PARTIAL DIFFERENTIAL EQUATIONS
AND MATHEMATICAL MORPHOLOGY

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ABSTRACT. – In the past few years, nonlinear parabolic PDEs have been introduced in image analysis. A complete classification of these equations is now established with the geometrical invariance properties that may be required. An important result is that there exists a unique second order parabolic equation which is invariant with respect to contrast changes and affine distortions. On the other hand, a classical result by Matheron yields a complete classification of morphological operators that is monotone, translation invariant and contrast invariant functions operators. In this paper, we prove that any adequately scaled and iterated affine invariant, morphological operator converges to the semi-group associated with the unique affine invariant PDE of the classification. In a second part, by using again Matheron’s characterization, we give a new proof of the convergence of other morphological operators, the weighted median filters, towards the Mean Curvature Motion. © Elsevier, Paris

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RéSUMÉ. – Les EDP non linéaires ont été introduites récemment dans l’analyse des images. Une classification de ces équations fondée sur leur invariance géométrique existe maintenant. Un résultat important est qu’il existe une seule équation parabolique du second ordre invariante par changement de contraste et par des transformations affines. Un résultat de Matheron clarifie tous les opérateurs morphologiques, c’est-à-dire les opérateurs monotones invariants par translation et invariants par changement de contraste. Dans cet article on montre que sous un changement d’échelle convenable tout opérateur morphologique invariant par affinité converge vers le semi-groupe associé à l’unique PDE de la classification invariante par transformations affines. Dans la seconde partie, utilisant la caractérisation de Matheron, nous donnons une nouvelle preuve de la convergence d’autres opérateurs morphologiques, les filtres à médiane pondérée, vers le mouvement à courbure moyenne. © Elsevier, Paris

1. Introduction

1.1. Mathematical morphology

In image analysis, one of the most basic tasks is to smooth an image \( u_0(x) \) for noise removal and shape simplification. Such a smoothing should preserve as much as possible the essential features of an image. This requirement is most easily formalized in terms of invariance. Two invariance requirements are basic in this context: given a smoothing operator \( T \), it should commute with contrast changes, that is, increasing functions. Indeed, for physical and technological reasons, most digital images are known up to a contrast change. The second obvious requirement is geometric invariance: since the position of the camera is in general arbitrary or unknown, the operator \( T \) should commute with translations, rotations, and, when possible, with affine and even projective transforms of the image plane.
**Definition 1.1.** Let $\mathcal{F}$ be a set of functions on $\mathbb{R}^N$ containing continuous functions and characteristic functions of level sets of elements of $\mathcal{F}$. We say that $\mathcal{T} : \mathcal{F} \rightarrow \mathcal{F}$ is morphological if and only if $\mathcal{T}$ is monotone (that is $u \leq v$ on $\mathbb{R}^N$ implies $\mathcal{T}u \leq \mathcal{T}v$ on $\mathbb{R}^N$), commutes with translations and continuous nondecreasing functions (contrast changes).

One of the basic results of Mathematical Morphology is the following result:

**Theorem 1.2 (Matheron).** Let $\mathcal{T}$ be an operator defined on a set of functions $\mathcal{F}$ as in Definition 1.1. Then $\mathcal{T}$ is morphological if and only if there exists a family $\mathcal{B}$ of subsets of $\mathbb{R}^N$ called structuring elements, such that:

\begin{equation}
\mathcal{T}u(x) = \inf_{B \in \mathcal{B}} \sup_{y \in B} u(x + y).
\end{equation}

In the same way there exists an other family $\mathcal{B}'$ such that

\begin{equation}
\mathcal{T}u(x) = \sup_{B \in \mathcal{B}'} \inf_{y \in B} u(x + y).
\end{equation}

### 1.2. Partial differential equations in image processing

One of the first and decisive attempts to explain the phenomenons involved in vision process were developed in Marr's book *Vision* ([15]). From his work, Witkin ([19]) and Koenderink ([14]) introduced in 1983 the concept of Scale Space. It consists in a family of operators $(\mathcal{T}_t)_{t \geq 0}$ over real valued functions in $\mathbb{R}^N$ ($N \geq 2$). In image processing, an image is modeled by a function $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$ representing the grey level at each point of the space. Let $u_t = \mathcal{T}_t u_0$; it corresponds to smoothed versions of the image depending upon a scale parameter $t$. A complete axiomatization was presented in [1] and all scale spaces were classified with respect to their geometrical invariance properties. It is then proved that $u_t$ is solution of a second order parabolic PDE. Among the relevant PDEs, we find the Mean Curvature Motion (MCM):

\begin{equation}
\frac{\partial u}{\partial t} = \Delta u - \frac{(D^2 u D u , D u)}{|D u|^2}.
\end{equation}

Another important result is that there exists a single Affine Morphological Scale Space (AMSS), i.e. commuting with nondecreasing functions, invariant by translation, grey level shift and affine mappings of $\mathbb{R}^N$. Moreover, this scale space is not projective invariant. Thus, in the frame of scale space theory, projective invariance is impossible. The corresponding PDE in the $N$-dimensional space is

\begin{equation}
\frac{\partial u}{\partial t}(x, t) = |D u| t \frac{1}{N+1} \left( \prod_{i=1}^{N-1} \lambda_i \right) \frac{1}{\lambda_1} H(\lambda_1, \cdots, \lambda_{N-1}),
\end{equation}

where $\lambda_i$ is the $i^{th}$ principal curvature of the level surface of $u(\cdot, t)$ at $x$ and $H$ is equal to 1 if and only if the $\lambda_i$ are all strictly positive, to -1 if they are strictly negative and 0 otherwise. The principal curvatures of $u$ are the eigenvalues of the second derivative $D^2 u$ restricted to the hyperplane orthogonal to $Du$, divided by $|Du|$. Of course, these curvatures are only defined when the gradient is different from 0.
1.3. Main results

In this paper, we prove that if we adequately scale Matheron morphological operators, then the iterated associated operators converge to the semi-group of a geometrical evolution PDE of the classification established in [1]. More precisely, from Matheron's theorem 1.2, we can also deduce (although this is not completely obvious) that $T$ is affine invariant if and only if the family $B$ is also affine invariant. Let $B$ be a family of structuring elements, let us introduce a scale parameter $s$ and consider the family of structuring elements $B_s$ obtained from $B$ by a simple dilation: $B_s = s^{1/N} B$ ($N$ being the space dimension). The real $s$ is thus a scale parameter linked to the size of the structuring elements. Let us now introduce the operator

$$ IS_s u(x) = \inf_{B \in B_s} \sup_{y \in B} u(x + y) $$

and the dual operator

$$ SI_s u(x) = \sup_{B \in B_s} \inf_{y \in B} u(x + y). $$

**Theorem 1.3.** Let $B$ an affine invariant closed (with respect to the Hausdorff distance) family of structuring elements which are closed, convex, symmetric with respect to 0, with measure 1 and let $u : \mathbb{R}^N \to \mathbb{R}$ be a $C^3$ function. Then we have:

$$ \lim_{s \to 0} \frac{IS_s u(x) - u(x)}{s^{N+1}} = c_B |p| (\lambda_1^+ \cdots \lambda_{N-1}^+) \frac{1}{N+1}, $$

where $p = |D u(x)|$ and $\lambda_1, \ldots, \lambda_{N-1}$ are the principal curvatures of the level surface going through $x$.

We have a similar result by replacing $IS_s$ by $SI_s$ (obtained by swapping inf and sup) and the $\lambda_i^+$ by $\lambda_i^-$. We shall need consistency results on the alternate operator $SI_s IS_s$. To this end, we prove that except at critical points, consistency is uniform. We then establish the link between consistency and convergence. Let $h = s^{1/(N+1)}$ and $T_h = SI_s IS_s$.

**Theorem 1.4.** Let $u_0 \in BUC(\mathbb{R}^N)$ (bounded and uniformly continuous). The approximate solutions $u_h$ defined by

$$ u_h(x, t) = T_h^t u_0(x) $$

converge towards the unique solution in $BUC(\mathbb{R}^N)$ of

$$ \frac{\partial u}{\partial t} = c_B |Du| \left( \prod_{i=1}^{N-1} \lambda_i \right)^{1/(N+1)} H(\lambda_1, \ldots, \lambda_{N-1}), $$

with initial data $u_0$. Here $H(\lambda_1, \ldots, \lambda_{N-1}) = -1$ if the $\lambda_i$ are all negative, 1 if they are all positive and 0 otherwise. Convergence is uniform on every compact set of $\mathbb{R}^N \times \mathbb{R}_+$. This result will be shown in Sections 2, 3 and 4, first by proving consistency in the three-dimensional case. We then generalize this result to any dimension and prove the convergence.
The last sections are devoted to a new proof that all properly rescaled and iterated weighted median filters, a class of isotropic morphological operators widely used in image processing, converge to Mean Curvature Motion. This result has already been proved by Ishii (see [13]), generalizing the proof by Barles and Georgelin ([12]) and Evans ([10]) and answering a conjecture of Bence, Merriman and Osher ([4]), but the tools used here are different and perhaps better adapted to the mathematical morphology theory. Precisely, let \( k \) a continuous radial probability density that decreases fast enough at infinity and define \( \mathcal{B} = \{ B \subset \mathbb{R}^N, \text{meas}_B > \frac{1}{2} \} \). Let also define the weighted median filter associated with the density \( k \) by

\[
\text{med}_k u(x) = \inf_{B \in \mathcal{B}} \sup_{y \in B} u(x + y).
\]

Scale \( k \) in \( k_h \) by \( k_h(x) = h^{-N} k(h^{-1} x) \), let \( T_h = \text{med}_{k_h} \) and define

\[
u\text{h}(x, t) = T_h^n u_0(x) \text{ if } nh^2 \leq t < (n + 1)h^2.
\]

We then prove that \( \nu_h \) converge uniformly on every compact set towards the solution of the Mean Curvature Motion defined by

\[
\frac{\partial u}{\partial t} - \left( \Delta u - \frac{D^2 u(Du, Du)}{|Du|^2} \right) = 0
\]

and with initial condition \( u_0 \). Except in [13], the proof of the convergence relies on consistency arguments (see [4], [10], [2] and [12] for Mean Curvature Motion and [12] for the affine invariant case).

In [7], Catté, Dibos and Koepfler already established a link between both point of views (Matheron’s Mathematical Morphology and geometrical PDEs) by proving that if \( \mathcal{B} \) is an isotropic family of segments centered at the origin with equal length, then adequately rescaled iterated Matheron filters converge to the viscosity solution of the Mean Curvature Equation. A more general result was presented in [18], where structuring elements were exhibited to approximate equations of the type

\[
\frac{\partial u}{\partial t} = |Du|^{(\text{curv}_u)^\gamma}
\]

in the plane, for all \( \gamma \geq 0 \). The affine invariant case in \( \mathbb{R}^2 \) was studied by Guichard and Morel ([12]) and Catté ([6]) for the AMSS curvature equation. A very fast morphological and affine invariant algorithm (but not relying on inf-sup schemes) is presented by Moisan in [17] to compute the solution of the AMSS Equation in the two-dimensional case. The extension of Guichard and Morel’s result to any dimension is interesting because three-dimensional images and even movies of three dimensional images are already available in the medical domain. These last ones can be considered as four-dimensional images, whereas two-dimensional movies can also be seen as three-dimensional images.

Thanks to [3], it is well known that convergence of iterated schemes to viscosity solutions of PDEs essentially relies on a consistency proof. The essential part of this paper is therefore dedicated to consistency of Matheron’s inf sup operators. The two major difficulties are that the operators are a priori non local and that we wish some uniform consistency results.
2. The three-dimensional case

Most of the time we will use only the $IS_s$ operator (cf Equation (1.5)), the $SI_s$ operator being deduced by the obvious relation

$$SI_s u = -IS_s (-u).$$

2.1. Pointwise consistency

In this Section, we will prove that the inf sup morphological operators are local. As a consequence, we will be able to estimate their action on regular functions by Taylor expansion. The technique will be similar to the two-dimensional case, except that we will have to assume that the used functions are more regular. We first examine the effect of $IS_s$ on quadratic forms (the same results for $SI_s$ will be deduced from $SI_s u = -IS_s (-u)$).

