_1b_2 - l_1b_3) - l_1(m_2b_1 + d_2) = 0, \\ (l_1m_2 - l_2m_1)\mathbf{x}_4 + m_1(m_2h_3 - l_2h_2)x_2 + m_2(m_1h_2 + l_1h_3)x_1 + m_2(d_1 + m_1b_2 - l_1b_3) - m_1(m_2b_1 + d_2) = 0. \end{cases} \tag{18}$$

If $l_1m_2 - m_1l_2 = 0$, two lines in the base are parallel. There is no solution and the pose of the platform cannot be determined.

3.1.5. Case PP-LP

In this case, one distance constraint is between a point in the platform and a line in the base, the other distance constraint is between a point in the platform and a point in the base. Let two distance constraints be $|P_2'B_2| = t_{22}$ and $|P_3'L_1| = d_{13}$. The equation system is as follows, where $d_1 = t_{22}^2 > 0$ and $d_2 = \pm d_{13}$.

$$\begin{cases} (h_2x_1 + x_3 - b_1)^2 + (h_2x_2 + x_4)^2 - d_1 = 0, \\ m_1(-h_3x_2 + x_3 - b_2) - l_1(h_3x_1 + x_4 - b_3) - d_2 = 0. \end{cases} \tag{19}$$

If $l_1 \neq 0$ and $l_1^2 \neq m_1^2$, equation system (19) can be reduced to triangular form (20) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \left(l_1^2 - m_1^2\right)\mathbf{x}_3^2 + 2\big(m_1(m_1h_3 - l_1h_2)x_2 + l_1(m_1h_3 + l_1h_2)x_1 + m_1^2b_2 - l_1^2b_1 + m_1(d_2 - l_1b_3)\big)\mathbf{x}_3 \\ \quad + \left(\left(l_1^2 - m_1^2\right)h_3^2 + 2l_1h_2(m_1h_3 - l_1h_2)\right)x_2^2 + 2(l_1h_2 - m_1h_3)(l_1h_3x_1 + d_2 - l_1b_3 + m_1b_2)x_2 \\ \quad + 2\big((l_1b_3 - d_2 - m_1b_2)h_3 - l_1h_2b_1\big)l_1x_1 - (m_1b_2 + d_2 - l_1b_3)^2 + l_1^2\left(b_1^2 - d_1 + h_2^2 - h_3^2\right) = 0, \\ l_1\mathbf{x}_4 - m_1x_3 + m_1h_3x_2 + l_1h_3x_1 + m_1b_2 - l_1b_3 + d_2 = 0. \end{cases} \tag{20}$$

If $l_1 = 0$, equation system (19) can be reduced to triangular form (21) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} m_1\mathbf{x}_3 - m_1h_3x_2 - m_1b_2 - d_2 = 0, \\ \mathbf{x}_4^2 + 2h_2x_2\mathbf{x}_4 + \left(2h_2^2 - h_3^2\right)x_2^2 - 2h_3(h_2x_1 + b_2 - b_1 + m_1d_2)x_2 \\ \quad - 2h_2(b_2 - b_1 + m_1d_2)x_1 - (b_2 - b_1 + m_1d_2)^2 - h_2^2 + d_1 = 0. \end{cases} \tag{21}$$

If $l_1^2 = m_1^2$, then $m_1 = \pm l_1$ and $l_1 = \pm\frac{\sqrt{2}}{2}$.

If $m_1 = l_1$ and $(h_3 - h_2)x_2 + (h_2 + h_3)x_1 - (b_1 - b_2 + b_3 - d_3) \neq 0$, we can get the following equation system, where $d_3 = \frac{d_2}{m_1}$.

$$\begin{cases} (h_2x_1 + x_3 - b_1)^2 + (h_2x_2 + x_4)^2 - d_1 = 0, \\ (-h_3x_2 + x_3 - b_2) - (h_3x_1 + x_4 - b_3) - d_3 = 0. \end{cases} \tag{22}$$

Equation system (22) can be reduced to triangular form (23) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} 2\big((h_3-h_2)x_2+(h_2+h_3)x_1-(b_1-b_2+b_3-d_3)\big)\mathbf{x}_3+2h_2(h_3-h_2)x_2^2 \\ \quad +2(h_2-h_3)\big(h_3x_1-(b_3-b_2-d_3)\big)x_2+2\big(h_3(b_3-b_2-d_3)-h_2b_1\big)x_1 \\ \quad -d_1-(b_3-b_2-d_3)^2+b_1^2+h_2^2-h_3^2=0, \\ 2\big((h_3-h_2)x_2+(h_2+h_3)x_1-(b_1-b_2+b_3-d_3)\big)\mathbf{x}_4-2h_2(h_3+h_2)x_2^2 \\ \quad +2h_3\big((h_2+h_3)x_1-(b_1-b_2+b_3-d_3)\big)x_2-2(h_2+h_3)(b_1-b_2+b_3-d_3)x_1-d_1 \\ \quad +(h_2+h_3)^2+(b_1-b_2+b_3-d_3)^2=0. \end{cases} \tag{23}$$

If $m_1=l_1$ and $(h_3-h_2)x_2+(h_2+h_3)x_1-(b_1-b_2+b_3-d_3)=0$, we can get no solution.

If $m_1=-l_1$ and $(h_2+h_3)x_2+(h_2-h_3)x_1-(b_1-b_2-b_3-d_3)\neq 0$, we can get the following equation system, where $d_3=\frac{d_2}{m_1}$.

$$\begin{cases} (h_2x_1+x_3-b_1)^2+(h_1x_2+x_4)^2-d_1=0, \\ (-h_3x_2+x_3-b_2)+(h_3x_1+x_4-b_3)-d_3=0. \end{cases} \tag{24}$$

Equation system (24) can be reduced to triangular form (25) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} 2\big((h_2+h_3)x_2+(h_2-h_3)x_1-(b_1-b_2-b_3-d_3)\big)\mathbf{x}_3-2h_2\big(h_2+h_3\big)x_2^2 \\ \quad +2(h_2+h_3)\big(h_3x_1-(b_2+b_3+d_3)\big)x_2+2\big(h_3(b_2+b_3+d_3)-h_2b_1\big)x_1-d_1 \\ \quad -(b_2+b_3+d_3)^2+b_1^2+h_2^2-h_3^2=0, \\ 2\big((h_2+h_3)x_2+(h_2-h_3)x_1-(b_1-b_2-b_3-d_3)\big)\mathbf{x}_4+2h_2(h_2-h_3)x_2^2 \\ \quad -2h_3\big((h_2-h_3)x_1-(b_1-b_2-b_3-d_3)\big)x_2+2(h_2-h_3)(b_1-b_2-b_3-d_3)x_1+d_1 \\ \quad -(h_2-h_3)^2-(b_1-b_2-b_3-d_3)^2=0. \end{cases} \tag{25}$$

If $m_1=-l_1$ and $(h_2+h_3)x_2+(h_2-h_3)x_1-(b_1-b_2-b_3-d_3)=0$, we could get no solution.

3.1.6. Case PP-PP

In this case, each of the two distance constraints is between a point in the platform and a point in the base. Let the distance constraints be $|P_3'B_3|=t_{33}$ and $|P_2'B_2|=t_{22}$. The equation system is as follows, where $d_1=t_{33}^2>0$ and $d_2=t_{22}^2>0$ respectively.

$$\begin{cases} (-h_3x_2+x_3-b_2)^2+(h_3x_1+x_4-b_3)^2-d_1=0, \\ (h_2x_1+x_3-b_1)^2+(h_2x_2+x_4)^2-d_2=0. \end{cases} \tag{26}$$

If $h_2x_2-h_3x_1+b_3\neq 0$ and $2(h_3(b_2-b_1)+b_3h_2)x_2+2(h_2(b_2-b_1)-h_3b_3)x_1+b_3^2+(b_2-b_1)^2+h_3^2+h_2^2\neq 0$, equation system (26) can be reduced to triangular form (27) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} 4\big(2\big(h_3(b_2-b_1)+b_3h_2\big)x_2+2\big(h_2(b_2-b_1)-h_3b_3\big)x_1+b_3^2+(b_2-b_1)^2+h_3^2+h_2^2\big)\mathbf{x}_3^2 \\ \quad +\big(c_{31}x_2^2+(c_{32}x_1+c_{33})x_2+c_{34}x_1+c_{35}\big)\mathbf{x}_3+c_{36}x_2^3+(c_{37}x_1+c_{38})x_2^2 \\ \quad +(c_{39}x_1+c_{310})x_2+c_{311}x_1+c_{312}=0, \\ 2(h_2x_2-h_3x_1+b_3)\mathbf{x}_4+2(h_3x_2+h_2x_1+b_2-b_1)x_3-2h_3b_2x_2+2(h_3b_3-h_2b_1)x_1 \\ \quad -d_2+d_1-h_3^2+h_2^2-b_3^2-b_2^2+b_1^2=0. \end{cases} \tag{27}$$

If $h_2x_2-h_3x_1+b_3=0$, we have the following equation system.

