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Existence Results for Impulsive Multivalued Semilinear Neutral Functional Differential Inclusions in Banach Spaces

M. Benchohra

Department of Mathematics, University of Sidi Bel Abbes, BP 89, 22000 Sidi Bel Abbes, Algeria
E-mail: benchohra@yahoo.com

J. Henderson

Department of Mathematics, Auburn University, Auburn, Alabama 36849-5310
E-mail: hendej2@mail.auburn.edu

and

S. K. Ntouyas

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail: sntouyas@cc.uoi.gr

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In this paper, the existence of mild solutions for first- and second-order impulsive semilinear neutral functional differential inclusions in Banach spaces is investigated. The results are obtained by using a fixed point theorem for condensing multivalued maps due to Martelli and semigroup theory. © 2001 Academic Press

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1. INTRODUCTION

The study of impulsive functional differential equations is linked to their utility in simulating processes and phenomena subject to short-time

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perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. That is why the perturbations are considered to take place "instantaneously" in the form of impulses. The theory of impulsive differential equations has seen considerable development; see the monographs of Bainov and Simeonov [1], Lakshmikantham *et al*. [12], and Samoilenko and Perestyuk [16], where numerous properties of their solutions are studied and detailed bibliographies are given.

This paper is devoted to extending existing results to a differential-inclusions scenario. To be precise, in [3], the authors used Schaefer's theorem to establish existence results for first- and second-order impulsive semilinear neutral functional differential equations in Banach spaces. The goal of this paper is to extend, via Martelli's fixed point theorem, the results of [3] to the differential inclusions context.

Section 3 deals with the existence of mild solutions for the first-order initial-value problem for semilinear neutral functional differential inclusions with impulsive effects given by

$$\frac{d}{dt}[y(t) - g(t, y_t)] \in Ay(t) + F(t, y_t), \quad t \in J = [0, b], \quad t \neq t_k, \ k = 1, \ldots, m, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in E; $F\colon J \times C([-r, 0], E) \longrightarrow 2^E$ is a bounded, closed, and convex-valued multivalued map; $g\colon J \times C([-r, 0], E) \longrightarrow E$ is a given function; $\phi \in C([-r, 0], E)\,(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, and $I_k \in C(E, E)$ $(k = 1, 2, \ldots, m)$ are bounded functions; $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively; and E is a real Banach space with norm $|\cdot|$.

For any continuous function y defined on $[-r, b] - \{t_1, \ldots, t_m\}$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], E)$ defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t.

In Section 4 we study second-order impulsive semilinear neutral functional differential inclusions of the form

$$\frac{d}{dt}[y'(t) - g(t, y_t)] \in Ay(t) + F(t, y_t), \quad t \in J = [0, b], \quad t \neq t_k, \ k = 1, \ldots, m, \tag{1.4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.5}$$

(1.6) $$\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-)), \quad k = 1, \ldots, m,$$

(1.7) $$y(t) = \phi(t), \qquad t \in [-r, 0], \quad y'(0) = \eta,$$

where A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in E; F, g, I_k, and ϕ are as in the problem (1.1)–(1.3); $\bar{I}_k \in C(E, E)$; and $\eta \in E$.

Other results on functional differential equations without impulsive effect can be found in the monograph of Erbe *et al.* [5], Hale and Verduyn Lunel [9], Henderson [10], and the survey paper of Ntouyas [15].

This paper is organized as follows. In Section 2 we recall briefly some basic definitions and preliminary facts which will be used throughout Sections 3 and 4. In Section 3 we establish existence theorems for (1.1)–(1.3), and in Section 4 we deal with (1.4)–(1.7). Our approaches are based on a fixed-point theorem for condensing multivalued maps due to Martelli [14].

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

$C([-r, 0], E)$ is the Banach space of all continuous functions from $[-r, 0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_J := \sup\{|y(t)| : t \in J\}.$$

A measurable function $y: J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see, for instance, Yosida [19]).

$L^1(J, E)$ denotes the Banach space of functions $y: J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\, dt \qquad \text{for all } y \in L^1(J, E).$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G: X \longrightarrow 2^X$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for each bounded set B of X (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$ there exists an openneighborhood M of x_0 such that $G(M) \subseteq N$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $CC(E)$ denotes the set of all nonempty compact, convex subsets of E. A multivalued map G: $J \longrightarrow CC(X)$ is said to be measurable if for each $x \in E$ the function $Y : J \longrightarrow \mathbb{R}$ defined by $Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$ is measurable.

An upper semicontinuous multivalued map G: $X \longrightarrow 2^X$ is said to be condensing [2] if for any subset $B \subset X$ with $\alpha(B) \neq 0$ we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness [2].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see the books by Deimling [4], Gorniewicz [8], and Hu and Papageorgiou [11].

