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A Note on k-Plane Integral Transforms*

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Let Π be a k-dimensional subspace of R^n, $n \geq 2$, and write $x = (x', x'')$ with x' in Π and x'' in the orthogonal complement $\Pi^\perp$. The k-plane transform of a measurable function f in the direction Π at the point x'' is defined by $Lf(\Pi, x'') = \int_\Pi f(x', x'')\, dx'$. In this article certain a priori inequalities are established which show in particular that if $f \in L^p(R^n)$, $1 \leq p < n/k$, then f is integrable over almost every translate of almost every k-space. Mapping properties of the k-plane transform between the spaces $L^p(R^n)$, $p \leq 2$, and certain Lebesgue spaces with mixed norm on a vector bundle over the Grassmann manifold of k-spaces in R^n are also obtained.

1. INTRODUCTION

Let Π be a k-dimensional subspace of R^n, $n \geq 2$, and write $x = (x', x'')$ with x' in Π and x'' in the orthogonal complement $\Pi^\perp$. The k-plane transform of a measurable function f in the direction Π at the point x'' is defined by

$$Lf(\Pi, x'') = L_\Pi f(x'') = \int_\Pi f(x', x'')\, dx', \tag{1.1}$$

provided the integral exists in the Lebesgue sense.

From the point of view of applications, the k-plane transform is of particular current interest in the following cases: $k = 1$, where it is the transform arising in radiographic reconstruction; $k = 2$, where it is the transform arising in nuclear magnetic-resonance reconstruction; and $k = n - 1$, where it is the Radon transform [2–5, 7].

It is easy to see that if $f \in L^1(R^n)$, then for any fixed Π the integral in (1.1) exists for almost every x'' in $\Pi^\perp$ and $\|L_\Pi f\|_{L^1(\Pi^\perp)} \leq \|f\|_{L^1(R^n)}$. On the other hand, again with Π fixed, it is easy to give examples of functions f which lie in all $L^p(R^n)$, $p > 1$, while the integral in (1.1) does not exist for any x''. With f fixed, however, such subspaces Π are exceptional. One purpose of this article is to establish certain a priori inequalities which show in particular that if $f \in L^p(R^n)$, $1 \leq p < n/k$, then f is integrable over almost every translate of almost every k-space Π. Such is not the case for $p \geq n/k$, as is shown by the function $f(x) =$

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$(2+|x|)^{-n/p}(\log(2+|x|))^{-1}$, which lies in $L^p(R^n)$ if $p > 1$, but is not integrable over any k-plane of dimension $\geq n/p$.

A second and related purpose of the article is to establish the identity $L^{\#}Lf = (2\pi)^k R_k f$, $L^{\#}$ being the formal adjoint of L, and R_k the Riesz potential of order k. The adjoint $L^{\#}$ is expressed explicitly, so that the identity contains an explicit formula for the inverse of the k-plane transform L.

Some of these results are given for $p = 2$ in [6, 9].

2. The Riesz Potential

The Riesz kernel of order α is the function

$$R_\alpha(x) = \frac{\Gamma((n-\alpha)/2)}{\pi^{n/2}\, 2^\alpha \Gamma(\alpha/2)} |x|^{\alpha-n}, \qquad 0 < \alpha < n. \tag{2.1}$$

The Riesz potential of a measurable function f is the convolution

$$R_\alpha f(x) = R_\alpha * f(x) = \int_{R^n} R_\alpha(y) f(x-y)\, dy, \tag{2.2}$$

whenever the integral exists in the Lebesgue sense.

We set

$$\nu_\alpha(x) = (1+|x|^2)^{(\alpha-n)/2} \tag{2.3}$$

and write $L^p(\nu_\alpha)$ for the L^p space with measure $\nu_\alpha(x)\, dx$. Since

$$R_k \nu_{\alpha-k}(x) \leq c\nu_k(x), \qquad \alpha < k, \tag{2.4}$$

it follows that

$$\| R_k f \|_{L^1(\nu_{\alpha-k})} \leq c \| f \|_{L^1(\nu_k)}, \qquad \alpha < k. \tag{2.5}$$

In particular, if $f \in L^1(\nu_k)$, then $R_k f$ is defined almost everywhere and lies in $L^1(\nu_{\alpha-k})$ for all $\alpha < k$. On the other hand, if $f \notin L^1(\nu_k)$, it is easily seen that $R_k f$ is defined nowhere.