A quadratic form equal to 0 at the origin can always be written in an orthonormal base of $\mathbb{R}^3$ in the form $px + ax^2 + by^2 + cz^2 + dxy + exz$ with $p \geq 0$. Indeed, it suffices to choose the first axis in the direction of the gradient and to diagonalize the second derivative in the hypersurface orthogonal to the gradient. If $p > 0$, the real numbers $\frac{2b}{p}$ and $\frac{2c}{p}$ represent the principal curvatures of the level surface passing through the origin. In order to prove consistency results, we shall proceed as follows:

1. We assume $p = 0$ (the gradient is equal to 0).
2. In the following $p \neq 0$ and we may assume $p > 0$ without loss of generality. We assume that one curvature at least is nonpositive that is $b = 0$ or $c = 0$.
3. We assume that both curvatures are positive.
   (a) We study $IS_s(x + y^2 + z^2)$.
   (b) Thanks to affine invariance, we extend the last result to $IS_s(px + by^2 + cz^2)$.
   (c) Finally, we study the most general case that is $IS_s(px + ax^2 + by^2 + cz^2 + dxy + exz)$.

In these three cases, we shall approach the inf sup with localized structuring elements.

4. Thanks to Taylor expansion and the locality of the structuring elements used for quadratic forms, we shall use the preceding results for $C^3$ functions.

In the following, we assume that $B$ is globally invariant by $SL(\mathbb{R}^3)$ (linear mappings with determinant 1), and that the structuring elements are closed, convex, symmetric with respect to the origin and with volume 1.

2.1.1. Study of $IS_s$ for quadratic forms

In order to simplify the notations we shall write $IS(px + ax^2 + by^2 + cz^2 + dxy + exz + fyz)$ instead of $IS((x, y, z) \mapsto px + ax^2 + by^2 + cz^2 + dxy + exz + fyz)(0)$. In the case $n = 3$, we set:

$$r = s^{1/5}.$$

We shall see in the next section how to choose the right exponent for any dimension.

1. Case of zero gradient. Let us first consider a quadratic form with zero gradient at the origin.
Lemma 2.1. — There exists a constant \( R \) such that for all \( s \), we have

\[
IS_s(ax^2 + by^2 + cz^2) \leq R^2 s^{\frac{4}{5}}(|a| + |b| + |c|).
\]

Proof. — Let us choose \( B \) in \( \mathcal{B} \), let be \( R \) its diameter and set \( B_s = s^\frac{4}{5}B \).

The result is then obvious. Indeed, we have:

\[
IS_s(ax^2 + by^2 + cz^2) \leq \sup_{x \in B_s} (ax^2 + by^2 + cz^2) \leq (|a| + |b| + |c|)R^2 s^{\frac{4}{5}}.
\]

Note that when \( s \) tends to 0, this element is strictly contained in the ball with radius \( r = s^\frac{4}{5} \). It will be relevant in the following. □

2. Case of a negative curvature. From now on, we will assume that the gradient is different from 0 at the origin. By choosing the first frame vector equal to \( Du/|Du| \) where \( u \) is here the quadratic form, we can assume that \( p = |Du| > 0 \).

Lemma 2.2. — Let assume \( p > 0 \) and \( b \leq 0 \) (or equivalently the curvature in the \( y \) direction is nonpositive at 0). Then, we obtain:

\[
IS_s(px + ax^2 + by^2 + cz^2 + dxz + exz) = O(s^{\frac{4}{5}}).
\]

Remark. — We have obviously the same result if \( c \leq 0 \).

Proof. — Let \( B \in \mathcal{B} \) and let us call \( B_r^s \) the structuring element defined by:

\[
B_r^s = \left( \begin{array}{c} \frac{s}{r} \frac{4}{5} \\ r \\ \left( \frac{s}{r} \right)^{\frac{1}{5}} \end{array} \right) B.
\]

We remark that \( \left( \frac{s}{r} \right)^{\frac{4}{5}} = r^{\frac{4}{5}} = o(r) \) (since \( s = r^5 \)). We can choose \( B \) such that \( B_r^s \) is contained in \( D(0, r) \) for \( s \) small enough. The fact that \( R^s \in \mathcal{B}_s \) is obvious. We then have

\[
IS_s(px + ax^2 + by^2 + cz^2 + dxz + exz) \leq K \left( \frac{s}{r} \right)^{\frac{4}{5}},
\]

where \( K \) depends on \( B, a, c, d, e \) but not upon \( s \). As \( s = r^5 \), we have \( \left( \frac{s}{r} \right)^{\frac{4}{5}} = s^{\frac{4}{5}} \) and the Lemma is proved. □

3. From now on, we will assume that the gradient is not equal to 0 at the origin and that both curvatures are strictly positive. We shall see that this case will give the asymptotic behavior of the \( \inf \sup \) operator.

(a) The study of \( IS_s(x + y^2 + z^2) \) will be interesting. We first extend a result true in the two-dimensional case.

Lemma 2.3. — Let \( c_B = \inf_{B \in \mathcal{B}} \sup_{x \in B} (x + y^2 + z^2) \). If the elements of \( \mathcal{B} \) are convex, closed, symmetric with respect to 0, and with measure 1, then \( c_B > 0 \).

Proof. — The measure of elements of \( \mathcal{B} \) being 1, we can choose \( 0 < a < 1 \) small enough such that \( B \in \mathcal{B} \) yields \( B \cap \partial D(0, a) \neq \emptyset \). This is possible as soon as the measure of
\( D(0, a) \) is smaller than 1. At an intersection point, we have \( x \geq 0 \) (we use the symmetry with respect to 0). Moreover, on \( D(0, a) \), \( |x| \geq x^2 \) since \( a < 1 \). Thus,

\[
\inf_{B \in \mathcal{B}} \sup_{x \in B} (x + y^2 + z^2) > a^2 > 0.
\]

This constant (which we denote \( c_B \)) only depends on the shape of \( \mathcal{B} \). \( \square \)

Remark. – The inf above is attained for a \( B_0 \in \mathcal{B} \). Indeed, we remark that for any structuring elements, the sup of \( x + y^2 + z^2 \) is attained at a point with \( x > 0 \) since the elements are symmetric with respect to the origin. Moreover \( x + y^2 + z^2 \) tends to infinity when \( |x| \to \infty \) and \( x > 0 \). Thus, a minimizing sequence of structuring elements is uniformly bounded (completely contained in a single ball). Since the structuring elements are closed, we can extract from any minimizing sequence a subsequence converging towards \( B_0 \in \mathcal{B} \) in the sense of Hausdorff distance (without any restriction, we assume that \( \mathcal{B} \) is closed for the Hausdorff distance). In the following, we shall always denote \( B_0 \) such an element (not unique a priori).

(b) In this paragraph, we shall see that affine invariance allows us to generalize the preceding result to \( IS_x(px + by^2 + cz^2) \).

**Lemma 2.4.** – We have

\[
IS_x(px + by^2 + cz^2) = c_B p \left( \frac{bc}{p^2} \right)^{\frac{1}{4}} s^\frac{3}{2}.
\]

Moreover, the inf sup is attained for a structuring element included in the ball with radius \( r = s^{\frac{3}{4}} \) for small enough scales.

**Proof.** – Let \( Q(x) = px + by^2 + cz^2 \), \( C(x) = x + y^2 + z^2 \) and finally \( A_x(p, b, c) \) the diagonal matrix (1):

\[
A_x(p, b, c) = \left( \begin{array}{ccc}
\frac{1}{2} (bc) & \frac{1}{2} p^{-\frac{1}{2}} \\
\frac{1}{2} h^{-\frac{3}{2}} & \frac{1}{2} p^{\frac{1}{2}} \\
\frac{1}{2} h^{-\frac{3}{2}} & \frac{1}{2} p^{\frac{1}{2}}
\end{array} \right).
\]

We remark that \( \det A_x = s \). Moreover, the mapping \( B \mapsto A_x B \) is bijective. A simple calculation yields \( Q(Ax) = ps^{\frac{3}{2}} \left( \frac{bc}{p^2} \right)^{\frac{1}{4}} C(x) \). Thus, we have:

\[
\inf_{B \in \mathcal{B}} \sup_{x \in B} Q(x) = \inf_{B \in \mathcal{B}} \sup_{x \in A_x B} Q(x) = \inf_{B \in \mathcal{B}} \sup_{x \in B} Q(A_x x) = ps^{\frac{3}{2}} \left( \frac{bc}{p^2} \right)^{\frac{1}{4}} \inf_{B \in \mathcal{B}} \sup_{x \in B} C(x).
\]

(1) Sometimes, we will write \( A(x) \), or \( A(D^2 u(x), Du(x)) \); we will precise when necessary.
If $B_0$ realizes the min of $\sup_{x \in B} C(x)$, then $A_rB_0$ minimizes $\sup_{x \in B} Q(x)$. We conclude that we can only consider a very well localized structuring element whose shape is well known when $s \to 0$. It is thinner and thinner in the gradient direction and is included in $D(0, r)$ when the scale is small enough (it is due to the asymptotic behavior of the matrix $A$ when the scale tends to 0). □

(c) Case of general quadratic forms. Until now we have only considered very specific quadratic forms. By diagonalizing the curvature tensor in the hypersurface orthogonal to the gradient, we can assume that there is no $yz$ term so that the most general case will be $IS_r(px + ax^2 + by^2 + cz^2 + dxy + exz)$. From now on, we will need the localized version of the inf sup operator. It will give a geometrical interpretation to the parameter $r$.

**Definition 2.5.** We define the localized operator $IS_r^s$ by:

$$IS_r^s u(x) = \inf_{B \in B} \sup_{x \in B \cap D(0, r)} u(x + y).$$

We can define in the same way $SI_r^s$ by:

$$SI_r^s u(x) = \sup_{B \in B} \inf_{x \in B \cap D(0, r)} u(x + y).$$

We have the obvious relation

$$IS_r^s u(x) \leq IS_r^s u(x).$$

It is also obvious that $IS_r^s$ is local; we shall prove that $IS_r^s$ and $IS_r^s$ are asymptotically equivalent when the scale tends to 0.

**Lemma 2.6.** There exists a function $G(p, a, b, c, d, e, s)$ bounded on every bounded subset of $\mathbb{R}^+ \times \mathbb{R}^5 \times [0, 1]$ such that for all $p > 0$, $b, c > 0$ and $s$ small enough, we have:

$$s^{-\frac{1}{2}} IS_r^s(px + ax^2 + by^2 + cz^2 + dxy + exz) = c_0(b + c + d + e) s^{-\frac{1}{2}} p + G(p, a, b, c, d, e, s) s^{-\frac{1}{2}}.$$

**Proof.** First, by using the symmetry of the structuring elements with respect to the origin, we can assume that $IS_r^s(px + ax^2 + by^2 + cz^2 + dxy + exz)$ is attained for $x \geq 0$ (since $p > 0$). Let $s \leq 1$. If $x \in D(0, r) = D(0, s^{\frac{1}{2}})$ and since we can assume $x \geq 0$, we have

$$x(p - s^{\frac{1}{2}}(|a| + |d| + |e|) + by^2 + cz^2) \leq px + ax^2 + by^2 + cz^2 + dxy + exz \leq x(p + s^{\frac{1}{2}}(|a| + |d| + |e|) + by^2 + cz^2),$$

so that

$$IS_r^s(px + ax^2 + by^2 + cz^2 + dxy + exz) \geq IS_r^s(px + ax^2 + by^2 + cz^2 + dxy + exz) \geq IS_r^s(x(p - s^{\frac{1}{2}}(|a| + |d| + |e|) + by^2 + cz^2)$$

since $x \in D(0, r)$ and $x \in D(0, s^{\frac{1}{2}})$, the last equality being true, since for small enough $s$, the used structuring element is included in $D(0, r)$. The function $s^{-\frac{1}{2}}((p - s^{\frac{1}{2}}(|a| + |d| + |e|))^{\frac{1}{2}} - p^{\frac{1}{2}})$ is bounded on...
every bounded subset of its definition domain. Indeed, for fixed \((a, d, e, s)\), it is a decreasing function of \(p\) and the value in \(p = s^\frac{1}{2}(|a| + |d| + |e|)\) is \(-(|a| + |d| + |e|)\).