For any $y \in C([-r, b], E)$ we define the set

$$S_{F,y} = \{v \in L^1(J, E) : v(t) \in F(t, y_t) \text{ for a.e. } t \in J\}.$$

Our main results are based on the following lemmas.

LEMMA 2.1 ([13]). *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying the Carathéodory conditions with the set of L^1-selections S_F nonempty, and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S_F : C(I, X) \longrightarrow CC(C(I, X)), \qquad y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}),$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

LEMMA 2.2 ([14]). *Let X be a Banach space and $N : X \longrightarrow CC(X)$ be a condensing map. If the set*

$$\mathcal{M} := \{y \in X : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

3. FIRST ORDER IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

In order to define the concept of a mild solution of (1.1)–(1.3) we consider the space

$$\begin{aligned}\Omega = \{y : [-r, b] \longrightarrow E : y_k \in C(J_k, E), k = 0, \dots, m \ \text{ and there exist}\\ y(t_k^-) \text{ and } y(t_k^+), \text{ with } y(t_k^-) = y(t_k),\ k = 1, \dots, m,\\ y(t) = \phi(t), \forall t \in [-r, 0]\},\end{aligned}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k}, k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}]$, $k = 0, \ldots, m$.

So let us start by defining what we mean by a mild solution of the problem (1.1)–(1.3).

DEFINITION 3.1. A function $y \in C([-r, b], E)$ is said to be a mild solution of (1.1)–(1.3) if $y(t) = \phi(t)$ on $[-r, 0]$; $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \ldots, m$; for each $0 \le t \le b$ the function $AT(t-s)g(s, y_s)$, $s \in [0, t)$, is integrable; there exists a $v \in L^1(J, E)$ such that $v(t) \in F(t, y_t)$ a.e on J; and

$$y(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, y_t) + \int_0^t AT(t-s)g(s, y_s)\,ds + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \qquad t \in J.$$

Let $v \in L^1(J, E)$ and consider the initial value problem (IVP) (3.1), (1.2), and (1.3), where

$$\frac{d}{dt}[y(t) - g(t, y_t)] = Ay(t) + v(t), \quad t \in J,\ t \neq t_k,\ k = 1, \ldots, m. \tag{3.1}$$

We need the following auxiliary result. Its proof is very simple, so we omit it.

LEMMA 3.2. *$y \in \Omega^1$ is a solution of* (3.1), (1.2), *and* (1.3) *if and only if $y \in \Omega$ is a solution of the impulsive integral equation*

$$y(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, y_t) + \int_0^t AT(t-s)g(s, y_s)\,ds + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \qquad t \in J.$$

We are now in a position to state and prove our existence result for the problem of (1.1)–(1.3). For the study of this problem we first list the following hypotheses.

(H1) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in E such that

$$|T(t)| \le M_1, \text{ for some } M_1 \ge 1, \quad \text{and} \quad |AT(t)| \le M_2,\ M_2 \ge 0,\ t \in J.$$

(H2) There exist constants $0 \le c_1 < 1$ and $c_2 \ge 0$ such that

$$|g(t, u)| \le c_1\|u\| + c_2, \qquad t \in J,\ u \in C([-r, 0], E).$$

(H3) There exist constants d_k such that $|I_k(y)| \le d_k$, $k = 1, \ldots, m$, for each $y \in E$.

(H4) $F: J \times C(J_0, E) \longrightarrow BCC(E)$; $(t, u) \longmapsto F(t, u)$ is measurable with respect to t for each $u \in C(J_0, E)$, u.s.c. with respect to u for each $t \in J$, and for each fixed $u \in C(J_0, E)$ the set

$$S_{F,u} = \left\{ g \in L^1(J, E) : \ g(t) \in F(t, u) \text{ for a.e. } \ t \in J \right\}$$

is nonempty.

(H5) $\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C([-r, 0], E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$\int_0^b \hat{m}(s)\, ds < \int_c^\infty \frac{d\tau}{\tau + \psi(\tau)},$$

where

$$c = \frac{1}{1 - c_1} \left\{ M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 M_2 b + c_2 + \sum_{k=1}^{m} d_k \right\}$$

and

$$\hat{m}(t) = \frac{1}{1 - c_1} \{M_2 c_1, M_1 p(t)\}.$$

(H6) The function g is completely continuous and for any bounded set $D \subseteq \Omega$ the set $\{t \to g(t, y_t) : y \in D\}$ is equicontinuous in Ω.

Remark 3.3. (i) If $\dim E < \infty$, then for each $u \in C([-r, 0], E)$ the set $S_{F,u}$ is nonempty (see Lasota and Opial [13]).