$$\begin{cases} x_1^2+x_2^2-1=0, \\ (-h_3x_2+x_3-b_2)^2+(h_3x_1+x_4-b_3)^2-d_1=0, \\ (h_2x_1+x_3-b_1)^2+(h_2x_2+x_4)^2-d_2=0, \\ h_2x_2-h_3x_1+b_3=0. \end{cases} \tag{28}$$

If $(b_3^2+(b_2-b_1)^2-h_3^2-h_2^2)h_2\neq 0$, equation system (28) can be reduced to triangular form (29) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \left(h_3^2+h_2^2\right)\mathbf{x}_1^2-2h_3\mathbf{x}_1b_3+b_3^2-h_2^2=0, \\ h_2\mathbf{x}_2-h_3x_1+b_3=0, \\ 2\left(\left(h_2^2+h_3^2\right)x_1-h_3b_3+h_2(b_2-b_1)\right)\mathbf{x}_3+2\left(h_2(h_3b_3-h_2b_1)-b_2h_3^2\right)x_1+2b_2h_3b_3 \\ \quad +\left(b_1^2-b_2^2-b_3^2+h_2^2-h_3^2+d_1-d_2\right)h_2=0, \\ \left(h_3^2+h_2^2\right)\left(2\left(h_3^2+h_2^2\right)(b_1-b_2)x_1-\left((b_2-b_1)^2-b_3^2+h_2^2+h_3^2\right)h_2+2(b_2-b_1)h_3b_3\right)\mathbf{x}_4^2 \\ \quad +(c_{41}x_1+c_{42})\mathbf{x}_4+c_{43}x_1+c_{44}=0. \end{cases} \tag{29}$$

If $2(b_3h_2+h_3(b_2-b_1))x_2-2(b_3h_3-h_2(b_2-b_1))x_1+(b_2-b_1)^2+b_3^2+h_2^2+h_3^2=0$, we have the following equation system.

$$\begin{cases} x_1^2+x_2^2-1=0, \\ (-h_3x_2+x_3-b_2)^2+(h_3x_1+x_4-b_3)^2-d_1=0, \\ (h_2x_1+x_3-b_1)^2+(h_2x_2+x_4)^2-d_2=0, \\ 2\left(b_3h_2+h_3(b_2-b_1)\right)x_2-2\left(b_3h_3-h_2(b_2-b_1)\right)x_1+(b_2-b_1)^2+b_3^2+h_2^2+h_3^2=0. \end{cases} \tag{30}$$

If $b_3^2+(b_2-b_1)^2-h_3^2-h_2^2\neq 0$ and $(b_3h_2+h_3(b_2-b_1))(d_2-d_1)(h_3^2+h_2^2)(b_3^2+(b_2-b_1)^2)\neq 0$, equation system (30) can be reduced to triangular form (31) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \left(h_2^2+h_3^2\right)\left(b_3^2+\left(b_2^2-b_1^2\right)\right)\mathbf{x}_1^2+c_{11}x_1+c_{12}=0, \\ 2\left(b_3h_2+h_3(b_2-b_1)\right)\mathbf{x}_2-2\left(b_3h_3-h_2(b_2-b_1)\right)x_1+(b_2-b_1)^2+b_3^2+h_2^2+h_3^2=0, \\ (c_{31}x_1+c_{32})\mathbf{x}_3+c_{33}x_1+c_{34}=0, \\ (c_{41}x_1+c_{42})\mathbf{x}_4+c_{43}x_1+c_{44}=0. \end{cases} \tag{31}$$

If $h_2x_2-h_3x_1+b_3=0$ and $2(b_3h_2+h_3(b_2-b_1))x_2-2(b_3h_3-h_2(b_2-b_1))x_1+(b_2-b_1)^2+b_3^2+h_2^2+h_3^2=0$, we have the following equation system.

$$\begin{cases} x_1^2+x_2^2-1=0, \\ (-h_3x_2+x_3-b_2)^2+(h_3x_1+x_4-b_3)^2-d_1=0, \\ (h_2x_1+x_3-b_1)^2+(h_2x_2+x_4)^2-d_2=0, \\ h_2x_2-h_3x_1+b_3=0, \\ 2\left(b_3h_2+h_3(b_2-b_1)\right)x_2-2\left(b_3h_3-h_2(b_2-b_1)\right)x_1+(b_2-b_1)^2+b_3^2+h_2^2+h_3^2=0. \end{cases} \tag{32}$$

If $(b_1-b_2)^2+b_3^2-h_3^2-h_2^2=0$ and $(b_2-b_1)\neq 0$, equation system (32) can be reduced to triangular form (33) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \mathbf{x}_1+c_{11}=0, \\ \mathbf{x}_2+c_{21}=0, \\ \mathbf{x}_3+c_{31}=0, \\ \mathbf{x}_4^2+c_{41}\mathbf{x}_4+c_{42}=0. \end{cases} \tag{33}$$

For case **DDA**, the degree of freedom of triangular form to each GSP is no more than two, so it is ruler and compass constructible.

3.2. Case DDD

For case **DDD**, the problem becomes more complexity. The reason is that the parameters increase and we have to solve an equation system consisting of three distance constraints simultaneously. We will classify real solutions to direct kinematics for each planar GSP with the method in [21].

3.2.1. Case PPP-LLL

In this case, each of the three distance constraints is between a point in the platform and a line in the base. Let the three constraints be $|\mathbf{P}_1'\mathbf{L}_3|=d_{13}$, $|\mathbf{P}_2'\mathbf{L}_2|=d_{22}$, and $|\mathbf{P}_3'\mathbf{L}_1|=d_{31}$.

If $(l_3m_2-l_2m_3)m_1h_3+(l_1m_3-l_3m_1)l_2h_2+(l_1m_2-l_2m_1)l_3h_1\neq 0$ and $(l_2m_3-m_2l_3)^2h_3^2+(l_1m_3-m_1l_3)^2h_2^2+(l_1m_2-m_1l_2)^2h_1^2+2(l_1m_2-m_1l_2)(l_1m_3-m_1l_3)((l_2m_3-m_2l_3)(h_1+h_2)h_3+(m_2m_3+l_2l_3)h_1h_2)\neq 0$ equation system consisting of the three constraints and $x_1^2+x_2^2-1=0$ can be reduced to triangular form (34) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \mathbf{x}_1^2 + c_{11}x_1 + c_{12} = 0, \\ \mathbf{x}_2 + c_{21}x_1 + c_{22} = 0, \\ \mathbf{x}_3 + c_{31}x_1 + c_{32} = 0, \\ \mathbf{x}_4 + c_{41}x_1 + c_{42} = 0. \end{cases} \tag{34}$$

Thus, the number of solution to above characteristic set is equal to the number of solution to $x_1^2 + c_{11}x_1 + c_{12} = 0$. It is clear that there is two solution if $c_{11}^2 - 4c_{12} > 0$, one solution if $c_{11}^2 - 4c_{12} = 0$, and no solution if $c_{11}^2 - 4c_{12} < 0$.

3.2.2. Case LLL-PPP

In this case, each of the three distance constraints is between a line in the platform and a point in the base. Let the constraints be $|\mathbf{L}_{33}\mathbf{B}_1| = d_{31}$, $|\mathbf{L}_{22}\mathbf{B}_2| = d_{22}$, $|\mathbf{L}_{11}\mathbf{B}_3| = d_{13}$.

If $(l_2m_3 - l_3m_2)m_1b_3 + (l_2m_3 - l_3m_2)l_1b_2 + (m_1l_3 - m_3l_1)l_2b_1 \neq 0$ and $m_2l_3 - l_2m_3 \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (35) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \mathbf{x}_1^2 + c_{11}x_1 + c_{12} = 0, \\ \mathbf{x}_2 + c_{21}x_1 + c_{22} = 0, \\ \mathbf{x}_3 + c_{31}x_1 + c_{32} = 0, \\ (c_{40}x_1 + c_{41})\mathbf{x}_4 + c_{42}x_1 + c_{43} = 0. \end{cases} \tag{35}$$

3.2.3. Case LLP-PPL

In this case, one of the three distance constraints is between a point in the platform and a line in the base. Each of the remaining two distance constraints is between a line in the platform and a point in the base. Let the constraints be $|\mathbf{B}_1\mathbf{L}_{33}| = d_{13}$, $|\mathbf{B}_2\mathbf{L}_{22}| = d_{22}$, and $|\mathbf{D}_{33}\mathbf{L}_1| = d_{31}$.

If $l_1 \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (36) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \mathbf{x}_1^4 + c_{11}x_1^3 + c_{12}x_1^2 + c_{13}x_1 + c_{14} = 0, \\ (c_{20}x_1 + c_{21})\mathbf{x}_2 + c_{22}x_1^2 + c_{23}x_1 + c_{24} = 0, \\ \left(x_1^2 + Ax_1 + B\right)\mathbf{x}_3 + c_{31}x_1^3 + c_{32}x_1^2 + c_{33}x_1 + c_{34} = 0, \\ \left(x_1^2 + Ax_1 + B\right)\mathbf{x}_4 + c_{41}x_1^3 + c_{42}x_1^2 + c_{43}x_1 + c_{44} = 0. \end{cases} \tag{36}$$

3.2.4. Case LLP-PPP

In this case, one of the three distance constraints is between a point in the platform and a point in the base. Each of the remaining two distance constraints is between a line in the platform and a point in the base. Let the constraints be $|\mathbf{B}_1\mathbf{P}'_1| = t_{11}$, $|\mathbf{B}_2\mathbf{L}_{22}| = d_{22}$ and $|\mathbf{B}_3\mathbf{L}_{11}| = d_{31}$.