With the Fourier transform on R^n given by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-i\langle x,\xi\rangle} f(x)\, dx, \tag{2.6}$$

the Fourier transform of R_α is given by

$$\hat{R}_\alpha(\xi) = (2\pi)^{-n/2} |\xi|^{-\alpha}, \tag{2.7}$$

so that

$$(R_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha} \hat{f}(\xi). \tag{2.8}$$

Conditions on f for the validity of (2.8) are discussed in some detail in [6]. Here the formula is needed only in simple cases.

According to (2.8) the inverse of the operator R_α, commonly denoted by Λ^α, is expressed in Fourier transforms by

$$(\Lambda^\alpha f)^\wedge(\xi) = |\xi|^\alpha \hat{f}(\xi). \tag{2.9}$$

3. Lower Dimensional Integrability

Let $d\Pi$ be the unique finite measure on the Grassmann manifold $G_{n,k}$ of k-spaces in R^n which is invariant under orthogonal transformations and normalized so that the measure of $G_{n,k}$ is $|S^{n-1}|/|S^{n-k-1}|$, the bars denoting the appropriate area measures on the spheres. If f is a function on R^n, then Lf is a function on the bundle $T(G_{n,k}) = \{(\Pi, x''): \Pi \in G_{n,k}, x'' \in \Pi^\perp\}$. A natural measure η is defined on $T(G_{n,k})$ by the formula

$$\int_{T(G_{n,k})} g(\Pi, x'')\, d\eta = \int_{G_{n,k}} \int_{\Pi^\perp} g(\Pi, x'')\, dx''\, d\Pi. \tag{3.1}$$

We consider L as an operator from measurable functions on R^n to measurable functions on $T(G_{n,k})$.

The following integration formulas are valid when f is a nonnegative measurable function on R^n or when either side is finite when f is replaced by its absolute value [6].

$$\int_{G_{n,k}} \int_{\Pi^\perp} |x''|^k f(x'')\, dx''\, d\Pi = \int_{R^n} f(x)\, dx, \tag{3.2}$$

$$|S^{n-k-1}| \int_{G_{n,k}} \int_{\Pi} |x'|^{n-k} f(x')\, dx'\, d\Pi = |S^{k-1}| \int_{R^n} f(x)\, dx. \tag{3.3}$$

Theorem 3.4. *The formal adjoint of L is given by*

$$L^\# g(x) = \int_{G_{n,k}} g(\Pi, P_{\Pi^\perp} x)\, d\Pi, \tag{3.5}$$

where $P_{\Pi^\perp}$ is the orthogonal projection in R^n on $\Pi^\perp$.

Proof. If f and g are nonnegative measurable functions on R^n and $T(G_{n,k})$, respectively, then

$$\langle Lf, g\rangle = \int_{G_{n,k}} \int_{\Pi^\perp} L_\Pi f(x'')\, g(\Pi, x'')\, dx''\, d\Pi$$

$$= \int_{R^n} f(x) \int_{G_{n,k}} g(\Pi, P_{\Pi^\perp} x)\, d\Pi\, dx = \langle f, L^\# g\rangle.$$

THEOREM 3.6. *If f is nonnegative and measurable on R^n, then*

$$(L^\# Lf)(y) = (2\pi)^k R_k f(y).$$

Proof. Formulas (1.1), (3.5), a change of variable $z' = x' - y'$ on $\Pi\mu$, and (3.3) give

$$(L^\# Lf)(y) = \int_{G_{n,k}} \int_{\Pi} f(x', P_{\Pi^\perp} y)\, dx'\, d\Pi$$

$$= \int_{G_{n,k}} \int_{\Pi} |z'|^{n-k} (|z'|^{k-n} f(z' + y))\, dz'\, d\Pi$$

$$= |S^{k-1}|\, |S^{n-k-1}|^{-1} \int_{R^n} f(y - z)\, |z|^{k-n}\, dz = (2\pi)^k R_k f(y).$$

As a consequence of (2.4), Theorem 3.6, and the simple identity

$$L_\Pi \nu_{\alpha-k}(x'') = c\nu_\alpha(x''), \tag{3.7}$$

we have the following a priori inequality.