(ii) If $\dim E = \infty$ and $u \in C([-r, 0], E)$ the set $S_{F,u}$ is nonempty if and only if the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t, u)\}$$

belongs to $L^1(J, \mathbb{R})$ (see Hu and Papageorgiou [11]).

THEOREM 3.4. *Assume that the hypotheses* (H1)–(H6) *hold. Then the IVP* (1.1)–(1.3) *has at least one solution on* $[-r, b]$.

Proof. Transform the problem into a fixed-point problem. Consider the operator $N: \Omega \longrightarrow 2^\Omega$ defined by:

$$N(y) = \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{ll} \phi(t), & t \in [-r, 0] \\ T(t)[\phi(0) - g(0, \phi)] + g(t, y_t) & \\ \quad + \int_0^t AT(t - s) g(s, y_s)\, ds & \\ \quad + \int_0^t T(t - s) v(s)\, ds & \\ \quad + \sum_{0 < t_k < t} I_k(y(t_k^-)), & t \in J. \end{array} \right. \right\},$$

where $v \in S_{F,y} = \{v \in L^1(J, E) : v(t) \in F(t, y_t) \text{ for a.e. } t \in J\}$.

Remark 3.5. Clearly from Lemma 3.2 the fixed points of N are solutions to (1.1)–(1.3).

We shall show that N satisfies the assumptions of Lemma 2.2. Using (H6) it suffices to show that the operator $N_1 : \Omega \longrightarrow 2^{\Omega}$, defined by

$$N_1(y) = \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{ll} \phi(t), & t \in [-r, 0] \\ T(t)\phi(0) + \int_0^t AT(t-s)g(s, y_s)\, ds & \\ \quad + \int_0^t T(t-s)v(s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t \in J. \end{array} \right. \right\},$$

where $v \in S_{F,y}$, is u.s.c. condensing with bounded, closed, and convex values. The proof will be given in several steps.

Step 1. $N_1(y)$ is convex for each $y \in \Omega$.
This is obvious since $S_{F,y}$ is convex (because F has convex values).

Step 2. N_1 maps bounded sets into bounded sets in Ω.
Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in N_1(y)$, $y \in B_q = \{y \in \Omega : \|y\|_{\Omega} \le q\}$, one has $\|N_1(y)\|_{\Omega} \le \ell$.
If $h \in N_1(y)$, then there exists a $v \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = T(t)\phi(0) + \int_0^t AT(t-s)g(s, y_s)\, ds + \int_0^t T(t-s)v(s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)).$$

By (H1)–(H3) and (H5) we have for each $t \in J$,

$$|h(t)| \le M_1\|\phi\| + M_2 b(c_1 q + c_2) + M_1 \sup_{y\in[0,q]} \psi(y)\Big(\int_0^t p(s)ds\Big) + \sum_{k=1}^{m} \sup\{|I_k(|y|)| : \|y\|_{\Omega} \le q\}.$$

Then for each $h \in N(B_r)$ we have

$$\|h\|_{\Omega} \le M_1\|\phi\| + M_2 b(c_1 q + c_2) + M_1 \sup_{y\in[0,\, q]} \psi(y)\Big(\int_0^b p(s)\, ds\Big) + \sum_{k=1}^{m} \sup\{|I_k(|y|)| : \|y\|_{\Omega} \le q\}. := \ell.$$

Step 3. N_1 maps bounded sets into equicontinuous sets of Ω.
Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$; and $B_q = \{y \in \Omega : \|y\|_{\Omega} \le q\}$ be a bounded set of Ω.

For each $y \in B_q$ and $h \in N_1(y)$, there exists $v \in S_{F,y}$ such that

$$h(t) = T(t)\phi(0) + \int_0^t AT(t-s)g(s,y_s)\,ds + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \qquad t \in J.$$

Thus

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| &\le |T(\tau_2) - T(\tau_1)|\|\phi\| \\
&\quad + \left|\int_0^{\tau_2} [AT(\tau_2 - s) - AT(\tau_1 - s)]g(s,y_s)\,ds\right| \\
&\quad + \left|\int_{\tau_1}^{\tau_2} AT(\tau_1 - s)g(s,y_s)\,ds\right| \\
&\quad + \left\|\int_0^{\tau_2} [T(\tau_2 - s) - T(\tau_1 - s)]v(s)\,ds\right\| \\
&\quad + \left\|\int_{\tau_1}^{\tau_2} T(\tau_1 - s)v(s)\,ds\right\| + \sum_{0<t_k<\tau_2-\tau_1} d_k \\
&\le \int_0^{\tau_2} |A[T(\tau_2 - s) - T(\tau_1 - s)]|(c_1 q + c_2)\,ds \\
&\quad + \int_{\tau_1}^{\tau_2} |AT(\tau_1 - s)|(c_1 q + c_2)\,ds \\
&\quad + \int_0^{\tau_2} |T(\tau_2 - s) - T(\tau_1 - s)|M_1 \\
&\qquad \times \sup_{y\in[0,q]} \psi(y)\Big(\int_0^b p(s)\,ds\Big)\,ds \\
&\quad + \int_{\tau_1}^{\tau_2} |T(\tau_1 - s)|M_1 \sup_{y\in[0,q]} \psi(y)\Big(\int_0^t p(s)\,ds\Big)\,ds \\
&\quad + \sum_{0<t_k<\tau_2-\tau_1} d_k.
\end{aligned}$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero.