If $l_1m_1l_2h_1(l_1m_2 - l_2m_1) \neq 0$ and $((b_1 - b_2)^2 + b_3^2)(4((m_1^2 - l_1^2)m_2l_2 - (m_2^2 - l_2^2)m_1l_1)b_1b_3 + 2(4m_1l_1m_2l_2 + (m_1^2 - l_1^2)(m_2^2 - l_2^2))b_1b_2 - b_3^2 - b_2^2 - b_1^2) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (37) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \mathbf{x}_1^4 + c_{11}x_1^3 + c_{12}x_1^2 + c_{13}x_1 + c_{14} = 0, \\ (c_{20}x_1 + c_{21})\mathbf{x}_2 + c_{22}x_1^2 + c_{23}x_1 + c_{24} = 0, \\ (Ax_1 + B)\mathbf{x}_3 + c_{31}x_1^3 + c_{32}x_1^2 + c_{33}x_1 + c_{34} = 0, \\ (Ax_1 + B)\mathbf{x}_4 + c_{41}x_1^3 + c_{42}x_1^2 + c_{43}x_1 + c_{44} = 0. \end{cases} \tag{37}$$

If $b_3 = 0$, $b_1 = b_2$, $l_1m_1l_2h_1(l_1m_2 - l_2m_1) \neq 0$, $(-m_1d_{22} + m_1m_2h_2 + m_2d_{31} + l_1m_2h_3) \neq 0$, and $(l_1m_2h_1 - l_2m_1h_1 + l_2l_1h_3 + l_2d_{31} - l_1d_{22} + l_1m_2h_2)^2 + (l_1h_3 + m_1h_2 + d_{31})^2m_2^2 - 2m_1d_{22}(l_1h_3 + m_1l_1^2h_2 + d_{31})m_2 + m_1^2d_{22} \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (34) with Wu–Ritt's characteristic set method [14,19].

3.2.5. Case LPP-PLL

In this case, one of the three distance constraints is between a line in the platform and a point in the base. Each of the remaining two distance constraints is between a point in the platform and a line in the base. Let the constraints be $|\mathbf{B}_1\mathbf{L}_{33}| = d_{13}$, $|\mathbf{P}'_2\mathbf{L}_2| = d_{22}$, $|\mathbf{P}'_3\mathbf{L}_1| = d_{31}$.

If $(h_2^2 + h_3^2)(m_2l_1 - m_1l_2) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (38) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \mathbf{x}_1^4 + c_{11}x_1^3 + c_{12}x_1^2 + c_{13}x_1 + c_{14} = 0, \\ (c_{20}x_1 + c_{21})\mathbf{x}_2 + c_{22}x_1^2 + c_{23}x_1 + c_{24} = 0, \\ (Ax_1 + B)\mathbf{x}_3 + c_{31}x_1^2 + c_{32}x_1 + c_{33} = 0, \\ (Ax_1 + B)\mathbf{x}_4 + c_{41}x_1^2 + c_{42}x_1 + c_{43} = 0. \end{cases} \tag{38}$$

3.2.6. Case PPP-LLP

In this case, one of the three distance constraints is between a point in the platform and a point in the base. Each of the remaining two distance constraints is between a point in the platform and a line in the base. Let the constraints be $|\mathbf{P}_1'\mathbf{B}_1| = t_{11}$, $|\mathbf{P}_2'\mathbf{L}_2| = d_{22}$, $|\mathbf{P}_3'\mathbf{L}_1| = d_{31}$.

If $(4(h_1 + h_2)(l_1m_2 - m_1l_2)((l_1m_2 - m_1l_2)h_1 + (m_2m_1 + l_2l_1)h_3) + h_2^2 + h_3^2)(h_2^2 + h_3^2) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (38) with Wu–Ritt's characteristic set method [14,19].

3.2.7. Case LPP-PLP

In this case, the three distance constraints are a constraint between a line in the platform and a point in the base, a constraint between a point in the platform and a line in the base, and a constraint between a point in the platform and a point in the base. Let the constraints be $|\mathbf{B}_1\mathbf{P}_1'| = t_{11}$, $|\mathbf{B}_2\mathbf{L}_{22}| = d_{22}$, $|\mathbf{P}_3'\mathbf{L}_1| = d_{31}$.

If $h_3^2 + h_1^2 \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (39) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \mathbf{x}_1^6 + c_{11}x_1^5 + c_{12}x_1^4 + c_{13}x_1^3 + c_{14}x_1^2 + c_{15}x_1 + c_{16} = 0, \\ \big(c_{20}x_1^2 + c_{21}x_1 + c_{22}\big)\mathbf{x}_2 + c_{23}x_1^3 + c_{24}x_1^2 + c_{25}x_1 + c_{26} = 0, \\ \big(Ax_1^3 + Bx_1^2 + Cx_1 + D\big)\mathbf{x}_3 + c_{31}x_1^4 + c_{32}x_1^3 + c_{33}x_1^2 + c_{34}x_1 + c_{35} = 0, \\ \big(Ax_1^3 + Bx_1^2 + Cx_1 + D\big)\mathbf{x}_4 + c_{41}x_1^4 + c_{42}x_1^3 + c_{43}x_1^2 + c_{44}x_1 + c_{45} = 0. \end{cases} \tag{39}$$

If $h_3^2 + h_1^2 = 0$, $t_{11} - b_1 = 0$, $m_2 = 0$ and $l_1b_3 + (b_1 - b_2)m_1 - d_{31} \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (38) with Wu–Ritt's characteristic set method [14,19].

3.2.8. Case LPP-PPP

In this case, one of the three distance constraints is between a line in the platform and a point in the base. Each of the remaining two distance constraints is between a point in the platform and a point in the base. Let the constraints be $|\mathbf{B}_1\mathbf{P}_1'| = t_{11}$, $|\mathbf{B}_2\mathbf{P}_2'| = t_{22}$ and $|\mathbf{B}_3\mathbf{L}_{11}| = d_{31}$.

If $m_1(b_3^2 + b_2^2)((b_2 - b_1)^2 + b_3^2) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (39) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

If $m_1 = 0$, $(b_3^2 + b_2^2)((b_2 - b_1)^2 + b_3^2) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (40) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \mathbf{x}_1^6 + c_{11}x_1^5 + c_{12}x_1^4 + c_{13}x_1^3 + c_{14}x_1^2 + c_{15}x_1 + c_{16} = 0, \\ (b_3x_1 - l_1d_{31} - h_3)(Ax_1 + B)\mathbf{x}_2 + c_{21}x_1^3 + c_{22}x_1^2 + c_{23}x_1 + c_{24} = 0, \\ \big(b_1x_1 - (h_2 + h_1)\big)(Ax_1 + B)\mathbf{x}_3 + c_{31}x_1^3 + c_{32}x_1^2 + c_{33}x_1 + c_{34} = 0, \\ \big(b_1x_1 - (h_2 + h_1)\big)(b_3x_1 - l_1d_{31} - h_3)(Ax_1 + B)\mathbf{x}_4 + c_{41}x_1^4 + c_{42}x_1^3 + c_{43}x_1^2 + c_{44}x_1 + c_{45} = 0. \end{cases} \tag{40}$$

If $(b_3^2 + b_2^2) = 0$, $l_1(l_1h_3 + d_{31}) \neq 0$ and $t_{11}^4 + 2(h_1 + h_2)(2(m_1h_2 + l_1h_3 + d_{31})m_1 - (h_2 + h_1))t_{11}^2 - 4(m_1 - 1)(m_1 + 1)(h_1 + h_2)^2h_3^2 + 4l_1(h_1 + h_2)^2(2d_{31} + (h_2 - h_1)m_1)h_3 + (h_1 + h_2)^2((h_1 + h_2)^2 - 4m_1^2h_1h_2 - 4h_1d_{31}m_1 + 4h_2d_{31}m_1 + 4d_{31}^2) \neq 0$, equation system consisting of the three constraints can be reduced to triangular form (41) with Wu–Ritt's characteristic set method [14,19], where c_{ij} are the polynomials in the parameters l_i, m_j, h_k, and d_t.

$$\begin{cases} \mathbf{x}_1^4 + c_{11}x_1^3 + c_{12}x_1^2 + c_{13}x_1 + c_{14} = 0, \\ (Ax_1 + B)\mathbf{x}_2 + c_{21}x_1^2 + +c_{22}x_1 + c_{23} = 0, \\ \big(b_1x_1 - (h_2 + h_1)\big)(Ax_1 + B)\mathbf{x}_3 + c_{31}x_1^3 + c_{32}x_1^2 + c_{33}x_1 + c_{34} = 0, \\ \big(b_1x_1 - (h_2 + h_1)\big)(Ax_1 + B)\mathbf{x}_4 + c_{41}x_1^3 + c_{42}x_1^2 + c_{43}x_1 + c_{44} = 0. \end{cases} \tag{41}$$

3.2.9. Case PPP-LPP

In this case, one of the three distance constraints is between a point in the platform and a line in the base. Each of the remaining two distance constraints is between a point in the platform and a point in the base. Let the constraints be $|\mathbf{P}_1'\mathbf{B}_1| = t_{11}$, $|\mathbf{P}_2'\mathbf{B}_2| = t_{22}$, $|\mathbf{P}_3'\mathbf{L}_1| = d_{31}$.