THEOREM 3.8. *For each $\alpha < k$ there is a constant c such that*

$$\int_{G_{n,k}} \| L_\Pi f \|_{L^1(\nu_\alpha)}\, d\Pi \leq c \| f \|_{L^1(\nu_k)}.$$

Proof. It suffices to establish the inequality when $f \geq 0$, in which case we have

$$\int_{G_{n,k}} \| L_\Pi f \|_{L^1(\nu_\alpha)}\, d\Pi = \int_{G_{n,k}} \langle L_\Pi f, \nu_\alpha\rangle\, d\Pi$$

$$= c \int_{G_{n,k}} \langle L_\Pi f, L_\Pi \nu_{\alpha-k}\rangle\, d\Pi = c\langle f, L^\# L \nu_{\alpha-k}\rangle = c\langle f, R_k \nu_{\alpha-k}\rangle$$

$$\leq c\langle f, \nu_k\rangle = c\| f \|_{L^1(\nu_k)}$$

Theorem 3.8 shows that if $f \in L^1(\nu_k)$, then Lf is defined almost everywhere on $T(G_{n,k})$ (i.e., for almost every k-space Π, f is integrable over almost all translates of Π) and Lf is locally integrable on $T(G_{n,k})$. For nonnegative functions the converse is true:

THEOREM 3.9. *If f is nonnegative, then Lf is defined almost everywhere and is locally integrable on $T(G_{n,k})$ if and only if $f \in L^1(\nu_k)$.*

Proof. It remains to establish the only if part. Choose $M > 0$ and let X_M be the characteristic function of the ball of radius M centered at the origin in R^n. Since Lf is locally integrable on $T(G_{n,k})$, Theorem 3.6 gives

$$\infty > \langle LX_M, Lf \rangle = \langle X_M, L^\# Lf \rangle = (2\pi)^k \int_{|x|<M} R_k f(x)\, dx.$$

Thus $R_k f(x)$ exists for almost every x with $|x| < M$. The remark following (2.5) now shows that $f \in L^1(\nu_k)$.

COROLLARY 3.10. *For $1 \leqslant p < n/k$ and $\alpha < k$,*

$$\int_{G_{n,k}} \| L_\Pi f \|_{L^1(\nu_\alpha)}\, d\Pi \leqslant c \| f \|_{L^p(R^n)},$$

Proof. Since $\nu_k \in L^q(R^n)$, $q > n/(n-k)$, the result follows from Theorem 3.8 and Holder's inequality.

Since the Riesz potential R_k is one to one on $L^1(\nu_k)$, Theorem 3.6 shows that the k-plane transform is also.

COROLLARY 3.11. *The k-plane transform is one to one on $L^1(\nu_k)$.*

Remark. Formulas (2.8) and (2.9) show that formally $f = \Lambda^k R_k f$. This, together with Theorem 3.6, gives the inversion formula

$$f = (2\pi)^{-k} \Lambda^k L^\# Lf, \tag{3.12}$$

for the k-plane transform. Various conditions for the validity of (3.12) can be found in [6, 9].

4. L^p MAPPING PROPERTIES

In this section we establish a priori estimates for the k-plane transform as a map between $L^p(R^n)$ and certain L^q spaces with mixed norm on the bundle $T(G_{n,k})$. The case $p = 2$ was treated in [6].

Let χ_M be the characteristic function of the ball of radius M centered at the origin in R^n. A simple computation gives $\|L_\Pi \chi_M\|_{L^q(\Pi^\perp)} = c_1 M^{(kq-n-k)/q}$. Since $\|\chi_M\|_{L^p(R^n)} = c_2 M^{n/p}$, it follows that an inequality of the type

$$\int_{G_{n,k}} \|L_\Pi f\|^r_{L^q(\Pi^\perp)}\, d\Pi \leqslant c\, \|f\|^r_{L^p(R^n)} \tag{4.1}$$

can hold only if $(kq - n - k)/q = n/p$, or equivalently only if $q = p(n-k)/(n-pk)$. We will establish (4.1) when $r = 2$, $p < n/k$, and $p \leqslant 2$. The restriction $p < n/k$ is necessary as is shown in the Introduction. The restrictions $r = 2$ and $p \leqslant 2$ are probably artifacts of the proof.

THEOREM 4.2. *Suppose that* $f \in L^p(R^n)$, $p \leqslant 2$, $p < n/k$. *Then*

$$\int_{G_{n,k}} \|L_\Pi f\|^2_{L^q(\Pi^\perp)}\, d\Pi \leqslant c\, \|f\|^2_{L^p(R^n)}, \qquad \textit{where} \qquad q = p(n-k)/(n-pk).$$

We begin with two reductions.