Conversely, if we choose the particular structuring element \(A_sB_0\) where \(A_s\) is here the matrix \(A_s(p + s^\frac{1}{2}(|a| + |d| + |e|), b, c)\) and \(B_0\) is always the same minimizing element, we obtain an element contained in \(D(0, r)\) for \(s\) small enough, and

\[
\begin{align*}
IS_s(px + ax^2 + by^2 + cz^2 + dxy + exz) &\leq \sup_{x \in A_sB_0} \left( px + ax^2 + by^2 + cz^2 + dxy + exz \right) \\
&< \sup_{x \in A_sB_0} \left( x(p + s^\frac{1}{2}(|a| + |d| + |e|)) + by^2 + cz^2 \right) \text{ since } x \in D(0, r).
\end{align*}
\]

Thus, we have:

\[
IS_s(px + ax^2 + by^2 + cz^2 + dxy + exz) < c_B(p + s^\frac{1}{2}(|a| + |d| + |e|))^{\frac{1}{2}}(b^+ c^+)\frac{1}{2} s^{\frac{1}{2}}
\]

which is the announced result since \(s^{-\frac{1}{2}}((p + s^\frac{1}{2}(|a| + |d| + |e|))^{\frac{1}{2}} - p^{\frac{1}{2}})\) is also bounded on every bounded subset of \(\mathbb{R}^+ \times \mathbb{R}^3 \times [0, 1]\).

2.1.2. Asymptotic behavior of \(IS_s\) on a \(C^3\) function

We obtained results concerning the effect of the \(\inf \sup\) operator on quadratic forms. Let us note that in [12], the point of view is different: thanks to a localizability property specific to dimension 2, it was proved that the element attaining the \(\inf \sup\) was close to a well localized element and the \(\sup\) on this second element is studied. With our method, we do not have this localization property, but we exhibited well localized elements approaching the \(\inf \sup\). Using Taylor expansion, it is now quite easy to prove the following theorem.

**Theorem 2.7.** Let \(\mathcal{B}\) a family of structuring elements which are closed, convex, symmetric with respect to 0, with measure equal to 1 and invariant by \(SL(\mathbb{R}^3)\). Let \(u\) be a \(C^3\) function. There exists a strictly positive constant \(c_B\) depending only on \(\mathcal{B}\) such that:

\[
\lim_{s \to 0} \frac{IS_s u(x) - u(x)}{s^{\frac{1}{2}}} = c_B |Du(x)| (\lambda_1^\frac{1}{2} \lambda_2^\frac{1}{2})^4,
\]

where \(p = |Du(x)|\), \(\lambda_1\) and \(\lambda_2\) are the principal curvatures of the level surface of \(u\) going through \(x\).

**Proof.** We can choose the frame axes such that for \(x_0 \in \mathbb{R}^3\), we have in a neighborhood of 0:

\[
u(x_0 + x) = u(x_0) + px + ax^2 + by^2 + cz^2 + dxy + exz + O(r^3)
\]

Indeed, we choose the first basis vector in the direction of the gradient if it is not zero. Then, we diagonalize the second derivative of \(u\) on the hypersurface orthogonal to the first chosen vector. If the gradient is null at \(x\), we choose the frame axes such that the second derivative relative to this frame is diagonal. Using the translation invariance and the invariance by addition of constants, we may assume that \(x_0 = 0\) and \(u(x_0) = 0\). If the gradient is null at \(x\) then Lemma 2.1 gives the result. Indeed, the structuring element used is included in \(D(0, r)\) for small enough \(s\), and the \(O(r^3)\) term due to Taylor expansion gives a \(O(s^{3/5})\) term, which is negligible in front of \(s^{\frac{1}{2}}\). In the same way, if one of the
curvatures is negative, Lemma 2.2 ensures Equation (2.14), since the structuring element is again well localized.

Let us now assume that the gradient is non zero and that both curvatures are strictly positive. Let us take a minimizing element $B_0$ of $\sup_{x \in B}(x + y^2 + z^2)$, and use the same $A_s$ matrix as in Lemma 2.6:

\[
\begin{align*}
(2.16) \quad IS_s u(0) & \leq \sup_{x \in A_s B_0} u(x) \\
& \leq \sup_{x \in A_s B_0} \left( px + ax^2 + by^2 + cz^2 + dxy + exz \right) + O(r^2),
\end{align*}
\]

since $A_s B_0$ is contained in $D(0, r)$ for small enough $s$. By using Lemma 2.6, we obtain

\[
(2.17) \quad IS_s u(0) \leq c_B (\lambda^+ \lambda^+_1)^{\frac{1}{2}} |Du(0)| s^{\frac{3}{2}} + O(s^\frac{3}{4}) + O(s^{\frac{9}{16}}),
\]

where $\lambda_1$ and $\lambda_2$ are the principal curvatures of the level surface going through the origin. In order to get a lower bound, we localize the $IS_s$ operator and obtain:

\[
(2.18) \quad IS_s u(0) \geq IS'_s u(0) = IS'_s \left( px + ax^2 + by^2 + cz^2 + dxy + exz + O(r^3) \right)
\]

and by using again the localization of the structuring elements of Lemma 2.6, we can conclude. Finally we have

\[
(2.19) \quad IS_s u(0) = c_B p \left( \frac{bc}{p^2} \right)^\frac{1}{2} s^{\frac{3}{2}} + O(s^\frac{9}{4})
\]

yielding the result. $\square$

2.1.3. Locality results

We cannot expect some locality results as strong as in the two-dimensional case. Some interesting and useful results are still true for smooth functions.

The following Corollary proves that the operator $IS_s$ (a priori non local) and the localized operator $IS'_s$ are asymptotically close to each other when the scale tends to 0.

**Corollary 2.8.** There exists a constant $\theta > 0$ such that for any $C^3$ function $u : \mathbb{R}^3 \to \mathbb{R}$ and $s$ small enough,

\[
(2.20) \quad IS_s u(0) = IS'_s u(0) + O(s^{\frac{1}{2} + \theta}).
\]

where $\theta > 0$.

**Proof.** The result is an obvious consequence of Lemma 2.6 and of Theorem 2.7. In the three different cases (zero gradient, a negative curvature, and non zero gradient and strictly positive curvatures), we know that we can find $\theta > 0$ such that

\[
IS_s u(0) = c_B p \left( \frac{b+c}{p^2} \right)^\frac{1}{2} s^{\frac{3}{2}} + O(s^{\frac{9}{16}}).
\]
If \( pb^+c^+ = 0 \), the fact that \( 0 \leq IS_u^p(0) \leq IS_u(0) = O(s^{1+\theta}) \) yields the result. If \( pb^+c^+ \neq 0 \), we saw in the proof of Theorem 2.7, that:
\[
\frac{c}{|B|^\frac{1}{r}} \left( \frac{b^+c^+}{r} \right) s^\frac{1}{r} + O(s^\frac{1}{r}) \leq IS_u^p(0) \leq IS_u(0) \leq \frac{c}{|B|^\frac{1}{r}} \left( \frac{b^+c^+}{r} \right) s^\frac{1}{r} + O(s^\frac{1}{r})
\]
and we conclude. □

We now give a version of a locality lemma, proved in [12] for Lipschitz functions in the two-dimensional case, but no longer true in higher dimensions. In higher dimensions, we need a little bit more of regularity.

**Corollary 2.9 (Locality).** – Let \( u \) be a \( C^3 \) function and \( v \) a Lipschitz function such that in the neighborhood of 0, we have,
\[
|u(x) - v(x)| \leq C|x|^3.
\]
Then we have:
\[
|IS_u(0) - IS_v(0)| = O(s^{1+\theta}).
\]

**Proof.** – Indeed, we know by Equation (2.20) that:
\[
IS_u(0) \geq IS_v(0) \quad \text{from (2.13)}
\]
\[
\geq IS_u^p(0) + O(r^3) \quad \text{by hypothesis}
\]
\[
= IS_u(0) + O(s^{1+\theta}) + O(s^\frac{1}{r}) \quad \text{from Theorem 2.7}.
\]
To find the reverse inequality, we choose structuring elements well adapted to \( u \).
\[
IS_u(0) < \sup_{x \in A_B} u(x)
\]
\[
\leq \sup_{x \in A_B} v(x) + O(s^\frac{1}{r})
\]
\[
\leq IS_v(0) + O(s^{1+\theta}) + O(s^\frac{1}{r}),
\]
where \( B_0 \) is still the same element and \( A_B \) is chosen as in the proof of Lemma 2.6 and Theorem 2.7. We used Equation (2.20) and Theorem 2.7 to get the result. □

We also have a maximum principle for functions, if one at least is regular.

**Corollary 2.10 (local comparison principle).** – Let \( u \) be a \( C^3 \) function and let \( v \) Lipschitz such that in \( D(0,r) \), \( u(x) < v(x) \). Then,
\[
IS_u(0) \leq IS_v(0) + O(s^{1+\theta}).
\]
Conversely, \( v(x) \leq u(x) \) in \( D(0,r) \) yields
\[
IS_v(0) \leq IS_u(0) + O(s^{1+\theta}).
\]

**Proof.** – The proof is already given by the preceding results. In the first case, we compare \( IS_u(0) \) and \( IS_v(0) \) by inequality (2.13). Since we only consider points inside \( D(0,r) \), \( u \leq v \). We can expand \( u \) by Taylor’s Formula and apply Lemma 2.6.

In the second case, we can choose structuring elements adapted to \( u \) as in the proof of Lemma 2.6. Since these elements are localized for small scales, we can compare \( u \) and \( v \) and get the result. □

**Remark.** – All these results are obviously valid for the \( SI_u \) operator.
Remark. — Our consistency results cannot follow the lines of [12] because they essentially use a locality property of affine invariant Matheron operators which is specific of dimension 2. We give in [5] a counterexample to this property in dimension 3.

2.2. Uniform consistency of \( \inf \sup \) operators

We proved that the \( IS_s \) is pointwise consistent with a differential operator. More precisely, for \( x \in \mathbb{R}^3 \),

\[
s^{-\frac{1}{2}}(IS_s u(x) - u(x)) = c_\|Du(x)\| \left( \lambda_2^+ \lambda_3^+ \right)^{\frac{1}{2}} + o(1),
\]

when \( s \) tends to 0. The function \( o(1) \) depends \textit{a priori} on \( x \), so that the consistency may be nonuniform in \( x \). We shall see that consistency is uniform as soon as \( x \) is \textit{biregular}, that is to say that neither the gradient nor the curvatures are zero at \( x \). In other cases, we will not be able to give better results than pointwise consistency. Uniform consistency shall be useful in the next subsection concerning alternate operators.

**Lemma 2.1** (Uniform consistency). — Let \( u \) a \( C^3 \) function and assume that \( x_0 \) is biregular. Then \( IS_s \) (or \( SI_s \)) is uniformly consistent in a neighborhood of \( x_0 \).

**Proof.** — As \( u \) is \( C^3 \), if we take \( x \in D(x_0, \frac{r_0}{2}) \) with a small enough \( r_0 \), we can ensure that \( Du \) is different from 0 and that both curvatures taken at \( x \) have the same sign as at \( x_0 \).

1. We assume that one curvature is negative. We take a structuring element \( B \) transformed by the linear mapping

\[
\left( \begin{array}{ccc}
\frac{2s}{r} \\
\frac{r}{3} \\
\frac{2s}{r}
\end{array} \right),
\]

where \( r \) is chosen such that \( r \leq r_0 \). It is not hard to see that we can choose \( B \) such that, when it is transformed by the above matrix, we get an element \( B^r \in B^s \) contained in \( D(0, \frac{r}{2}) \). When we center this element at \( x \in D(x_0, \frac{r}{2}) \), we still obtain an element included in \( D(x_0, r) \). Thus, if \( x \in D(x_0, \frac{r}{2}) \) we have:

\[
(2.24) \quad IS_s u(x) \leq \sup_{y \in x + B_r^s} u(y) \leq u(x) + \sup_{y \in B_r^s} ((Du(x), y) + \frac{1}{2} (D^2 u(x)y, y) + O(r^3)).
\]

The term \( O(r^3) \) is uniform over \( D(x_0, r_0) \) since \( u \) is \( C^3 \). Using again the same arguments as in the proof of Lemma 2.2, we deduce the uniform consistency of the scheme.

2. Let us now assume that both principal curvatures are strictly positive. As \( u \) is \( C^3 \), we can choose \( r_0 \) small enough such that for any \( x \in D(x_0, \frac{r_0}{2}) \), we have \( |Du(x)| \geq \frac{1}{2} |Du(x_0)| > 0 \). We can assume a relation of the same kind for the curvatures. By Lemma 2.6, there exists a scale \( 0 < s_0 \leq (r_0/2)^3 \) such that for all \( x \in D(x_0, \frac{r_0}{2}) \) and for all \( s \leq s_0 \), the useful structuring elements at \( x \) are included in
\[ x + D(0, r) \subset x + D(0, \frac{r_0}{2}) \subset D(x_0, r_0). \]

\[ w + D(0, r) \subset w + D(0, \frac{r_0}{2}) \subset D(w_0, r_0). \]

We use again Equation (2.24) with a uniform \( O(r^2) \) term. Lemma 2.6 yields the conclusion, and uniform consistency of \( IS_u \) is proved for biregular points. \( \square \)

Uniform consistency gives a stronger version of the local comparison principle as soon as the considered point is biregular.