If $(h_1^2 + h_3^2) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (39) with Wu–Ritt's characteristic set method [14,19].

If $(h_1^2 + h_3^2) = 0$ and $t_{11}^4 - 2b_1(b_1 l_1^2 - 2l_1 m_1 b_3 - b_1 m_1^2 + 2m_1^2 b_2 + 2m_1 d_{31})t_{11}^2 + b_1^4 + 4m_1(l_1 b_3 - m_1 b_2 - d_{31})b_1^3 + 4(l_1 b_3 - m_1 b_2 - d_{31})^2 b_1^2 \neq 0$, equation system consisting of the three constraints can be reduced to triangular form (42) with Wu–Ritt's characteristic set method [14,19].

3.2.10. Case PPP-PPP

In this case, each of the three distance constraints is between a point in the platform and a point in the base. Let the constraints be $|\mathbf{P}_1'\mathbf{B}_1| = t_{11}$, $|\mathbf{P}_2'\mathbf{B}_2| = t_{22}$, $|\mathbf{P}_3'\mathbf{B}_3| = t_{33}$.

If $(b_2^2 + b_3^2)(b_3^2 + (b_2 - b_1)^2)(h_2^2 + h_3^2)(h_1^2 + h_3^2) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (39) with Wu–Ritt's characteristic set method [14,19].

If $(h_1^2 + h_3^2) = 0$, $(b_2^2 + b_3^2)((b_3^2 + (b_2 - b_1)^2)t_{11}^4 - 2b_1(b_2(b_2 - b_1)^2 - t_{33}^2(b_2 - b_1) + b_3^2 b_2)t_{11}^2 + (b_3^2 + b_2^2)b_1^4 - 2b_2(b_2^2 + b_3^2 - t_{33}^2)b_1^3 + (b_2^2 + b_3^2 - t_{33}^2)^2 b_1^2) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (42) with Wu–Ritt's characteristic set method [14,19].

$$\begin{cases} \mathbf{x}_1^4 + c_{11}x_1^3 + c_{12}x_1^2 + c_{13}x_1 + c_{14} = 0, \\ (c_{20}x_1 + c_{21})\mathbf{x}_2 + c_{22}x_1^2 + c_{23}x_1 + c_{24} = 0, \\ \big(Ax_1^2 + Bx_1 + C\big)\mathbf{x}_3 + c_{31}x_1^2 + c_{32}x_1 + c_{33} = 0, \\ \big(Ax_1^2 + Bx_1 + C\big)\mathbf{x}_4 + c_{41}x_1^2 + c_{42}x_1 + c_{43} = 0. \end{cases} \tag{42}$$

If $(b_2^2 + b_3^2) = 0$, $(h_3^2 + h_1^2)((h_2^2 + h_3^2)t_{11}^4 - 2(h_1 + h_2)(h_1 h_3^2 + h_1 h_2^2 + h_2 t_{33}^2)t_{11}^2 + (h_1 + h_2)^2(h_3^4 + (h_1^2 + h_2^2 - 2t_{33}^2)h_3^2 + (h_2 h_1 + t_{33}^2)^2)) \neq 0$, equation system consisting of the three constraints and $x_1^2 + x_2^2 - 1 = 0$ can be reduced to triangular form (36) with Wu–Ritt's characteristic set method [14,19].

3.3. Classification of real solutions to planar GSP

For **DDD** planar GSPs, with Wu–Ritt's characteristic set method, we can reduce them to triangular form consisting of one quadratic and three linear equations shown as equation systems (34), (35), one quartic and three linear equations shown as equation systems (36), (37), (38), (41), (42), or an equation of degree six and three linear equations shown as equation systems (39), (40). So the number of the real solutions to the triangular form is equal to that of the nonlinear equation. Equation systems (34), (35) are ruler and compass constructible. With the method in [21], we can obtain the conditions to get real solution of direct kinematics for remaining triangular forms.

For equation $x_1^4 + b_1x_1^3 + b_2x_1^2 + b_3x_1 + b_4 = 0$, the discriminant sequence is $\{D_1, D_2, D_3, D_4\}$ shown in Appendix A. We can obtain the following conclusions [21].

1. There is no real solution if the revised sign of the discriminant sequence is any one of the set $\{[1,-1,-1,1], [1,-1,1,-1], [1,-1,0,0], [1,-1,1,1], [1,-1,1,0], [1,1,-1,1]\}$;
2. There is one real solution if the revised sign of the discriminant sequence is any one of the set $\{[1,-1,-1,0], [1,0,0,0], [1,1,-1,0]\}$;
3. There are two real solutions if the revised sign of the discriminant sequence is any one of the set: $\{[1,-1,-1,-1], [1,1,-1,-1], [1,1,0,0], [1,1,1,-1]\}$;
4. There are three real solutions if the revised sign of the discriminant sequence is $[1,1,1,0]$;
5. There are four real solutions if the revised sign of the discriminant sequence is $[1,1,1,1]$.

For equation $x_1^6 + c_1x_1^5 + c_2x_1^4 + c_3x_1^3 + c_4x_1^2 + c_5x_1 + c_6 = 0$, the discrimination sequence is $\{D_1, D_2, D_3, D_4, D_5, D_6\}$ shown in Appendix B. We can obtain the following conclusions [21].

1. There is no real solution if the revised sign of the discriminant sequence is any one of the set $\{[1,-1,-1,-1,1,-1], [1,-1,-1,1,-1,-1], [1,-1,-1,1,1,-1], [1,-1,1,-1,-1,-1], [1,-1,1,-1,-1,1], [1,-1,1,-1,1,-1], [1,-1,1,1,-1,-1], [1,-1,1,1,-1,1], [1,1,-1,-1,1,-1], [1,1,-1,1,-1,-1], [1,1,-1,1,-1,1], [1,-1,1,1,1,-1], [1,1,-1,1,1,-1], [1,1,1,-1,1,-1], [1,-1,1,-1,1,1], [1,-1,-1,1,-1,1], [1,-1,-1,1,-1,0], [1,-1,1,1,-1,0], [1,1,-1,1,-1,0], [1,-1,1,-1,-1,0], [1,-1,1,-1,1,0], [1,-1,1,1,0,0], [1,-1,-1,1,0,0], [1,1,-1,1,0,0], [1,-1,1,-1,0,0], [1,-1,0,0,0,0], [1,-1,1,0,0,0]\}$;
2. There is one real solution if the revised sign of the discriminant sequence is any one of the set $\{[1,-1,-1,-1,1,0], [1,-1,1,1,1,0], [1,1,1,-1,1,0], [1,-1,-1,1,1,0], [1,1,-1,1,1,0], [1,1,-1,-1,1,0], [1,-1,-1,0,0,0], [1,1,-1,0,0,0], [1,0,0,0,0,0]\}$;

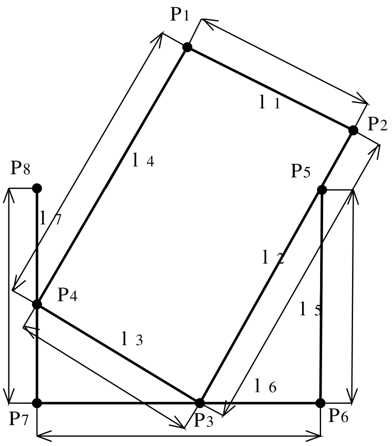

Fig. 4. An example of planar DDD GSPs.

3. There are two real solutions if the revised sign of the discriminant sequence is any one of the set $\{[1,-1,-1,-1,-1,1]$, $[1,-1,-1,-1,0,0]$, $[1,-1,-1,-1,1,1]$, $[1,-1,-1,1,1,1]$, $[1,-1,1,1,1,1]$, $[1,1,-1,-1,-1,1]$, $[1,1,-1,-1,0,0]$, $[1,1,-1,-1,1,1]$, $[1,1,-1,1,1,1]$, $[1,1,0,0,0,0]$, $[1,1,1,-1,-1,1]$, $[1,1,1,-1,0,0]$, $[1,1,1,-1,1,1]$, $[1,1,1,1,-1,1]\}$;
4. There are three real solutions if the revised sign of the discriminant sequence is any one of the set $\{[1,-1,-1,-1,-1,0]$, $[1,1,-1,-1,-1,0]$, $[1,1,1,-1,-1,0]$, $[1,1,1,0,0,0]$, $[1,1,1,1,-1,0]\}$;
5. There are four real solutions if the revised sign of the discriminant sequence is any one of the set $\{[1,-1,-1,-1,-1,-1]$, $[1,1,-1,-1,-1,-1]$, $[1,1,1,-1,-1,-1]$, $[1,1,1,1,-1,-1]$, $[1,1,1,1,0,0]$, $[1,1,1,1,1,-1]\}$;
6. There are five real solutions if the revised sign of the discriminant sequence is $[1,1,1,1,1,0]$;
7. There are six real solutions if the revised sign of the discriminant sequence is $[1,1,1,1,1,1]$.