(a) It suffices to prove the theorem when $p \geqslant 2n/(n+k)$. Indeed, the theorem is obvious for $p = 1$, $q = 1$, and the intermediate values of p and q are taken care of by the interpolation theorem for L^p spaces with mixed norms [1], since $T(G_{n,k})$ is locally a product space.

(b) For a given value of p it suffices to prove the theorem for f in $L_0^\infty(R^n)$, i.e., bounded with compact support. Indeed, if $f \geqslant 0$ we can approximate by a nonnegative increasing sequence in $L_0^\infty(R^n)$ and use the monotone convergence theorem. The point of this assumption is that there is then no difficulty with the validity of (2.8) either for f itself or for $L_\Pi f$. (See [6, Lemma 4.1].)

In the course of the proof we shall need the Fourier transform relationship

$$\begin{aligned} (L_\Pi f)^\wedge(\xi'') &= (2\pi)^{(k-n)/2} \int_{\Pi^\perp} e^{-i\langle x'', \xi''\rangle} L_\Pi f(x'')\, dx'' \\ &= (2\pi)^{k/2} \hat{f}(\xi'') \qquad \text{for } \xi'' \in \Pi^\perp. \end{aligned} \tag{4.3}$$

Proof. Under conditions (a) and (b), set

$$\alpha = n(2-p)/2p, \qquad \beta = k/2 - \alpha = (p(n+k) - 2n)/2p. \tag{4.4}$$

Then

$$\frac{1}{2} = \frac{1}{p} - \frac{\alpha}{n} \qquad \text{and} \qquad \frac{1}{q} = \frac{1}{2} - \frac{\beta}{n-k}. \tag{4.5}$$

According to (2.9), Parseval's theorem, (4.3), (4.4), (3.2), and (2.8),

$$\int_{G_{n,k}} \left\| \bigwedge^{\beta} L_{\Pi} f \right\|^2_{L^2(\Pi^\perp)} d\Pi = \int_{G_{n,k}} \int_{\Pi^\perp} | \, | \xi'' |^{\beta} (L_{\Pi} f)\hat{}\, (\xi'')|^2 \, d\xi'' \, d\Pi$$

$$= (2\pi)^k \int_{G_{n,k}} \int_{\Pi^\perp} | \xi'' |^k \, | \, | \xi'' |^{-\alpha} \hat{f}(\xi'')|^2 \, d\xi'' \, d\Pi$$

$$= (2\pi)^k \int_{R^n} | \, | \xi |^{-\alpha} \hat{f}(\xi)|^2 \, d\xi = (2\pi)^k \| R_{\alpha} f \|^2_{L^2(R^n)} \,. \tag{4.6}$$

Now, (4.5) and Sobolev's inequality [8] in $\Pi^\perp$ give

$$\| L_{\Pi} f \|_{L^q(\Pi^\perp)} = \left\| R_{\beta} \bigwedge^{\beta} L_{\Pi} f \right\|_{L^q(\Pi^\perp)} \leqslant c \left\| \bigwedge^{\beta} L_{\Pi} f \right\|_{L^2(\Pi^\perp)}, \quad \text{for a.e. } \Pi, \tag{4.7}$$

while (4.5) and Sobolev's inequality in R^n give

$$\| R_{\alpha} f \|_{L^2(R^n)} \leqslant c \| f \|_{L^p(R^n)} \,. \tag{4.8}$$

Squaring (4.7), integrating over $G_{n,k}$, and applying (4.6) and (4.8) give the desired inequality. (If $\alpha = 0$, then (4.8) is not needed.)

By duality we obtain an a priori inequality for $L^{\#}$.

THEOREM 4.9. *Assume that g is a measurable function on $T(G_{n,k})$, $p \geqslant 2(n-k)/n$, and $p > 1$. Then*

$$\| L^{\#} g \|^2_{L^q(R^n)} \leqslant c \int_{G_{n,k}} \| g(\Pi, x'') \|^2_{L^p(\Pi^\perp)} \, d\Pi \qquad \textit{with} \qquad q = pn/(n-k).$$

The assumption $p > 1$ is necessary, but the assumption $p \geqslant 2(n-k)/n$ appears to be an artifact of the proof of Theorem 4.2.

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