**Lemma 2.12.** Let \( u \) a \( C^3 \) function and \( \nu \) Lipschitz. Let \( x_0 \) be a biregular point of \( u \). Assume that in \( D(x_0, r_0) \), \( u \leq \nu \). Then, for any \( x \) in a neighborhood of \( x_0 \),

\[ IS_x u(x) \leq IS_x \nu(x) + O(s^{\frac{1}{2} + \theta}), \]

the term \( O(s^{\frac{3}{2}}) \) being uniform with respect to \( x \) in the neighborhood of \( x_0 \).

In the same way, if \( \nu \leq u \) in \( D(x_0, r_0) \), then

\[ IS_x \nu(x) \leq IS_x u(x) + O(s^{\frac{1}{2} + \theta}), \]

the term \( O(s^{\frac{3}{2}}) \) being uniform with respect to \( x \) near \( x_0 \).

**Proof.** Let us assume that \( u \leq \nu \) in \( D(x_0, r_0) \). As \( x_0 \) is a biregular point, we can also assume that in \( D(x_0, r_0) \), \( Du \) is nonzero and the curvatures have the same sign as \( \nu \).

1. If one of the curvatures is negative all over \( D(x_0, r_0) \), we choose \( 0 < s_0 \leq \left( \frac{r_0}{2} \right)^{-\frac{1}{5}} \) such that for any \( x \in D(x_0, r_0) \) the structuring element element used in Lemma 2.2 translated by \( x \) is uniformly included in \( D(x_0, r_0) \). We can conclude as in Corollary 2.10.

2. We now assume that both curvatures are positive. As above, we choose \( x \in D(x_0, \frac{r_0}{2}) \) and \( 0 < s_0 \leq \left( \frac{r_0}{2} \right)^{-\frac{1}{5}} \) such that for any \( s < s_0 \) the adapted structuring element for \( u \) at point \( x \) is included in \( x + D(0, r_0) \cap D(x_0, r_0) \).

\[ IS_x u(x) \geq IS_x^{s_0/2} u(x) \quad \text{from (2.13)} \]

\[ \geq IS_x^{s_0/2} u(x) \quad \text{by hypothesis} \]

\[ \geq u(x) + IS_x^{s_0/2} (px + ax^2 + by^2 + cz^2 + dxy + exz) + O(s^{\frac{1}{2}}) \]

by expanding \( u \) by Taylor’s Formula

\[ = u(x) + IS_x^{s_0/2} (x(p - s^{\frac{1}{2}}(|a| + |d| + |e|)) + by^2 + cz^2) + O(s^{\frac{3}{2}}) \]

from Lemma 2.6

\[ = IS_x u(x) + O(s^{\frac{1}{2} + \frac{\theta}{2}}) + O(s^{\frac{3}{2}}). \]

In the last inequality, both approximation terms are uniform thanks to Lemma 2.6 and to the good localization of the used structuring elements. Of course \( \frac{1}{2} + \frac{1}{10} = \frac{3}{5} \), but we prefer to keep two different terms in order to show that they do not come from the same approximation. This arithmetic coincidence will no longer be true in higher dimensions.

If we assume \( \nu \leq u \), we take again the elements \( \Lambda_x B_0 \) where \( \Lambda_x = \Lambda_x (p \mid s^{\frac{1}{2}}(|a| + |d| + |e|), b, c) \). We obtain elements uniformly contained in \( D(0, s^{1/5}) \) near \( x_0 \) because this point is biregular. Hence,

\[ IS_x \nu(x) \leq \sup_{y \in x + \Lambda_x B_0} \nu(y) \]

\[ \leq \sup_{y \in x + \Lambda_x B_0} u(y) \]

\[ = IS_x u(x) + O(s^{\frac{1}{2} + \frac{\theta}{2}}) + O(s^{\frac{3}{2}}), \]

with uniform \( O(s^{\frac{1}{2} + \frac{\theta}{2}}) \) and \( O(s^{\frac{3}{2}}) \) terms. \( \square \)
2.3. Alternate schemes

The alternate operator \( S_l IS_s \) is useful because it is tangent to the AMSS differential operator and it is morphological. The proof of its consistency needs the uniform consistency of \( IS_s \) which is true near bi-regular points. So we can expect a good result of consistency for the alternate scheme near such points.

**Theorem 2.13.** Let \( u \) be a \( C^3 \) function and \( x_0 \) a bi-regular point of \( u \). Then,

\[
\lim_{s \to 0} \frac{S_l IS_s u(x_0) - u(x_0)}{s^{\frac{1}{2}}} = c_B|Du(x_0)|g(u)(x_0)
\]

where \( g(u)(x) = \text{sign}(\lambda_1)(G(u)(x)^+)^{\frac{1}{4}} \), where \( G(u)(x) \) is the Gaussian curvature of the level surface of \( u \) passing through \( x \).

Moreover the consistency of the scheme is uniform near \( x_0 \).

**Proof.** It is clear that we shall use Uniform Comparison Principle (Lemma 2.12).

There exists \( s_0 \) and a \( x_0 \) centered ball (say \( D(x_0, \frac{3}{2}) \)) such that for any \( s \leq s_0 \) and \( x \in D(x_0, \frac{3}{2}) \) we have:

\[
IS_s u(x) = u(x) + c_B|Du(x)|(\lambda^+_1 \lambda^+_2)^{\frac{1}{4}} s^{\frac{1}{2}} + H(D^3 u(x), D^2 u(x), Du(x), x, s) s^{\frac{1}{4} + \theta},
\]

the function \( H \) being bounded on a neighborhood of \( x_0 \). We now choose \( s \) such that \( s \leq 1 \) and \( s^{\frac{1}{2}} \leq \frac{s_0}{2} \) and set:

(i) \( M(s) = \sup_{y \in B(x_0, r)} |H(y)| \),

(ii) \( h_1(s) = \sup_{y \in B(x_0, r)} c_B|Du(y)|(\lambda^+_1(y) \lambda^+_2(y))^{\frac{1}{4}} s^{\frac{1}{2}} \),

(iii) \( h_2(s) = \inf_{y \in B(x_0, r)} c_B|Du(y)|(\lambda^+_1(y) \lambda^+_2(y))^{\frac{1}{4}} s^{\frac{1}{2}} \).

We obtain

\[
h_2(s) - M s^{\frac{1}{4} + \theta} + u(x) \leq IS_s u(x) \leq u(x) + h_1(s) + M s^{\frac{1}{4} + \theta}
\]

for \( x \) close to \( x_0 \). By using the Uniform Comparison Principle (Lemma 2.12) for \( S_l \), near \( x_0 \), we obtain:

\[
h_2(s) \leq IS_s u(x_0) \leq O(s^{\frac{1}{4} + \theta}) \leq S_l IS_s u(x_0) \leq h_1(s) + S_l u(x_0) + O(s^{\frac{1}{4} + \theta}),
\]

the \( O(s^{\frac{3}{4}}) \) term still being uniform with respect to \( s \) in a neighborhood of \( x_0 \). When \( s \) tends to 0, we get

\[
\lim_{s \to 0} \frac{S_l IS_s u(x) - S_l u(x)}{s^{\frac{1}{2}}} = c_B|Du(x_0)|(\lambda^+_1 \lambda^+_2)^{\frac{1}{4}}.
\]

By using \( S_l u = -IS_s(-u) \), we deduce

\[
\lim_{s \to 0} \frac{S_l u(x_0) - u(x_0)}{s^{\frac{1}{2}}} = -c_B|Du(x_0)|(\lambda^-_1 \lambda^-_2)^{\frac{1}{4}}.
\]

We take the sum of the two preceding equalities and we obtain the result. With our proof, uniform consistency is obvious. \( \square \)
2.4. Summary

We obtained pointwise consistency results for $IS_s$ and $SI_s$ applied to $C^3$ functions. Moreover, near a biregular point, the consistency is uniform. At such a point, we also proved that the $SI_s IS_s$ operator is consistent, and this consistency is also uniform. When the point is not biregular, we have not prove any result for the alternate scheme. The advantage of the alternate scheme is that it is fully morphological and tangent to the $AMSS$ operator. Furthermore, this operator is invariant by the transformation $u \mapsto -u$, contrary to the differential operator associated to $IS_s$ and $SI_s$. This transformation corresponds to a contrast inversion of the image. Therefore the invariance property of the $AMSS$ operator may be interesting. It is also clear that the following operator

$$M_s = \frac{1}{2}(IS_s + SI_s)$$

is also tangent to the differential operator $AMSS$ but it is not morphological. It has been used for instance in [7] to construct a scheme converging to Mean Curvature Motion.

3. The $N$-dimensional case

We shall now extend the results obtained in $\mathbb{R}^3$ to any dimension. The proof is essentially the same as in the three dimensional case, so we just indicate how to choose the localization parameters. The main consistency result is:

**Theorem 3.1.** Let $\mathcal{B}$ an affine invariant family of structuring elements which are closed, convex, symmetric with respect to $0$, with measure $1$ and let $u : \mathbb{R}^N \to \mathbb{R}$ be a $C^3$ function. Then

$$\lim_{s \to 0} \frac{IS_s u(x) - u(x)}{s^{N+1}} = c_\mathcal{B} |Du(x)| \left(\lambda_1^+ \cdots \lambda_{N-1}^+\right)^{\frac{N}{N+1}},$$

where $p = |Du(x)|$ and $\lambda_1, \ldots, \lambda_{N-1}$ are the principal curvatures of the level surface going through $x$.

**Proof.** The proof will be based on the same arguments as in the three dimensional case. We start studying $IS_s$ on quadratic forms. We first assume that the gradient is zero at the origin, then that one of the $N - 1$ principal curvatures is nonpositive. Next, we study $IS_s(x + y_1^2 + \cdots + y_{N-1}^2)$, $IS_s(px + b_1y_1^2 + \cdots + b_{N-1}y_{N-1}^2)$ and the case of a general quadratic form. Each time, we shall construct well localized structuring elements approaching the inf sup. Finally, we will be able to extend these results to $C^3$ functions. Let $s = r^{N+2}$. The proof of the following Lemmas are the same as in Section 2.

**Lemma 3.2 (Lemma 2.1 bis)**

$$IS_s(ax^2 + b_1y_1^2 + \cdots + b_{N-1}y_{N-1}^2) = O(s^{\frac{N}{N+1}}) = o(s^{\frac{N}{N+1}}).$$

This estimate is obtained for a structuring element with a diameter similar to $s^{\frac{1}{N+2}} = o(s^{\frac{1}{N+2}}) = o(r)$. 

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Lemma 3.3 (Lemma 2.2 bis). Let us assume that \( b_1 \leq 0 \), then:

\[
I_{S^*}(px + b_1 y_1^2 + \cdots + b_{N-1} y_{N-1}^2) = O(s^{2/(N+1)}) = o(s^{2/(N+1)}).
\]

The inf sup is again approached for a structuring element asymptotically included in \( D(0, r) \).

Proof. We change the matrix used in Lemma 2.2 into

\[
\begin{pmatrix}
\frac{s}{r} \\
\frac{s}{r} \\
\vdots \\
\frac{s}{r}
\end{pmatrix}
\]

and the result follows.

Lemma 3.4 (Lemma 2.3 bis). Set

\[ c_B = I_{S^*}(x + y_1^2 + \cdots + y_{N-1}^2) \]

if the structuring elements are closed, convex, symmetric with respect to the origin and with measure 1, then \( c_B > 0 \).

Let \( D_0 \) a minimizing structuring element attaining \( c_B \).

Lemma 3.5 (Lemma 2.4 bis). Assume that \( p > 0 \) and \( b_1, \ldots, b_{N-1} > 0 \). Then,

\[
I_{S^*}(px + b_1 y_1^2 + \cdots + b_{N-1} y_{N-1}^2) = c_B s^{2/(N+1)} (\lambda_1 \cdots \lambda_{N-1})^{1/(N+1)},
\]

where \( \lambda_i = 2b_i/p \) is the \( i \)th principal curvature. Moreover, the inf sup is attained for a localized structuring element.