Example 1. The problem in Fig. 4 can be reduced into merging two rigid bodies $p_1p_2p_3p_4$ and $p_5p_6p_7p_8$. We take $p_5p_6p_7p_8$ as the base and $p_1p_2p_3p_4$ the platform. The constraints are $|l_7p_4|=0$, $|l_6p_3|=0$ and $|p_5l_2|=0$, which is an **LPP-PLL** case. Let $p_7=(0,0)$. The parametric equations for lines l_6 and l_7 are $p=(0,0)+u_1(1,0)$ and $p=(0,0)+u_2(0,1)$. Let point p_3 be the origin of the moving coordinate system. Then $p_3=(x_3,x_4)$. Let $|p_6p_7|=b_2$, $|p_5p_6|=b_3$, $|p_2p_3|=h_2$ and $|p_3p_4|=h_3$. Thus the coordinates for points p_4 and p_5 are $p_4=(-x_2h_3+x_3, x_1h_3+x_4)$ and $p_5=(b_2,b_3)$. The parametric equation of line l_2 is $p=(x_3,x_4)+u_3(x_1,x_2)$.

The equation system is

$$\begin{cases} x_1^2+x_2^2-1=0,\\ \bigl|x_2(b_2-x_3)-x_1(b_3-x_4)\bigr|=0,\\ |-h_3x_2+x_3|=0,\\ |x_4|=0. \end{cases} \tag{43}$$

Equation system (43) can be reduced to triangular form (44) with Wu–Ritt's characteristic set method [14,19] under the variable order $x_1<x_2<x_3$ if $b_2\neq 0$, $b_3\neq 0$ and $h_3\neq 0$.

$$\begin{cases} h_3^2\mathbf{x}_1^4-2b_3h_3x_1^3+\bigl(b_3^2+b_2^2-2h_3^2\bigr)x_1^2+2h_3b_3x_1-b_2^2+h_3^2=0,\\ b_2\mathbf{x}_2+h_3x_1^2-b_3x_1-h_3=0,\\ b_2\mathbf{x}_3+h_3^2x_1^2-h_3b_3x_1-h_3^2=0,\\ \mathbf{x}_4=0. \end{cases} \tag{44}$$

The discriminant sequence of equation $h_3^2\mathbf{x}_1^4-2b_3h_3x_1^3+(b_3^2+b_2^2-2h_3^2)x_1^2+2h_3b_3x_1-b_2^2+h_3^2=0$ is $\{D_1,D_2,D_3,D_4\}$, where $D_1=h_3^4$, $D_2=-h_3^6(2b_2^2-b_3^2-4h_3^2)$, $D_3=-h_3^6b_2^2(b_3^4+2b_2^2b_3^2+4b_3^2h_3^2+b_2^4-2h_3^2b_2^2)$ and $D_4=-b_2^4h_3^6(-b_2^4h_3^2+b_2^6+16h_3^4b_2^2+b_3^6+3b_2^4b_3^2+8h_3^2b_3^4-20h_3^2b_3^2b_2^2+3b_2^2b_3^4)$.

If we take $p_8=(0,33)$, $h_2=30$, $b_2=15$ and $b_3=3$, we can get $D_1=h_3^4$, $D_2=h_3^6(2h_3-21)(2h_3+21)$, $D_3=h_3^6(23h_3^2-3042)$, $D_4=-h_3^6(16h_3^4-10\,053h_3^2+1\,423\,656)$. Thus, the number of real solution is based on the value of parameter h_3.

If we take $h_3=20$, the revised sign of the discriminant sequence is $[1,1,1,1]$. So it has four real solutions shown in Fig. 5, where the solutions to (x_1,x_2,x_3,x_4) are

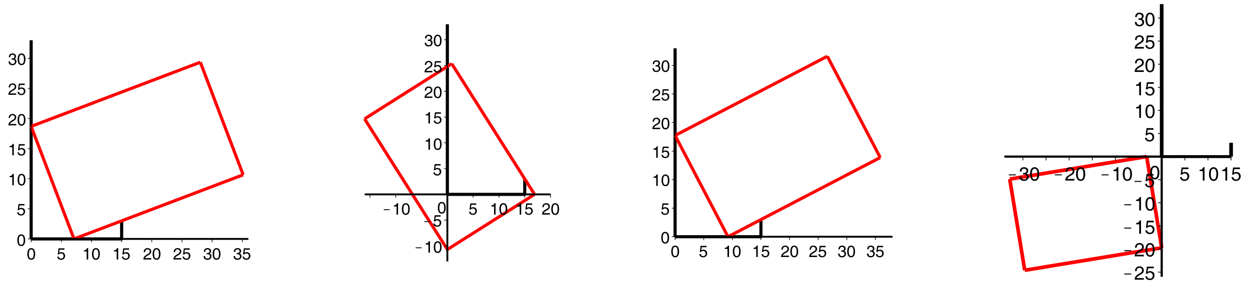

Fig. 5. Four solutions to planar DDD GSPs in Fig. 4 when $h_3 = 20$.

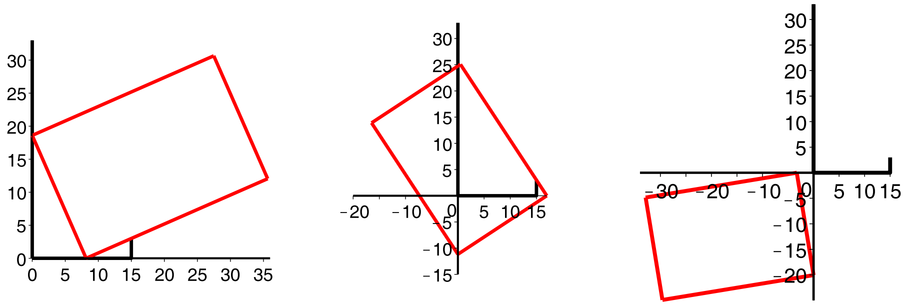

Fig. 6. Three solutions to planar DDD GSPs in Fig. 4 when $h_3 = \frac{3\sqrt{2234+170\sqrt{17}}}{8}$.

$$\left(\frac{2+3\sqrt{6}}{10}, \frac{6-\sqrt{6}}{10}, 2(6-\sqrt{6}), 0\right), \qquad \left(\frac{2-3\sqrt{6}}{10}, \frac{6+\sqrt{6}}{10}, 2(6+\sqrt{6}), 0\right),$$
$$\left(\frac{-1+3\sqrt{39}}{20}, \frac{3+\sqrt{39}}{20}, 3+\sqrt{39}, 0\right), \qquad \left(\frac{-1-3\sqrt{39}}{20}, \frac{3-\sqrt{39}}{20}, 3-\sqrt{39}, 0\right).$$

If we take $h_3 = \frac{3\sqrt{2234+170\sqrt{17}}}{8}$, the revised sign of the discriminant sequence is $[1, 1, 1, 0]$. So it has three real solutions shown in Fig. 6, where the solutions to (x_1, x_2, x_3, x_4) are

$$\left(\frac{\sqrt{2234+170\sqrt{17}}(121+15\sqrt{17})}{10\,816}, \frac{\sqrt{2234+170\sqrt{17}}(175-23\sqrt{17})}{10\,816}, \frac{45-3\sqrt{17}}{4}, 0\right),$$
$$\left(\frac{\sqrt{228\,475-50\,115\sqrt{17}}}{676} - \frac{\sqrt{2234+170\sqrt{17}}(2029+475\sqrt{17})}{281\,216},\right.$$
$$\frac{\sqrt{2234+173\sqrt{17}}(1035+173\sqrt{17})}{281\,216} + \frac{(5+\sqrt{17})\sqrt{228\,475-50\,115\sqrt{17}}}{2074},$$
$$\left.\frac{15+3\sqrt{17}}{4} + \frac{(15+3\sqrt{17})\sqrt{(2234+170\sqrt{17})(228\,475-50\,115\sqrt{17})}}{21\,632}, 0\right),$$
$$\left(-\frac{\sqrt{228\,475-50\,115\sqrt{17}}}{676} - \frac{\sqrt{2234+170\sqrt{17}}(2029+475\sqrt{17})}{281\,216},\right.$$
$$\frac{\sqrt{2234+173\sqrt{17}}(1035+173\sqrt{17})}{281\,216} - \frac{(5+\sqrt{17})\sqrt{228\,475-50\,115\sqrt{17}}}{2074},$$
$$\left.\frac{15+3\sqrt{17}}{4} - \frac{(15+3\sqrt{17})\sqrt{(2234+170\sqrt{17})(228\,475-50\,115\sqrt{17})}}{21\,632}, 0\right).$$

If we take $h_3 = \frac{21}{2}$, the revised sign of the discriminant sequence is $[1, -1, -1, -1]$. So it has two real solutions shown in Fig. 7, where the solutions to (x_1, x_2, x_3, x_4) are

$$(-0.9845022807, -0.1753717746, -1.841403635, 0), \qquad (0.9726431468, 0.2323043457, 2.439195628, 0).$$

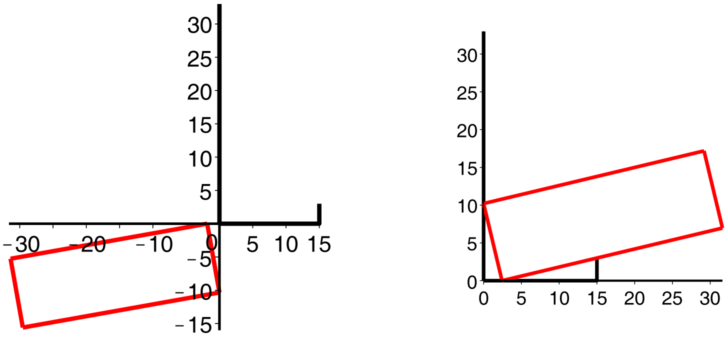

Fig. 7. Two solutions to planar DDD GSPs in Fig. 4 when $h_3 = \frac{21}{2}$.