Proof. We change the term of the matrix \( A_s \) of Lemma 2.4 corresponding to the gradient direction into

\[
s^{2/(N+1)} (b_1 \cdots b_{N-1})^{1/(N+1)},
\]

The term corresponding to the \( j \)th curvature (1 \( \leq j \leq N - 1 \)) becomes

\[
s^{1/(N+1)} b_1^{\frac{N}{N+1}} \cdots b_j^{\frac{N}{N+1}} \cdots b_{N-1}^{\frac{N}{N+1}} p^{\frac{1}{N+1}}.
\]

A minimizing structuring element is \( A_s B_0 \) (it is not unique \( a \) priori). We easily check that this element is asymptotically included in \( D(0, r) \).

Lemma 3.6 (Lemma 2.6 bis). There exists a function \( G(p, (b_i), (c_i)) \) bounded on every compact subset of \( R^*_+ \times R^{N-1} \times R^{N-1} \) such that:

\[
I_{S^*}(px + ax^2 + \sum_{i=1}^{N-1} b_i y_i^2 + \sum_{i=1}^{N-1} c_i x y_i) = c_B s^{2/(N+1)} \left( \prod_{i=1}^{N-1} b_i^{\frac{1}{N+1}} \right) s^{\frac{1}{N+1}}
\]

\[ + G(p, (b_i), (r_i)) s^{\frac{1}{N+1}} + s^{\frac{1}{N+1}}. \]
The proof is exactly the same as in Lemma 2.6. Uniform consistency near a biregular point is also a direct consequence. We extend the preceding results to $C^3$ functions thanks to Taylor expansion. Indeed, the structuring elements involved in the preceding Lemmas are all included in $D(0, r)$ when the scale tends to 0. We check that a function $O(r^3)$ is $o(s^{2/(N+1)})$; it is obvious since $r = s^{1/(N+2)}$. □

All the locality results are easily extended. Uniform consistency near a biregular point is proven in the same way. We also get the same results for the alternate operator $SI_s IS_s$.

**Theorem 3.7 (Lemma 2.13 bis).** Let $B$ a family of structuring elements invariant by $SL(R^N)$ with elements that are closed, convex symmetric with respect to the origin and with Lebesgue measure equal to 1. There exists $c_B > 0$ such that for any $C^3$ function $u$,

\[
\lim_{s \to 0} \frac{SI_s IS_s u(x) - u(x)}{s^{N+1}} = c_B |Du(x)| (\lambda_1 \cdots \lambda_{N-1})^{\frac{1}{N+1}},
\]

where the $(\lambda_i)$ are the principal curvatures of the level surface passing through $x$.

### 4. Convergence of the Affine inf sup Scheme

#### 4.1. Notations

From now on, we set $h = s^{2/(N+1)}$ and write $T_h$ instead of $T_{s^{2/(N+1)}}$. Here $T_h$ stands for $SI_h IS_h$. The aim of this section is the following: We want to prove that by letting $nh \to t$ while $n \to +\infty$, the iterated filtered function $(T_h)^n u$ converges towards the viscosity solution of the parabolic equation

\[
\frac{\partial u}{\partial t} = |Du| \left( \prod_{i=1}^{N-1} \lambda_i \right)^{\frac{N}{N+1}} H(\lambda_1, \ldots, \lambda_{N-1}),
\]

where $H = -1$ if the curvatures are all strictly negative, $H = 1$ if they are all strictly positive and $H = 0$ otherwise. We eliminate the power of $t$ on the right-hand side of Equation (1.4) by setting $t' = t^\alpha$ with an appropriate $\alpha > 0$. In the following discussion, we prefer to write Equation (4.1) in its generic form

\[
\frac{\partial u}{\partial t}(x, t) = F(D^2 u(x, t), Du(x, t)).
\]

It will simplify the statements. From the result exposed in Section 3, the operator $T_h$ is tangent to the differential operator $F(D^2 u(x, t), Du(x, t))$, that is

\[
\lim_{h \to 0} \frac{T_h u(x) - u(x)}{h} = F(D^2 u(x, t), Du(x, t)).
\]

This relation is true for the alternate operator only at biregular points. Let $u_0$ be an original image, and for $h > 0$ define the approximate solution $u_h(x, t)$ of (4.2) with initial condition $u_0 \in BUC(R^N)$ by

\[
\forall n \in N, \forall t \in [(n - 1)h, nh], \quad u_h(x, t) = ((T_h)^n u_0)(x).
\]

We shall prove that, up to a rescaling, the approximate solutions $u_h$ converge towards the unique viscosity solution of the AMS5 Equation (4.1).
We recall the definition of viscosity solution. An overview of this theory is available in [9]. In the following definition, we denote by $C^\infty_c$ the space of infinitely differentiable compactly supported functions.

**Definition 4.1.** Let $u$ a continuous function. We say that $u \in C^\infty_c$ is a viscosity subsolution (resp. supersolution) of:

$$
\begin{align*}
\frac{\partial u}{\partial t}(x, t) - F(D^2 u(x, t), Du(x, t)) &= 0, \\
\end{align*}
$$

if and only if for any $\varphi \in C^\infty_c$, if $u - \varphi$ has a strict maximum at $(x, t)$ (resp. strict minimum) then

$$
\begin{align*}
\frac{\partial \varphi}{\partial t}(x, t) - F(D^2 \varphi(x, t), D\varphi(x, t)) &\geq 0 \quad (\text{resp. } \geq 0).
\end{align*}
$$

We say that $u$ is viscosity solution if $u$ is both sub and supersolution.

**Remark.** The notion of viscosity solution is local. There are many variants to the definition above; for instance we can replace strict maximum by local maximum or local strict maximum, and $C^\infty_c$ by $C^2$ (in the case of second order equations) without changing the results. For further details, see [9].

In [12], it was proved that the solution of (4.1) is unique in $BUC(\mathbb{R}^N)$ (bounded and uniformly continuous). Thus, in this paper, we are not concerned with problems of existence and uniqueness. Convergence will be deduced from consistency but problems at critical points will prevent us to use directly the results in [3]. For these points, we need some particular results.

**Lemma 4.2.** Let $a(x) = |x|^2$. Then,

$$
\lim_{(x, h) \to (0, 0)} \frac{T_h a(x) - a(x)}{h} = 0.
$$

The main point here is that the limit is taken for $x \to 0$ and $h \to 0$ independently. The $T_h$ operator can either be $IS_h$, $SI_h$, or $SI_h IS_h$.

**Proof.** First, it suffices to study $IS_a a$. Indeed, $SI_a a = a$, so the lemma is obvious if $T_h = SI_a$. If we prove the result for $IS_a$, then it is true for the alternate operator. Indeed, $IS_a a \geq a$. Thus, $SI_a IS_a a \geq SI_a a = a$, since $SI_a$ is monotone and $SI_a a = a$. Moreover, $SI_a IS_a a \leq IS_a a$. Hence, it suffices to prove the result for $IS_a$.

First, we prove in the following Lemma that we can assume that the elements of $\mathcal{B}$ are all hypercubes with the same volume. Indeed, if $B \in \mathcal{B}$, we can find a minimum hypercube containing $B$ such that on each side of the cube, there is at least a point belonging to $B$. Thus, the inf sup on this new family, is larger than the inf sup on the original family.

**Lemma 4.3.** There exists $K_N > 0$ only depending upon the dimension $N$, such that for any structuring element $B \in \mathcal{B}$, $B$ is included in a hypercube whose measure is $K_N$, and on each side of the cube, there is a point of $B$.

**Proof.** This Lemma is very intuitive. Let $a_1 \in B$ such that $|u_1|$ is maximal. This is possible since $B$ is closed convex with Lebesgue measure equal to 1, so it is bounded. We choose the first frame axis in the direction of $a_1$. Next, in the $N - 1$-dimensional affine...
spaces orthogonal to \( a_1 \), we choose \( a_2 \in B \) such that the norm of the component of \( a_2 \) orthogonal to \( a_1 \) is maximal. We choose the second axis in the direction of the component of \( a_2 \) orthogonal to \( a_1 \). We choose \( a_3 \) in the same way: In all the \( N - 2 \)-dimensional affine spaces we take the point of \( B \) with the largest component orthogonal to the space spanned by \( a_1 \) and the component of \( a_2 \) orthogonal to \( a_1 \). This process ends when we have chosen exactly \( N \) points \( a_1, \ldots, a_N \). The polyhedron \( 0a_1 \cdots a_N \) is included in \( B \) by convexity. Its Lebesgue measure is \( \prod_{i=1}^N |a_i| \) up to a multiplicative constant depending on the dimension \( N \). Thus, \( \prod_{i=1}^N |a_i| < c_n \). To conclude, we note that the hypercube whose sides have the directions of the frame and contain \( a_1, \ldots, a_N \) contains \( B \) and has Lebesgue Measure \( \prod_{i=1}^N |a_i| \) up to a multiplicative constant depending on the dimension \( N \). So the lemma is proved.

As we only need an upper bound for \( IS_a \), we shall replace the family of elements \( B \) by the family of rectangles constructed in Lemma 4.3. By an adapted scaling, we shall also assume that their Lebesgue measure is \( s^{1/N} \). By using the radial symmetry, we see that studying \( IS_a(x) \) is just the same as studying:

\[
IS_a(2rx + x^2 + y_1^2 + \cdots + y_{N-1}^2),
\]

where \( r = |x| \). When \( r = 0 \), the symmetry implies that the structuring element attaining the Inf-Sup is also symmetric. Thus, the structuring element realizing the inf sup is the cube with side length equal to \( s^{1/N} \). When \( r > 0 \), the symmetry implies that the cube reaching the inf sup is symmetric with respect to directions \( y_1, \ldots, y_{N-1} \) and that its axes coincide with the frame axes. We also see that the gradient term implies that the element is thinner in the gradient direction than in the \( N - 1 \) other directions. More precisely, we assume this width equal to \( x \), so the lengths in the \( N - 1 \) other directions are given by \( xy^{N-1} = \frac{s}{2} x \). Then, the Sup of \( 2rx + x^2 + y_1^2 + \cdots + y_{N-1}^2 \) on this element is given by:

\[
2rx + x^2 + (N - 1)\left(\frac{s}{2N}x\right)^{\frac{2}{N-1}}.
\]

A simple calculation shows that the derivative of this function with respect to \( x \) is strictly positive for \( x \geq \frac{1}{2} s^{\frac{1}{N-1}} \). Hence, we can assume that \( 0 < x \leq \frac{1}{2} s^{\frac{1}{N-1}} \), that is the element is thinner than the cube in the direction of the gradient. Thus,

\[
\inf_{B \in B} \sup_{B \in B} \left(2rx + x^2 + \sum_{i=1}^{N-1} y_i^2\right) = \inf_{B \in B} \sup_{B \cap \{x \leq \frac{1}{2} s^{\frac{1}{N-1}}\}} \left(2rx + x^2 + \sum_{i=1}^{N-1} y_i^2\right)
\]

\[
\leq \inf_{B \in B} \sup_{B \cap \{x \leq \frac{1}{2} s^{\frac{1}{N-1}}\}} \left((2r + \frac{1}{2} s^{\frac{1}{N}})x + \sum_{i=1}^{N-1} y_i^2\right)
\]

\[
= K_N \left(2r + \frac{1}{2} s^{\frac{1}{N}}\right)^{\frac{2}{N-1}}s^{\frac{1}{N-1}}
\]

\[
= K_N \left(2r + \frac{1}{2} h^{\frac{1}{N}}\right)^{\frac{2}{N-1}}h,
\]

where \( K_N \) is a constant that only depends on the dimension. Hence, we have

\[
0 \leq \frac{IS_a(x) - a(x)}{h} \leq K_N \left(2r + \frac{1}{2} h^{\frac{1}{N}}\right)^{\frac{2}{N-1}}
\]
and the Lemma is proved, since the term on the right-hand side of the inequality tends to 0 when \( r \) and \( h \) tend to 0 independently. \( \Box \)

We now use the preceding Lemma to find a kind of consistency result when the gradient of the analysed function is equal to 0.

**Lemma 4.4.** Let \( \varphi \in C^3 \) and bounded such that \( D\varphi(0) = 0 \). Then,

\[
\lim_{(x,h) \to (0,0)} \frac{T_h \varphi(x) - \varphi(x)}{h} = 0.
\]

In this equation, the limit is taken for \( x \to 0 \) and \( h \to 0 \) independently.