Because there is one real solution if the revised sign of the discriminant sequence is any one of the set $\{[1, -1, -1, 0], [1, 0, 0, 0], [1, 1, -1, 0]\}$, let $D_4 = 0$. Thus we could obtain four different solutions to h_3, which are

$$-\frac{3\sqrt{2234 + 170\sqrt{17}}}{8}, \qquad \frac{3\sqrt{2234 + 170\sqrt{17}}}{8}, \qquad -\frac{3\sqrt{2234 - 170\sqrt{17}}}{8}, \qquad \frac{3\sqrt{2234 - 170\sqrt{17}}}{8}.$$

For each solution, the revised sign of the discriminant sequence is $[1, 1, 1, 0]$. This means equation system (44) has three solutions and the case of containing one real solution will not appear.

4. Conclusions

The classification of direct Kinematics for the planar *generalized Stewart platform* (GSP) consisting of two rigid bodies connected by three constraints between three pairs of points or lines in the base and the moving platform respectively is introduced. The purpose of classifying direct kinematics for these new types of planar Stewart platforms is to find new and better parallel mechanisms. We solve planar direct Kinematics, and get all the solutions to the problem instead of only one solution. We also give the conditions to which type of planar GSPs is ruler and compass constructible and the detailed classification of direct kinematics for every planar GSP. We obtain the explicit conditions on the parameters to have a given number of real solutions for sixteen forms of planar GSPs. For **DDA** GSPs, we are able to give the explicit conditions on the parameters to all of the possible degenerate cases.

Appendix A. The discrimination sequence for a quantic equation

The discriminant sequence of $x_1^4 + b_1x_1^3 + b_2x_1^2 + b_3x_1 + b_4 = 0$ is $\{D_1, D_2, D_3, D_4\}$, where $D_1 = 1$, $D_2 = -8b_2 + 3b_1^2$, $D_3 = 14b_2b_3b_1 - 4b_2^3 + 16b_4b_2 - 3b_1^3b_3 + b_1^2b_2^2 - 6b_1^2b_4 - 18b_3^2$, $D_4 = -6b_1^2b_4b_3^2 - 4b_1^2b_2^3b_4 - 192b_3b_4^2b_1 + 144b_2b_4^2b_1^2 + 144b_2b_4b_3^2 + 18b_2b_3^3b_1 + 256b_4^3 + 18b_1^3b_3b_4b_2 - 80b_3b_1b_4b_2^2 - 4b_1^3b_3^3 - 27b_1^4b_4^2 + b_1^2b_2^2b_3^2 - 4b_2^3b_3^2 + 16b_2^4b_4 - 128b_4^2b_2^2 - 27b_3^4$.

Appendix B. The discrimination sequence for an equation of degree six

For equation $x_1^6 + c_1x_1^5 + c_2x_1^4 + c_3x_1^3 + c_4x_1^2 + c_5x_1 + c_6 = 0$ the discrimination sequence is $\{D_1, D_2, D_3, D_4, D_5, D_6\}$, where

$$\begin{aligned}
D_1 &= 1, \\
D_2 &= -12c_2 + 5c_1^2, \\
D_3 &= 24c_4c_2 + 24c_2c_3c_1 - 8c_2^3 - 10c_1^2c_4 - 5c_1^3c_3 + 2c_1^2c_2^2 - 27c_3^2, \\
D_4 &= 64c_2^3c_5c_1 - 120c_5c_3c_2^2 + 120c_1^2c_6c_4 - 70c_1^3c_4c_5 - 18c_1^2c_4c_3^2 + 60c_1^3c_3c_6 + 40c_1^4c_3c_5 - 24c_1^2c_2^2c_6 - 16c_1^3c_2^2c_5 \\
&\quad - 8c_1^2c_2^3c_4 + 3c_1^2c_2^2c_3^2 - 288c_2c_6c_4 + 306c_3^2c_5c_1 - 720c_3c_5c_4 - 336c_3c_4^2c_1 - 168c_3c_1c_4c_2^2 + 38c_1^3c_3c_4c_2 \\
&\quad - 224c_4^2c_2^2 + 384c_4^3 - 81c_3^4 + 96c_2^3c_6 + 32c_2^4c_4 - 12c_2^3c_3^2 + 300c_2c_5^2 - 12c_1^3c_3^3 - 125c_1^2c_5^2 - 45c_1^4c_4^2 \\
&\quad + 324c_6c_3^2 + 328c_2c_4c_5c_1 - 288c_2c_3c_1c_6 - 162c_2c_3c_1^2c_5 + 244c_2c_4^2c_1^2 + 324c_2c_4c_3^2 + 54c_2c_3^3c_1, \\
D_5 &= -1344c_2c_6c_4^3 + 256c_4^5 - 192c_2^4c_6^2 + 16c_2^4c_4^3 + 72c_2^5c_5^2 - 128c_4^4c_2^2 - 1296c_2c_6^3 - 27c_1^4c_4^4 + 160c_1^5c_5^3 \\
&\quad + 540c_1^2c_6^3 + 81c_3^5c_5 - 27c_3^4c_4^2 + 1728c_6^2c_4^2 + 276c_3^2c_1c_4c_2^2c_5 + 296c_3c_1c_4c_2^3c_6 - 1872c_4c_5c_1c_6c_2^2 \\
&\quad - 558c_3c_1^2c_5c_6c_2^2 + 3024c_4^2c_5c_6c_1 + 1875c_5^4 + 14c_1^2c_2^3c_4c_5c_3 - 2214c_1^2c_4c_6c_5c_3 - 62c_1^3c_3^2c_4c_2c_5 \\
&\quad - 66c_1^3c_3c_4c_2^2c_6 + 130c_1^4c_3c_5c_6c_2 - 200c_1^4c_4c_6^2 + 1620c_3^2c_5^2c_4 - 192c_3c_4^4c_1 + 3240c_3c_5c_6^2 - 1600c_4c_5^3c_1
\end{aligned}$$

$$
\begin{aligned}
&+648c_2c_4^2c_3c_6c_1+1620c_2c_3^2c_1c_6c_5-602c_2c_3c_1^2c_5c_4^2+216c_2c_3^2c_1^2c_4c_6-1452c_2c_4c_5^2c_1c_3\\
&+1704c_2c_4c_1^3c_5c_6-1134c_6c_5c_3^3+216c_3c_1c_6^2c_2^2-80c_3c_1c_4^3c_2^2-108c_3^3c_1c_6c_2^2-424c_3c_1c_5^2c_2^3\\
&-72c_2^3c_6c_5c_3+24c_2^3c_5c_1c_4^2+112c_2^4c_5c_1c_6-56c_2^4c_4c_5c_3-648c_4c_3^2c_6c_2^2-644c_4^2c_1^2c_6c_2^2+700c_4c_1^2c_5^2c_2^2\\
&+432c_4^2c_2^2c_5c_3+38c_1^3c_4^2c_3c_6-174c_1^3c_3^2c_6c_5+117c_1^4c_3c_5c_4^2+18c_1^3c_3c_4^3c_2+24c_1^3c_3^3c_6c_2+97c_1^3c_3c_5^2c_2^2\\
&-40c_1^4c_3^2c_4c_6-6c_1^3c_2^2c_5c_4^2-28c_1^3c_2^3c_5c_6+122c_1^3c_4c_5^2c_3+120c_1^4c_4^2c_6c_2-132c_1^4c_4c_5^2c_2-300c_1^5c_4c_5c_6\\
&+16c_1^2c_2^4c_4c_6-3c_1^2c_2^2c_3^3c_5+c_1^2c_2^2c_3^2c_4^2-6c_1^2c_2^3c_3^2c_6+18c_1^2c_4c_3^3c_5+160c_2c_5c_1c_4^3+384c_2c_3^2c_1^2c_5^2\\
&-54c_2c_3^4c_1c_5+18c_2c_3^3c_1c_4^2-630c_2c_3c_1^3c_6^2-486c_2c_4c_3^3c_5-1512c_2c_5c_6^2c_1-1188c_2c_5^2c_1^2c_6+828c_2c_4c_6^2c_1^2\\
&-270c_4c_3^3c_6c_1+828c_3^2c_5c_1c_4^2+1620c_3c_5^2c_6c_1-2808c_3c_1c_6^2c_4-64c_2^5c_4c_6+12c_2^3c_3^3c_5-4c_2^3c_3^2c_4^2\\
&+24c_2^4c_3^2c_6+248c_2^3c_6^2c_1^2+432c_4c_6^2c_2^2-616c_4c_5^2c_2^3+1420c_5^3c_1c_2^2+558c_3^2c_2^2c_5^2+1080c_5^2c_6c_2^2-6c_1^2c_3^2c_4^3\\
&-4c_1^2c_2^2c_4^3-18c_1^2c_2^4c_5^2+1265c_1^2c_5^3c_3-52c_1^2c_5^2c_4^2+837c_1^2c_6^2c_3^2-36c_1^3c_5c_4^3-88c_1^4c_3^2c_5^2+12c_1^3c_3^4c_5\\
&-4c_1^3c_3^3c_4^2+125c_1^5c_3c_6^2-50c_1^4c_2^2c_6^2+56c_1^2c_6c_4^3+330c_1^3c_5c_6^2+220c_1^4c_5^2c_6-2400c_2c_5^3c_3+592c_4^2c_2^3c_6\\
&+648c_3^2c_6c_4^2-621c_5^2c_1c_3^3+144c_2c_4^4c_1^2+1440c_2c_5^2c_4^2+144c_2c_3^2c_4^3-5400c_4c_6c_5^2-324c_2c_6^2c_3^2\\
&+162c_3^4c_6c_2-1004c_2c_5^3c_1^3-1344c_3c_5c_4^3+1512c_2c_4c_6c_5c_3,
\end{aligned}
$$

$$
\begin{aligned}
D_6 ={}& 2808c_2c_3^3c_4^2c_6c_5+1500c_5^4c_6c_2^2+16c_2^4c_4^3c_5^2-64c_2^4c_4^4c_6+9216c_4c_6^3c_2^4+512c_2^5c_4^2c_6^2-192c_2^4c_5^2c_6^2\\
&+108c_2^3c_3^4c_6^2+16c_2^3c_3^3c_5^3-8640c_2^3c_6^3c_3^2-17\,280c_4^2c_6^3c_2^2-128c_4^4c_2^2c_5^2+512c_4^5c_2^2c_6-4352c_4^3c_2^3c_6^2\\
&+2250c_5^5c_2^2c_1+43\,200c_6^4c_2^2c_1^2-900c_4c_5^4c_2^3+540c_1^2c_6^3c_5^2-32\,400c_1^2c_6^4c_4-27c_1^2c_2^4c_5^4+1500c_1^4c_6^3c_4^2\\
&-192c_1^2c_6^2c_4^4+256c_1^2c_2^5c_6^3-50c_1^2c_5^4c_4^2+2000c_1^2c_5^5c_3+320c_1^4c_6c_5^4+410c_1^3c_6^2c_5^3+560c_4^2c_2^2c_5^3c_3\\
&-5428c_4^2c_1^2c_5^2c_6c_2^2-4536c_4c_3^2c_5^2c_6c_2^2-3456c_4c_6^3c_1c_3c_2^2-4464c_3^2c_1c_5c_2^3c_6^2+10\,152c_3c_1^2c_5c_4c_6^2c_2^2\\
&-1584c_3^2c_1c_4^2c_2^2c_6c_5-682c_3c_1^2c_5^3c_6c_2^2+356c_3^2c_1c_4c_2^2c_5^3+3272c_3c_1c_4c_2^3c_5^2c_6+2808c_3^3c_1c_4c_2^2c_6^2\\
&-108c_3^3c_1c_5^2c_6c_2^2+16\,632c_3c_1c_5^2c_6^2c_2^2-2496c_3c_1c_4^2c_2^3c_6^2+15\,264c_5c_1c_4^2c_6^2c_2^2-2496c_4^3c_2^2c_6c_5c_3\\
&+320c_3c_1c_4^4c_2^2c_6-13\,040c_4c_5^3c_1c_6c_2^2-80c_3c_1c_4^3c_2^2c_5^2-640c_2^4c_5c_1c_4c_6^2+320c_2^4c_4^2c_6c_5c_3-96c_2^3c_5c_1c_6c_4^3\\
&-72c_2^3c_3^3c_4c_6c_5-5760c_4c_5c_2^3c_3c_6^2+1020c_1^4c_4^2c_5^2c_6c_2+1980c_1^3c_4c_5c_6^2c_3^2-128c_1^4c_3^2c_5^4+108c_1^3c_3^5c_6^2\\
&+16c_1^3c_3^4c_5^3-1350c_1^3c_3^3c_6^3-36c_1^3c_5^3c_4^3-27c_1^4c_4^4c_5^2+27\,000c_1^3c_3c_6^4+108c_1^4c_4^5c_6-1600c_2c_5^5c_1^3\\
&-32\,400c_2c_6^3c_5^2+62\,208c_2c_6^4c_4-1600c_3c_5^3c_4^3+27\,000c_3c_6^2c_5^3+43\,200c_4^2c_6^2c_5^2-2500c_4c_5^5c_1\\
&-22\,500c_4c_6c_5^4+38\,880c_5c_6^4c_1-46\,656c_6^5-4464c_3^2c_5c_1c_6c_4^3+3942c_5^2c_1c_4c_6c_3^3+768c_3c_4^5c_1c_6\\
&+6912c_3c_5c_6c_4^4-77\,760c_3c_4c_5c_6^3+46\,656c_3c_1c_6^3c_4^2+2250c_3c_5^4c_6c_1-192c_3c_4^4c_1c_5^2-22\,896c_2c_4c_5c_1c_6^2c_3^2\\
&-2412c_2c_3^2c_1^2c_5^2c_4c_6+3272c_2c_3c_1^2c_5c_6c_4^3+324c_2c_3^4c_1c_4c_6c_5+10\,152c_2c_5^2c_1c_4^2c_6c_3-5760c_2c_4^3c_3c_1c_6^2\\
&-3456c_2c_4^2c_6^2c_5c_3-640c_2c_5c_1c_6c_4^4-4536c_2c_3^2c_1^2c_4^2c_6^2+3942c_2c_3^3c_1^2c_5c_6^2+19\,800c_2c_3c_1^3c_6^3c_4\\
&-12\,330c_2c_3c_1^3c_6^2c_5^2+1980c_2c_3^2c_1c_6c_5^3-72c_2c_3^3c_1c_6c_4^3+18c_2c_3^3c_1c_5^2c_4^2-746c_2c_3c_1^2c_5^3c_4^2\\
&+8748c_2c_4c_6^2c_1^2c_5^2+19\,800c_2c_4c_6c_5^3c_3-2050c_2c_4c_5^4c_1c_3-13\,040c_2c_4^2c_1^3c_6^2c_5+9768c_2c_4c_1^3c_6c_5^3\\
&-31\,320c_2c_5c_6^3c_1^2c_3+31\,968c_2c_5c_6^3c_1c_4+1020c_1^3c_3^2c_5c_2^2c_6^2+560c_1^4c_3^2c_5^2c_4c_6-2050c_1^4c_3c_5c_4c_6^2c_2\\
&+356c_1^3c_3^2c_4^2c_2c_6c_5-630c_1^4c_3c_5c_6c_4^3+160c_1^4c_3c_5^3c_6c_2-80c_1^3c_3^2c_4c_2c_5^3-746c_1^3c_3c_4c_2^2c_5^2c_6\\
&-630c_1^3c_3^3c_4c_2c_6^2-72c_1^3c_3^4c_4c_6c_5+24c_1^3c_3^3c_5^2c_6c_2+560c_1^3c_3c_4^2c_2^2c_6^2-682c_1^3c_5^2c_4^2c_6c_3+16\,632c_1^2c_4^2c_6^2c_5c_3\\
&-108c_1^2c_3^3c_4^2c_6c_5-192c_1^2c_2^4c_5c_3c_6^2+18c_1^2c_2^3c_4c_5^3c_3+144c_1^2c_2^4c_4c_5^2c_6+144c_1^2c_2^3c_4c_6^2c_3^2-6c_1^2c_2^3c_3^2c_5^2c_6\\
&-4c_1^2c_2^2c_3^2c_6c_4^3+c_1^2c_2^2c_3^2c_5^2c_4^2-12\,330c_1^2c_4c_6c_5^3c_3-72c_1^3c_3c_4^4c_2c_6+18c_1^3c_3c_4^3c_2c_5^2+160c_1^3c_2^3c_5c_4c_6^2\\
&-80c_1^2c_2^3c_4^2c_6c_5c_3+24c_1^3c_2^2c_5c_6c_4^3+18c_1^2c_2^2c_3^3c_4c_6c_5-27c_1^2c_2^2c_3^4c_6^2-4c_1^2c_2^2c_3^3c_5^3-6c_1^2c_3^2c_5^2c_4^3\\
&+24c_1^2c_3^2c_6c_4^4-6480c_2c_6^3c_1^2c_4^2-10\,560c_2c_6c_5^2c_4^3-1700c_2c_6c_1^2c_5^4-27\,540c_2c_5^2c_6^2c_3^2+3888c_2c_4c_6^3c_3^2\\
&-1800c_2c_6^2c_1c_5^3-4860c_2c_4c_3^4c_6^2-630c_2c_4c_3^3c_5^3+560c_2c_3^2c_1^2c_5^4-486c_2c_3^5c_1c_6^2-72c_2c_3^4c_1c_5^3\\
&+21\,384c_2c_3^3c_1c_6^3+160c_2c_5^3c_1c_4^3+144c_2c_3^2c_5^2c_4^3-576c_2c_3^2c_6c_4^4+144c_2c_4^4c_1^2c_5^2-77\,760c_2c_3c_6^4c_1\\
&-576c_2c_4^5c_1^2c_6+15\,552c_5c_3^2c_6^3c_1+5832c_4^2c_3^3c_1c_6^2+21\,384c_4c_6^2c_5c_3^3-6318c_5c_1c_6^2c_3^4-486c_3^5c_4c_6c_5
\end{aligned}
$$