**Proof.** Since \( \varphi \) is \( C^3 \) and bounded, we can find \( K > 0 \) such that for all \( y \)

\[
\varphi(y + x) \leq \varphi(x) + px + K|y|^2,
\]

where \( p = |D\varphi(x)| \), and as usual, we chose the \( x \) direction as the direction of the gradient at \( x \) (we also denote \( y = (x, y_1, \ldots, y_{N-1}) \)). If \( p = 0 \) then we know that we can find a constant \( C \) such that

\[
T_h \varphi(x) \leq \varphi(x) + Ch^{N+1},
\]

(the structuring element attaining the inf sup is a “cube” see Lemma 4.2).

If \( p \neq 0 \), then Lemma 4.2 tells us that

\[
\lim_{(x,h) \to (0,0)} \frac{1}{h} IS_h(px + K|y|^2) = 0.
\]

Indeed, \( x \to 0 \) and \( \varphi \) is regular, thus \( p \to 0 \) when \( h \) tends to 0. Thus,

\[
\limsup_{(x,h) \to (0,0)} \frac{1}{h} (T_h \varphi(x) - \varphi(x)) \leq 0.
\]

Conversely, we write

\[
\varphi(y + x) \geq \varphi(x) + px - K|y|^2,
\]

then

\[
IS_s \varphi(x + y) \geq IS_s(\varphi(x) + px - K|y|^2) = \varphi(x) + px - K|y|^2.
\]

In order to apply \( SI_s \), we use again Lemma 4.2, the identity \( SI_s(-u) = -IS_s(u) \), and the symmetry of the structuring elements with respect to the origin. More precisely,

\[
SI_s IS_s \varphi(x) \geq \varphi(x) - IS_s(-px + K|y|^2) = \varphi(x) - IS_s(px + K|y|^2).
\]

From Lemma 4.2, we get:

\[
\liminf_{(x,h) \to (0,0)} \frac{T_h \varphi(x) - \varphi(x)}{h} \geq 0.
\]

Finally,

\[
\lim_{(x,h) \to (0,0)} \frac{T_h \varphi(x_h) - \varphi(x_h)}{h} = 0. \quad \Box
\]
4.2. Main convergence theorem

We now prove the convergence result, by using the results in [3] saying that any monotone, stable and consistent scheme converges to the unique solution of the associated PDE if it satisfies a strong maximum principle. This last condition is satisfied for the AMSS equation.

**Theorem 4.5.** — Let $u_0$ in $BUC(\mathbb{R}^N)$. The approximate solutions $u_h$ (with $T_h = SI_h IS_h$) defined by Equation (4.3) converge towards the unique solution of

\begin{equation}
\frac{\partial u}{\partial t} = \alpha_0 |Du| \left( \prod_{i=1}^{N-1} \lambda_i \right)^{\frac{1}{N-1}} H(\lambda_1, \cdots, \lambda_{N-1}),
\end{equation}

with initial data $u_0$. Here, $H(\lambda_1, \cdots, \lambda_{N-1}) = -1$ if the $\lambda_i$ are all negative, 1 if they are all positive and 0 otherwise. Convergence is uniform on every compact set of $\mathbb{R}^N \times \mathbb{R}_+$.

**Proof.** — We directly use the results in [3]. The hypotheses are satisfied thanks to consistency and Lemma 4.4. We have no consistency results for the alternate scheme if the gradient is different from zero and one principal curvature is null but this is easily solved by locally adding a small quadratic form to the test functions and using the continuity of $F$ with respect to its first argument. \(\square\)

5. Motion by mean curvature

We saw in the first sections the use of Matheron's formalism to approximate the solution of the AMSS. By using inf sup operators, we shall now recover some results already proved by Ishii in [13]. The purpose is to generate Mean Curvature Motion (MCM). It is again described by a parabolic equation and associated to a scale space, much used in image processing

\begin{equation}
\frac{\partial u}{\partial t} - \left( \Delta u - \frac{(D^2 u Du, Du)}{|Du|^2} \right) = 0.
\end{equation}

The initial interest of MCM is the problem of minimal surfaces. In the two dimensional case, $\Delta u - (D^2 u Du, Du)/|Du|^2$ can also be written $(D^2 u Du^\perp, Du^\perp)/|Du|^2$, where $Du^\perp$ is a direction orthogonal to the gradient. This expression is the curvature of the level line of $u$ at the considered point. Therefore, the level lines of $u$ move according to their Mean Curvature. The reader may find complementary information about Mean Curvature Motion in [4], [7], [8], [10], [11], [13]. In [7], inf sup operators were already used to construct Mean Curvature Motion: The structuring elements where segments centered at the origin with length proportional to the scale. Another algorithm was proposed by Bence, Merriman and Osher in [4] to move sets by Mean Curvature and the convergence was also proved in [10] and [2]. The main idea is to use the Heat Kernel. Indeed, let $C$ a subset of $\mathbb{R}^N$ ($N \geq 2$). If we compute the solution of the heat equation for $t \in [0, h]$:

\[
\begin{cases}
  \frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) \\
  u(x, 0) = \chi_C(x)
\end{cases}
\]
and define

\[ T_h C = \left\{ x \in \mathbb{R}^N : u(x, h) \geq \frac{1}{2} \right\} ; \]

then if we let \( h \to 0 \) and \( nh \to t \), the set \( T_h^t C \) “tends” to a set \( C_t \). This set is the image of the initial set \( C \) by Mean Curvature Motion in the viscosity sense (see [9]). In [13], Ishii proposed a generalized scheme using a more general kernel. This kernel is assumed to be “decreasing” at infinity, but it does not need to decrease as fast as the Gaussian. The same scheme as above, used with this new kernel, is also proved to converge towards \( MCM \). We shall interpret these kernels as morphological operators. Indeed, let \( k \) a radial, nonnegative, continuous function satisfying

\[
\frac{1}{V_c(x)} k(x) \, dx = 1, \quad R^N, \\
\frac{1}{V_c(x)} k(x) \, dx = 1, \quad R^N,
\]

associated with the probability measure

\[
\text{meas}_k(E) = \int_E k(x) \, dx.
\]

We assume as well that \( k \) is small at infinity, that is

\[
\gamma > 3, \quad \text{of course, this condition is easily satisfied for the Heat kernel. We then define the weighted median filter associated with the probability measure } k \text{ by:}
\]

\[
\text{med}_k u(x) = \inf \left\{ \lambda, \text{meas}_k \{u(x) \leq \lambda\} \leq \frac{1}{2} \right\}.
\]

It is easy to see that this operator is morphological (monotone, contrast invariant and translation invariant). Therefore, according to Matheron’s Theorem 1.2 (see [16]), there exists a family of structuring elements \( B \) such that

\[
\text{med}_k u(x) = \inf \sup \left\{ u(x + y) : B \in \mathcal{B} , y \in B \right\}.
\]

The family is explicit in [12]. Set

\[
\mathcal{B} = \left\{ B \subset \mathbb{R}^N , \text{meas}_k(B) \geq \frac{1}{2} \right\},
\]

in [12], it is proved that

\[
\text{med}_k u(x) = \inf \sup \left\{ u(x + y) : B \in \mathcal{B} , y \in B \right\}.
\]

Since the probability measure \( k \) is radial, the family \( \mathcal{B} \) is isotropic invariant and so is \( \text{med}_k \). Let introduce a scale parameter \( h \) and shrink the weight function \( k \) into \( k_h \) with

\[
k_h(x) = \frac{1}{h^N} k\left(\frac{x}{h}\right),
\]

so that \( k_h \) still defines a probability measure but its variance tends to \( 0 \) when the scale tends to \( 0 \). We can define \( \mathcal{B}_h \) by replacing \( k \) by \( k_h \) in Equation (5.7); in other terms, \( \mathcal{B}_h = h\mathcal{B} \). Even though we shrink the weight function, the process is \textit{a priori} non local since the elements of \( \mathcal{B}_h \) are unbounded. We shall prove in the following, that when the scale tends to \( 0 \), the operator applied to \( C^0 \) functions is local. The locality of the operator will be guaranteed by the fact that the weight function \( k \) is small at infinity. As in the case of the AMSS, we shall first prove that the inf sup operator is consistent with the differential operator of Mean Curvature Motion. Next, we shall prove the convergence.
6. Consistency of the weighted median filter

First, we will describe more precisely the action of the $T_h$ defined by Equation (5.8) on some specific quadratic forms. We will then introduce two local operators $I^*_h$ and $S^*_h$ such that $I^*_h \leq T_h \leq S^*_h$. We shall prove that for quadratic forms, $I^*_h$ and $S^*_h$ are asymptotically equivalent when $h \to 0$. By applying these results on Taylor expansion of $C^3$ functions, we shall obtain the final consistency result:

$$T_h u(x) = u(x) + c(k)h^2(u)h^2 + o(h^2), \text{ if } Du(x) \neq 0.$$  

6.1. Preliminary heuristic study

We suppose that the weight function $k$ is radial. Let introduce the real function $f$ such that $k(x) = f(|x|)$. It is interesting to search minimal conditions on $k$ to get Mean Curvature Motion. In [13], Ishii makes the only assumption that $f$ is summable and that $\int f(r)(r^N + r^{N-2}) dr < +\infty$. In this paper, we suppose that $k$ is continuous, and that it is decreasing outside a ball that is: there exists $R > 0$ such that $f$ is decreasing over $[R, +\infty]$. These hypotheses may seem strong enough, but in fact they are not very restrictive. From a technical point of view, these assumptions will allow to apply Lebesgue’s dominated convergence theorem. When $h$ tends to 0 what is the asymptotic behavior of $\tilde{T}_h$:

$$\tilde{T}_h(x + hb_1y_1^2 + \cdots + hb_{N-1}y_{N-1}^2).$$

With a simple heuristic argument, we shall see that isotropic invariance implies that this quantity only depends on the mean curvature of the quadratic form. Indeed, let us denote $m(h)$ the inf sup and follow the argumentation developed in [12]. By definition the level set $x + hb_1y_1^2 + \cdots + hb_{N-1}y_{N-1}^2 = m(h)$ separates $\mathbb{R}^N$ into two parts with the same measure with respect to the probability measure so that for all $h > 0$, we have

$$m(h) - m(h, y_1, \ldots, y_{N-1}) dy_1 \cdots dy_{N-1} = \frac{1}{2}.$$

When $h = 0$, the symmetry of $k$ implies $m(h) = 0$, and the level surface associated with the inf sup is the hypersurface $x = 0$. We then define two functions $\varphi(m, h)$ and $\psi(m, h)$ depending of the $N - 1$ parameters $b_1, \ldots, b_{N-1}$, by

$$\varphi(m, h) = \int_{\mathbb{R}^{N-1}} y_1^2 k\left(m - h \sum_{i=1}^{N-1} b_i y_i^2, y_1, \ldots, y_{N-1}\right) dy_1 \cdots dy_{N-1}$$

and

$$\psi(m, h) = \int_{\mathbb{R}^{N-1}} k\left(m - h \sum_{i=1}^{N-1} b_i y_i^2, y_1, \ldots, y_{N-1}\right) dy_1 \cdots dy_{N-1}.$$  

By using the radial symmetry of $k$, if we differentiate formally Equation (6.1), we find

$$m'(h)\psi(m(h), h) - \left(\sum_{i=1}^{N-1} b_i\right)\varphi(m(h), h) = 0.$$  

(2) To simplify the notations, for a quadratic form $Q$ we will note $T_h Q$ instead of $T_h Q(0)$ as we already did it above.
We take \( h = 0 \) and set

\[
c(k) = \frac{1}{2} (N - 1) \int_0^{+\infty} r^N f(r) \, dr - \int_0^{+\infty} r^{N-2} f(r) \, dr.
\]

We obtain an explicit expression of \( m'(0) \)

\[
m'(0) = c(k) \frac{2}{N-1} \sum_{i=1}^{N-1} b_i.
\]

Note that the term after \( c(k) \) in Equation (6.5) is by definition the mean curvature of the quadratic form, so that

\[ m'(0) = c(k) \kappa, \]

where \( \kappa \) is the mean curvature of the quadratic form at the origin. The formal differentiation we made to compute \( m'(0) \) is now to be justified. Let us consider the ordinary differential equation

\[
\dot{m}'(h) = \frac{\varphi(\dot{m}(h), h)}{\psi(\dot{m}(h), h)} \quad \text{and} \quad \dot{m}(0) = 0,
\]

where \( \varphi \) and \( \psi \) are defined in (6.2) and (6.3) and the \( b_i \) are considered as parameters. These two functions are continuous and near 0, \( \psi \) is strictly positive: This is a consequence of the hypotheses upon \( k \). We separate the integrals in the definition of \( \varphi \) and \( \psi \) into two parts; if we choose \( R > 0 \) large enough, we can make the integral outside \( D(0, R) \) as small as we want, since we assumed that \( f \) is nonincreasing far from the origin and that it is small at infinity. Inside \( D(0, R) \), continuity is a consequence of Lebesgue's convergence Theorem. By Peano's Theorem, we then know that Equation (6.6) admits a solution, and by the definition of \( m \), we know that it is unique so that \( \dot{m} = m \). Thus,

\[
m(h) = c(k) \frac{2}{N-1} \left( \sum_{i=1}^{N-1} b_i \right) h + o(h).
\]

### 6.2. Notations, introduction of two local operators

We saw in the preceding section that the behavior of the filter \( T_h \) is quite simple on very specific quadratic forms: The second derivative is zero in the direction of the gradient and there are no cross-terms \( xy_i \) involving the gradient direction (the other ones \( y_i y_j \) can always be eliminated by diagonalizing the second derivative in the hypersurface orthogonal to the gradient). Unfortunately, it will not be as easy as soon as the quadratic form contains cross-terms. One way to solve this problem is to find local versions of the filter \( T_h \) that can be more easily described than \( T_h \) itself. We set \( r = h^\alpha \) where \( \frac{1}{2} < \alpha < 1 \) will be determined later in the analysis. Let

\[
I_h^r u(x) = \inf_{B \in B_h} \sup_{y \in B \cap D(0, r)} u(x + y)
\]
and

\[ S_h^r u(x) = \inf_{B \subseteq B_h} \sup_{y \in B} u(x + y). \]  

The notation is redundant since \( r \) can be expressed as a function of \( h \), but as in the AMSS case, it aims at insisting on the geometric interpretation of this parameter. It is obvious that for any \( x \in \mathbb{R}^N \)

\[ I_h^r u(x) \leq T_h u(x) \leq S_h^r u(x) \]  

since in \( I_h^r \), we take the sup on smaller elements, and in \( S_h^r \) we take the inf on a smaller family of structuring elements.

6.3. Generalization to any quadratic form

Let us define a quadratic form

\[ Q(x) = px + ax^2 + \sum_{i=1}^{N-1} b_i y_i^2 + \sum_{i=1}^{N-1} c_i x y_i, \]

where \( x = (x, y_1, \ldots, y_{N-1}) \in \mathbb{R}^N \). The case where the gradient equal to 0 will be treated separately. The case with a gradient term will be studied via \( I_h^r \) and \( S_h^r \). Each time, we shall assume without any restriction, that the gradient term \( p \) is strictly positive. Let

\[ Q(x) = px + ax^2 + \sum_{i=1}^{N-1} b_i y_i^2 + \sum_{i=1}^{N-1} c_i x y_i. \]

**Lemma 6.1.** Let \( p > 0 \).

\[ I_h^r \left( px + ax^2 + \sum_{i=1}^{N-1} b_i y_i^2 + \sum_{i=1}^{N-1} c_i x y_i \right) \geq c(k) \kappa h^2 + o(h^2), \]

where \( \kappa \) is the mean curvature at the origin.

**Proof.** First, we get rid with the \( N - 1 \) cross-terms \( x y_i \) by remarking that

\[ c_i x y_i \geq -|c_i| \left( \frac{x^2}{\varepsilon} + \varepsilon y_i^2 \right), \]

where \( \varepsilon = h^\theta, \theta \) being a small parameter that will be determined later. Thus,

\[ px + ax^2 + \sum_{i=1}^{N-1} b_i y_i^2 + \sum_{i=1}^{N-1} c_i x y_i \geq px + a(\varepsilon) x^2 + \sum_{i=1}^{N-1} b_i(\varepsilon) y_i^2, \]

where

\[
\begin{cases}
a(\varepsilon) = a - \varepsilon^{-1} \sum_i |c_i| \\
b_i(\varepsilon) = b_i + \varepsilon |c_i|, \quad 1 \leq i \leq N - 1.
\end{cases}
\]
As in [12] we eliminate the \( x^2 \) term by using morphological invariance. Let \( g(s) \) the real function

\[
g(s) = s - \frac{a}{p^2} s^2,
\]

(where \( a \) stands for \( a(\varepsilon) \)). For \( s < (2a)^{-1} p^2 = O(\varepsilon) \), \( g \) is increasing so we can use it as a contrast change. Indeed, in the ball with radius \( r \), we have:

\[
px + a(\varepsilon)x^2 + \sum_{i=1}^{N-1} b_i(\varepsilon)y_i^2 = O(r) + O(\varepsilon^{-1} r^2).
\]

If we choose \( \theta \) small enough, \( r = o(\varepsilon) \) when \( h \) tends to 0, and the quadratic form takes its values in an interval where \( g \) is invertible. We then have

\[
(6.12) \quad I_k'(px + ax^2 + \sum_i b_i y_i^2 + \sum_i c_i x y_i) \\
\geq g^{-1}(I_k'(g(px + a(\varepsilon)x^2 + \sum_i b_i(\varepsilon)y_i^2)))
\]

\[
(6.13) \quad \text{using morphological invariance} \\
\geq g^{-1}(I_k'(px + \sum_i b_i y_i^2 - \frac{a}{p^2}(ax^2 + \sum_i b_i y_i^2)^2 - 2\frac{a}{p}x(ax^2 + \sum_i b_i y_i^2)))
\]

(in the last inequality, we did not write the dependence of \( a \) and \( b_i \) upon \( c \)). In the ball of radius \( r \), we have

\[
\frac{a(\varepsilon)}{p^2}(a(\varepsilon)x^2 + \sum_i b_i(\varepsilon)y_i^2)^2 = O(\varepsilon^{-3} r^4)
\]

and

\[
\frac{a(\varepsilon)}{p}x(a(\varepsilon)x^2 + \sum_i b_i(\varepsilon)y_i^2) = O(\varepsilon^{-2} r^3).
\]

We now choose \( \theta \) small enough such that \( 3\theta - 2\theta > 2 \), yielding \( \varepsilon^{-2} r^3 = o(h^2) \). This is feasible since we assume \( \alpha > \frac{2}{3} \). We are then led to study

\[
I_k'(px + \sum b_i(\varepsilon)y_i^2),
\]

since

\[
I_k'(px + ax^2 + \sum_i b_i y_i^2 + \sum_i c_i x y_i) \geq g^{-1}(I_k'(px + \sum_i b_i(\varepsilon)y_i^2 + o(h^2))) \\
= g^{-1}(I_k'(px + \sum_i b_i(\varepsilon)y_i^2) + o(h^2)).
\]

We use the same kind of arguments as in our preliminary study. Indeed,

\[
I_k'(px + \sum b_i y_i^2) = \inf_{B \in B_\varepsilon} \sup_{x \in B \cap D(0, r)} (px + \sum b_i y_i^2) \\
= h \inf_{B \in B_\varepsilon} \sup_{x \in B \cap D(0, \xi)} (px + h \sum b_i y_i^2).
\]
Here \( hbi \) stands in fact for \( hbi(\varepsilon) = hbi - h^{1+\theta}|c_i| \). Define \( m_1(h) \) by Equation (6.1) where \( b_i \) is replaced by \( hbi(\varepsilon) \). We can also define two functions \( \varphi_1 \) and \( \psi_1 \) as in Equations (6.2) and (6.3) with \( b_i = hbi(\varepsilon) \). Note that the first derivative of the function \( hbi(\varepsilon) \) at \( h = 0 \) is equal to \( hbi \). By the same argument as above, we prove that \( \varphi_1 \) and \( \psi_1 \) are continuous so that by Peano's Theorem, \( m_1 \) is solution of the ordinary differential equation

\[
m_1'(h) = \frac{\varphi_1(m_1(h), h)}{\psi_1(m_1(h), h)} \quad \text{and} \quad m_1(0) = 0.
\]

We then find that \( m_1(h) = c(k)phc + o(h) \) where \( \kappa \) is the mean curvature of the quadratic form at the origin. Since \( h = o(r) \), this value is attained inside \( D(0, r) \). To see this, we take \( y_1 = \cdots = y_{N-1} = 0 \). The Inf-Sup is attained for \( x = m_1/p \). Thus Equation (6.12) becomes

\[
I_h^{\gamma}(Q) \geq g^{-1}(c(k)\kappa ph^2 + o(h^2)).
\]

Remarking that \( g^{-1}(s) = s + O(\varepsilon^{-1}s^2) \), we conclude that

\[
I_h^{\gamma}(Q) \geq c(k)\kappa ph^2 + o(h^2) \quad \Box.
\]

Conversely, to find an upper bound of \( T_hQ \), we study \( S_h^\gamma Q \).

**Lemma 6.2.** We have

\[
(6.14) \quad S_h^\gamma(px + ax^2 + \sum_{i=1}^{N-1} b_i y_i^2 + \sum_{i=1}^{N-1} c_i x y_i) \leq c(k)\kappa h^2 + o(h^2).
\]

**Proof.** We use the same argument as for the study of \( I_h^{\gamma} \). We first eliminate the cross-terms by using

\[
c_i x y_i \leq |c_i| \left( \frac{x^2}{\varepsilon} + \varepsilon y_i^2 \right).
\]

The same calculation as above with the constraint change eliminates the \( x^2 \) term and, after a short computation, we are led to study

\[
S_h^\gamma(px + h \sum_i b_i(\varepsilon)y_i^2).
\]

The used structuring elements are included in \( D(0, r) \), but this ball is bigger and bigger for the shrinking probability measure \( k_h \). In fact, there is less and less difference between the whole space and \( D(0, r) \) when we shrink the probability measure. More precisely,

\[
(6.15) \quad S_h^\gamma(px + \sum_i b_i(\varepsilon)y_i^2) = \inf_{B \in \mathcal{B}} \sup_{x \in D(0, \frac{r}{h})} \left( px + h \sum_i b_i(\varepsilon)y_i^2 \right).
\]

For \( h > 0 \), denote \( m_2(h) \) the Inf-Sup attained in Equation (6.15). By definition, for any \( h \)

\[
\text{meas}_k \left\{ x \in \mathbb{R}^N \cap D \left( 0, \frac{r}{h} \right), x \leq p^{-1}(m_2(h) - h \sum_i b_i(h)y_i^2) \right\} = \frac{1}{2}.
\]
that is
\[ \int_{R^{N-1}} \int_{-\infty}^{p^{-1}(m_2(h)-h) - h} \sum y_i^2 \chi_D(0, \frac{h}{r}) k(x) \, dx = \frac{1}{2}. \]

We now use the fact that \( k \) is small at infinity to minimize the effect of \( \chi_D(0, \frac{h}{r}) \).

Since \( K = \int_{R^N} |x|^\gamma k(x) \, dx < +\infty \),

\[ \int_{|x|>a} k(x) \, dx \leq \frac{K}{a^\gamma}. \]

Then, for any \( h \geq 0 \)

\[ \frac{1}{2} = \int_{R^{N-1}} \int_{-\infty}^{p^{-1}(m_2(h)-h) - h} \sum y_i^2 \chi_D(0, \frac{h}{r}) k(x) \, dx \quad \text{by definition} \]

\[ \leq \int_{R^{N-1}} \int_{-\infty}^{p^{-1}(m_2(h)-h) - h} \sum y_i^2 k(x) \, dx \quad \text{since } k \text{ is nonnegative} \]

\[ \leq \frac{1}{2} + K \left( \frac{h}{r} \right)^\gamma \quad \text{from Equation (6.16)} \]

\[ = \frac{1}{2} + Kh^{\gamma(1-\alpha)}. \]

Formally, if \( \gamma(1-\alpha) > 1 \), the first derivative of the function defined on the right-hand side of (6.18) at \( h = 0 \) must be equal to 0. As we assumed \( \alpha > \frac{2}{3} \), this implies \( \gamma > 3 \).