$$
\begin{aligned}
&+162c_3^4c_5^2c_6c_2-9720c_3^2c_6c_5^2c_4^2+1020c_3^2c_5^3c_1c_4^2-128c_1^2c_2^4c_4^2c_6^2-13\,824c_6^4c_2^3+108c_2^5c_5^4-1024c_2^6c_6^3\\
&+256c_1^5c_5^5+3125c_1^6c_6^4+34\,992c_3^2c_6^4-8748c_6^3c_3^4+108c_3^5c_5^3+729c_3^6c_6^2+256c_5^2c_4^5-1024c_6c_4^6\\
&+4816c_4^2c_2^3c_5^2c_6+8208c_4^2c_2^2c_6^2c_3^2-630c_3c_1c_5^4c_2^3+6912c_3c_1c_6^3c_2^4-9720c_3^2c_1^2c_6^3c_2^2+768c_2^5c_5c_3c_6^2\\
&+144c_2^4c_5^3c_1c_6-72c_2^4c_4c_5^3c_3-576c_2^5c_4c_5^2c_6-576c_2^4c_4c_6^2c_3^2+24c_2^4c_3^2c_5^2c_6-10\,560c_2^3c_6^3c_1^2c_4\\
&+248c_2^3c_6^2c_1^2c_5^2-120c_2^3c_6c_5^3c_3+16c_2^3c_3^2c_6c_4^3-4c_2^3c_3^2c_5^2c_4^2+24c_2^3c_5^3c_1c_4^2-6480c_4c_5^2c_6^2c_2^2+4816c_4^3c_1^2c_6^2c_2^2\\
&+1020c_4c_1^2c_5^4c_2^2+46\,656c_5c_6^3c_2^2c_3-21\,888c_5c_6^3c_2^3c_1+5832c_5c_3^3c_2^2c_6^2+15\,600c_5c_1^3c_6^3c_2^2-120c_1^3c_4^3c_3c_6^2\\
&+144c_1^3c_5c_6c_4^4+825c_1^4c_3^2c_4^2c_6^2+144c_1^3c_3c_5^4c_2^2-900c_1^4c_3^3c_5c_6^2-1600c_1^3c_3c_6^3c_2^3+2250c_1^4c_3^2c_6^3c_2\\
&-3750c_1^5c_3c_6^3c_4+2000c_1^5c_3c_6^2c_5^2-208c_1^3c_3^2c_6c_5^3+16c_1^3c_3^3c_6c_4^3-4c_1^3c_3^3c_5^2c_4^2+144c_1^4c_3c_5^3c_4^2-36c_1^3c_2^3c_5^3c_6\\
&+2000c_1^4c_2^2c_6^3c_4-50c_1^4c_2^2c_6^2c_5^2-6c_1^3c_2^2c_5^3c_4^2-1700c_1^4c_4c_6^2c_5^2+160c_1^3c_4c_5^4c_3+2250c_1^5c_4^2c_6^2c_5\\
&-900c_1^4c_4^3c_6^2c_2-1600c_1^5c_4c_6c_5^3-192c_1^4c_4c_5^4c_2+2250c_1^4c_5c_6^3c_3-1800c_1^3c_5c_6^3c_4-2500c_1^5c_5c_6^3c_2\\
&+248c_1^2c_6c_5^2c_4^3+15\,417c_1^2c_5^2c_6^2c_3^2-27\,540c_1^2c_4c_6^3c_3^2+162c_1^2c_4c_3^4c_6^2+24c_1^2c_4c_3^3c_5^3-4c_1^2c_2^3c_4^3c_5^2\\
&+16c_1^2c_2^3c_4^4c_6-31\,320c_3c_5^2c_1c_6^2c_4+3125c_5^6-13\,824c_6^3c_4^3-21\,888c_5c_1c_6^2c_4^3+15\,600c_4^2c_5^3c_6c_1\\
&+2000c_2c_5^4c_4^2-3750c_2c_5^5c_3-8640c_3^2c_6^2c_4^3+2250c_3^2c_5^4c_4-1350c_6c_5^3c_3^3-22\,500c_2c_6^4c_1^4+825c_3^2c_2^2c_5^4\\
&+108c_3^4c_6c_4^3-900c_5^4c_1c_3^3+9216c_2c_6^2c_4^4-27c_3^4c_5^2c_4^2.
\end{aligned}
$$

References

[1] B. Dasgupta, T.S. Mruthyunjaya, The Stewart platform manipulator: A review, Mechanism and Machine Theory 35 (2000) 15–40.

[2] J. Duffy, Statics and Kinematics with Applications to Robotics, Cambridge University Press, 1996.

[3] J.C. Faugère, D. Lazard, Combinatorial classes of parallel manipulators, Mechanism and Machine Theory 30 (1995) 765–776.

[4] X.S. Gao, K. Jiang, Survey on geometric constraint solving, Journal of Computer-Aided Design & Computer Graphics 16 (2004) 385–396 (in Chinese).

[5] X.S. Gao, D. Lei, Q. Liao, G. Zhang, Generalized Stewart–Gough platforms and their direct kinematics, IEEE Transactions on Robotics 21 (2005) 141–151.

[6] X.S. Gao, Q. Lin, G. Zhang, A C-tree decomposition algorithm for 2D and 3D geometric constraint solving, Computer-Aided Design 38 (2006) 1–13.

[7] G.M. Gosselin, J.P. Merlet, The direct kinematics of planar parallel manipulators: Special architectures and number of solutions, Mechanism and Machine Theory 29 (1994) 1083–1097.

[8] V.E. Gough, Automobile stability, control, and tyre performance, in: Proceedings of Automobile Division, Institute of Mechanical Engineering, 1956, pp. 392–394.

[9] D. Kapur, Automated geometric reasoning: Dixon resultants, Gröbner bases and characteristic sets, in: D. Wang (Ed.), Automated Deduction in Geometry, in: LNAI, vol. 1360, 1998, pp. 1–36.

[10] D. Lazard, Generalized Stewart platform: How to compute with rigid motions?, in: IMACS Symposium on Symbolic Computation, Lille, June 1993, pp. 85–88.

[11] D. Lazard, J.P. Merlet, The (true) Stewart platform has 12 configurations, in: Proceedings of IEEE International Conference on Robotics and Automation, 1994, pp. 2160–2165.

[12] H.L. Liu, T.Z. Zhang, H.S. Ding, Forward solution of the 3-RPR planar parallel mechanism with Wu's method, Journal of Beijing Institute of Technology 20 (2000) 565–569 (in Chinese).

[13] J.P. Merlet, Parallel Robots, Kluwer, Dordrecht, 2000.

[14] B. Mishra, Algorithmic Algebra, Springer, Berlin, 1993.

[15] B. Mourrain, Enumeration problems in geometry, robotics and vision, in: L. Gonzalez, T. Recio (Eds.), Algorithms in Algebraic Geometry and Applications, in: Progress in Mathematics, Birkhäuser Verlag, Basel, Switzerland, 1996, pp. 285–306.

[16] G.R. Pennock, D.J. Kassner, Kinematic analysis of a planar eightbar linkage: Application to a platform-type robot, Journal of Mechanical Design 114 (1992) 87–95.

[17] D. Stewart, A platform with six degrees of freedom, Proceedings of the Institute of Mechanical Engineering 180 (1965) 371–386.

[18] H. Su, C.L. Collins, J.M. McCarthy, Classification of RRSS linkages, Mechanism and Machine Theory 37 (2002) 1413–1433.

[19] W.T. Wu, Basic Principles of Mechanical Theorem Proving in Geometries, Springer, Berlin, 1995.

[20] G.F. Zhang, X.S. Gao, Planar generalized Stewart platforms and their direct kinematics, in: LNAI, vol. 3763, 2006, pp. 198–211.

[21] L. Yang, X.R. Hou, Z.B. Zeng, A complete discrimination system for polynomials, Science in China (Series E) 39 (1996) 628–646.