We are led again to

\[ m_2(h) = c(k)\kappa h + o(h). \]

Again, we made a formal differentiation in Equation (6.18) that we now justify. We can define two functions \( \varphi_2 \) and \( \psi_2 \) by:

\[ \psi_2(m, h) = \int_{R^{N-1}} k \left( m - h \sum_{i=1}^{N-1} b_i(h) y_i^2, y_1, \ldots, y_{N-1} \right) \chi_D(0, \frac{h}{r}) \, dy_1 \cdots dy_{N-1} \]

and

\[ \varphi_2(m, h) = \int_{R^{N-1}} y_i^2 k \left( m - h \sum_{i=1}^{N-1} b_i(h) y_i^2, y_1, \ldots, y_{N-1} \right) \chi_D(0, \frac{h}{r}) \, dy_1 \cdots dy_{N-1} \]

and by

\[ \psi_2(m, 0) = \int_{R^{N-1}} k(m, y_1, \ldots, y_{N-1}) \, dy_1 \cdots dy_{N-1} \]

and

\[ \varphi_2(m, 0) = \int_{R^N} y_i^2 k(m, y_1, \ldots, y_{N-1}) \, dy_1 \cdots dy_{N-1}. \]
By using Lebesgue’s theorem and the hypotheses on $k$, it is easy to check that these functions are continuous; we finally use Peano’s theorem to prove that $m_2$ has a derivative at 0 and that it verifies

$$m'_2(h) = \frac{\varphi_2(m_2(h), h)}{\psi_2(m_2(h), h)} \quad \text{and} \quad m_2(0) = 0.$$ 

Moreover $b_i(h)$ has a derivative at 0 and $b'_i(0) = b_i$. Then the differentiation we made in Equation (6.18) is justified and $m'_2(0) = c(k)\kappa$. We conclude exactly as in Lemma 6.1 that

$$s^p_h \left( px \sum_{i+1} b_i(h)y_i^2 \right) = c(k)h^2 + o(h^2)$$

and we compose by the inverse of the contrast change $g$ as above. 

**Corollary 6.3.** - Let $Q(x) = px + ax^2 + \sum_{i=1}^{N-1} b_i y_i^2 + \sum_{i=1}^{N-1} c_i x y_i$. Let $p > 0$ and $\kappa$ be the mean curvature of $Q$ at the origin that is $\kappa = 2p^{-1} \sum_i b_i$. Then,

$$T_h Q = c(k)psh^2 + o(h^2).$$

**Proof.** - It is a direct consequence of Lemmas 6.1 and 6.2. Indeed, as soon as $\gamma > 3$, we can choose $\frac{2}{3} < \alpha < 1$ such that $\gamma(1-\alpha) > 1$. From Lemmas 6.1 and 6.2, we obtain

$$c(k)psh^2 + o(h^2) \leq T_h Q \leq s^p_h Q \leq c(k)psh^2 + o(h^2)$$

and the proof is complete. 

6.4. **Case of a $C^3$ function**

Thanks to $I^p_h$ and $S^p_h$, we proved that the weighted median filter applied to quadratic forms is local when the scale tends to 0. The result is also true for $C^3$ functions.

**Theorem 6.4.** - Assume that the probability measure $k$ verifies $\int |x|^\gamma k(x) \, dx < +\infty$ with $\gamma > 3$. We also assume that $\tau = h^\alpha$ with $\frac{2}{3} < \alpha < 1$. Then,

$$T_h u(x) = u(x) + c(k)\kappa(u)h^2 + o(h^2) \text{ if } Du(x) \neq 0,$$

where $c(k)$ is a constant only depending on $k$.

**Proof.** - First, by translation invariance and grey level shift invariance, we can assume that $x = 0$ and $u(x) = 0$. The result is a simple consequence of Taylor expansion. As $u$ is $C^3$, we can write for $x \in D(0,r)$

$$u(x) = px + ax^2 + \sum_{i=1}^{N-1} b_i y_i^2 + \sum_{i=1}^{N-1} c_i x y_i + O(r^3),$$

with well chosen axes. We also have $O(r^3) = o(h^2)$. We then use our two local operators $I^p_h$ and $S^p_h$ and we immediately get the result from Corollary 6.3. 

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It is clear that if \( Du(0) \neq 0 \), then the consistency is uniform around 0. This means that Equation (6.22) is true with a \( o(h^2) \) term independent of \( x \) when \( x \) is close enough to 0; indeed the inverse of \( p = \phi(0) \) appeared only because of the contrast change used in the proof of Lemmas 6.1 and 6.2. As \( u \) is a smooth function, in a neighborhood of 0 we can find \( c > 0 \) such that \( |Du| \geq c > 0 \) in this neighborhood, and the estimations we found are then uniform. When the gradient is equal to 0, we will not have a uniform result, but thanks to Lemma 7.1 below and proved in [2], the following lemma will provide a sufficient result to establish the convergence of the iterated median filter.

**Lemma 6.5.** Let \( \phi \) a \( C^3 \) function. Let \( x_0 \in \mathbb{R}^N \) such that

\[
\left\{ \begin{array}{l}
\phi(x_0) = 0 \\
D^2\phi(x_0) = 0,
\end{array} \right.
\]

and let \( x_h \to x_0 \) when \( h \to 0 \). Then,

\[
\lim_{h \to 0} \frac{T_h \phi(x_h) - \phi(x_h)}{h^2} = 0.
\]

**Proof.** Let assume \( |x| < r \). By Taylor expansion, we have

\[
\phi(x_h + x) = \phi(x_h) + (D\phi(x_h), x) + (D^2\phi(x_h)x, x) + O(r^3)
\]

the last term \( O(r^3) \) being uniform with respect to \( h \). Set \( D\phi(x_h) = p_h \). We write

\[
\phi(x_h + x) \leq \phi(x_h) + p_h x + K_h |x|^2 + O(r^3),
\]

with well chosen frame axes. From the hypotheses, we can also assume that \( p_h \geq 0 \) and \( K_h > 0 \) and also \( p_h \to 0 \) and \( K_h \to 0 \) when \( h \) tends to 0. Then,

\[
T_h \phi(x_h) - \phi(x_h) \leq \frac{1}{h^2} S_h (p_h x + K_h |x|^2) + O\left(\frac{r^3}{h^2}\right).
\]

We remark that the level surfaces of \( p_h x + K_h |x|^2 \) are spheres centered at the point

\[
\left(-\frac{p_h}{2K_h}, 0, \ldots, 0\right).
\]

The zero level surface is completely included in the half-space \( \{x \leq 0\} \) and its interior has a \( k \)-measure strictly less than \( \frac{1}{2} \). Thus, we can assume that the inf sup of the right-hand term in (6.23) is attained with \( x \geq 0 \). We are then led to study

\[
\frac{1}{B \subset D(0,r)} \sup_{B \in B_h} \sup_{x \in B^D(x \geq 0)} (p_h x + K_h |x|^2),
\]

since we do not change the operator (for this particular function) by restricting the structuring element to the half space \( \{x \geq 0\} \). As \( x \geq 0 \) and \( |x| \leq r \), we have \( x^2 \leq r x \).
By replacing \( r \) by its value \( h^\alpha \) where \( \alpha \) is chosen as above, we find

\[
\inf_{B \subset D(0,r)} \sup_{x \in B \setminus \{x \geq 0\}} \frac{(p_h x + K_h |x|^2)}{h^2} \\
\leq \inf_{B \subset B(0,r)} \sup_{x \in B \setminus \{x \geq 0\}} \left( \frac{(p_h + K_h h^\alpha) x + K_h (y_1^2 + \cdots + y_{N-1}^2)}{p_h + K_h h^\alpha} \right) \\
= h(p_h + K_h h^\alpha) \inf_{B \subset D(0,r)} \sup_{x \in B \setminus \{x \geq 0\}} \left( x + \frac{K_h h}{p_h + K_h h^\alpha} (y_1^2 + \cdots + y_{N-1}^2) \right) \\
- h(p_h + K_h h^\alpha) \left( c_k (N - 1) K_h + o \left( \frac{K_h h}{p_h + h^\alpha K_h} \right) \right) \\
= h^2 (c_k (N - 1) K_h + o(K_h)).
\]

We can then divide by \( h^2 \) and let \( h \) tend to 0. As \( K_h \) tends to 0,

\[ \limsup_{h \to 0} T_h \varphi(x_h) - \varphi(x_h) \leq 0. \]

Conversely,

\[ \frac{T_h \varphi(x_h) - \varphi(x_h)}{h^2} \geq \frac{1}{h^2} \inf_{B \subset B(0,r)} \sup_{x \in B \setminus \{x \geq 0\}} \left( \frac{(p_h + K_h h^\alpha) x - K_h (y_1^2 + \cdots + y_{N-1}^2)}{p_h + K_h h^\alpha} \right) \]

with the same hypotheses on \( p_h \) and \( K_h \) as above. We remark that the level surfaces of the quadratic form now considered are spheres centered at \((x, 0, \ldots, 0)\). As the zero level sphere is included in \( \{x \geq 0\} \) and its interior has a \( k \)-measure strictly less than \( \frac{1}{2} \), we can assume that the Inf-Sup is attained with \( x \leq 0 \).

In \( D(0,r) \) and when \( x \leq 0 \), we have \(-x^2 \geq rx\). This allows us to eliminate the \( x^2 \) term in the quadratic form, we get:

\[
\frac{1}{h^2} \inf_{B \subset B(0,r)} \sup_{x \leq 0} \left( \frac{(p_h x - K_h |x|^2)}{h^2} \right) \\
\geq \frac{1}{h^2} \inf_{B \subset B(0,r)} \sup_{x \leq 0} \left( \frac{-(x - K_h h^\alpha (y_1^2 + \cdots + y_{N-1}^2) + O \left( \frac{r^3}{h^2} \right)}{h^2} \right) \\
= \frac{1}{h^2} \inf_{B \subset B(0,r)} \sup_{x \leq 0} \left( \frac{x - \frac{K_h h}{p_h + K_h h^\alpha} (y_1^2 + \cdots + y_{N-1}^2)}{h^2} \right) + O \left( \frac{r^3}{h^2} \right) \\
= -c_k (N - 1) K_h + o(K_h) + o(1).
\]

We let \( h \) tend to 0 and get

\[ \liminf_{h \to 0} \frac{T_h \varphi(x_h) - \varphi(x_h)}{h^2} \geq 0 \]

Finally, from (6.24) and (6.26)

\[ \lim_{h \to 0} \frac{T_h \varphi(x_h) - \varphi(x_h)}{h^2} = 0. \]
7. Convergence of the weighted iterated median filter

We know that $T_h$ is consistent with the differential operator of Mean Curvature. We shall now prove that the iterated operator $T_h^n$ tends to the nonlinear semi-group generated by Mean Curvature Motion (MC&M). Precisely, if $u_0$ is a bounded Lipschitz function mapping $\mathbb{R}^N$ in $\mathbb{R}$, for $h > 0$, we define the approximate solutions $u_h$ as

\begin{equation}
 u_h(x, t) = T_h^n u(x) \text{ if } nh^2 \leq t < (n + 1)h^2.
\end{equation}

We prove the convergence of the scheme. The existence is proved by semi-group theory in [10]. As in the case of the AMSS equation, we use consistency to prove the convergence. Problems at critical points are treated with Lemma 6.5, thanks to the following Lemma proved in [2], saying that at a critical point, the second derivative of the test function can also be assumed equal to 0.

\textbf{Lemma 7.1.} - Let $u$ a continuous function. Then $u$ is subsolution (resp supersolution) of

\[ \frac{\partial u}{\partial t} - \left( \Delta u - \frac{(D^2u Du, Du)}{|Du|^2} \right) = 0 \]

if and only if for all $\varphi \in C^2(\mathbb{R}^N \times \mathbb{R}^+)$ and $x$ such that $u - \varphi$ has a maximum (resp. a minimum) at $x$,

\[ \begin{cases}
 \frac{\partial \varphi}{\partial t} (x) - \left( \Delta \varphi - \frac{(D^2 \varphi D \varphi, D \varphi)}{|D \varphi|^2} \right) < 0 \quad (\text{resp. } > 0) \text{ if } D \varphi(x) \neq 0, \\
 \frac{\partial \varphi}{\partial t} (x) \leq 0 \quad (\text{resp. } \geq 0) \text{ if } D \varphi(x) = 0 \text{ and } D^2 \varphi(x) = 0.
\end{cases} \]

In other terms, nothing is required at points where the gradient is equal to 0.

\textbf{Theorem 7.2.} - Let $u_h$ be the approximate solutions defined as above. Then these solutions converge uniformly on every compact set of $\mathbb{R}^N \times \mathbb{R}^+$ towards a function $u$ which is solution of the Mean Curvature Motion.

\textbf{Proof.} - The proof is exactly the same as in the AMSS case, using consistency at noncritical points and Lemmas 6.5 and 7.1 at critical points. Again convergence is a consequence of [3]. \(\square\)

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