The equicontinuities for the cases $\tau_1 < \tau_2 \le 0$ and $\tau_1 \le 0 \le \tau_2$ are obvious.

As a consequence of Step 2, Step 3, and (H6) together with the Arzela–Ascoli Theorem we can conclude that $N : \Omega \longrightarrow 2^{\Omega}$ is a compact multivalued map and, therefore, a condensing multivalued map.

Step 4. N has a closed graph.

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that for each $t \in J$,

$$h_n(t) = T(t)[\phi(0) - g(0, \phi(0))] + g(t, y_{nt}) + \int_0^t AT(t-s)g(s, y_{ns})\, ds + \int_0^t T(t-s)v_n(s)ds + \sum_{0<t_k<t} I_k(y_n(t_k)).$$

We must prove that there exists $v_* \in S_{F, y_*}$ such that for each $t \in J$,

$$h_*(t) = T(t)[\phi(0) - g(0, \phi(0))] + g(t, y_{*t}) + \int_0^t AT(t-s)g(s, y_{*s})\, ds + \int_0^t T(t-s)v_*(s)ds + \sum_{0<t_k<t} I_k(y_*(t_k)).$$

Since the functions $g(t, \cdot)$, $t \in J$, I_k, $k = 1, \ldots, m$, are continuous we have that

$$\Big\| \Big(h_n - T(t)[\phi(0) - g(0, \phi(0))] - g(t, y_{nt}) - \int_0^t AT(t-s)g(s, y_{ns})ds - \sum_{0<t_k<t} I_k(y_n(t_k)) \Big) - \Big(h_* - T(t)[\phi(0) - g(0, \phi(0))] - g(t, y_{*t}) - \int_0^t AT(t-s)g(s, y_{*s})ds - \sum_{0<t_k<t} I_k(y_*(t_k)) \Big) \Big\|_\Omega \longrightarrow 0, \quad \text{as } n \to \infty.$$

Consider the linear continuous operator

$$\Gamma : L^1(J, E) \longrightarrow C(J, E),$$
$$v \longmapsto \Gamma(v)(t) = \int_0^t T(t-s)v(s)\, ds.$$

From Lemma 2.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - T(t)[\phi(0) - g(0, \phi(0))] - g(t, y_{nt}) - \int_0^t AT(t-s)g(s, y_{ns})\, ds - \sum_{0<t_k<t} I_k(y_n(t_k)) \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 2.1 that

$$h_*(t) - T(t)[\phi(0) - g(0, \phi(0))] - g(t, y_{*t}) - \int_0^t AT(t-s)g(s, y_{*s})\, ds - \sum_{0<t_k<t} I_k(y_*(t_k)) = \int_0^t T(t-s)v_*(s)ds$$

for some $v_* \in S_{F,y_*}$.

Step 5. Now it remains to show that the set

$$\mathcal{M} := \{y \in \Omega : \lambda y \in N(y), \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \mathcal{M}$. Then $y \in \lambda N(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$,

$$y(t) = \lambda^{-1} T(t)[\phi(0) - g(0, \phi(0))] + \lambda^{-1} g(t, y_t) + \lambda^{-1} \int_0^t A(t-s) g(s, y_s)\, ds$$
$$+ \lambda^{-1} \int_0^t T(t-s) v(s)\, ds + \lambda^{-1} \sum_{0<t_k<t} I_k(y(t_k)).$$

This implies by (H1)–(H3) and (H5) that for each $t \in J$ we have

$$|y(t)| \leq M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|y_t\| + c_2$$
$$+ M_2 c_1 \int_0^t \|y_s\|\, ds + M_2 c_2 b + \int_0^t p(s)\psi(\|y_s\|)\, ds + \sum_{k=1}^m d_k.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \qquad 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$ that

$$\mu(t) \leq M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2 c_1 \int_0^t \mu(s)\, ds$$
$$+ M_2 c_2 b + \int_0^t p(s)\psi(\mu(s))\, ds + \sum_{k=1}^m d_k.$$

Thus

$$\mu(t) \leq \frac{1}{1-c_1}\Bigg\{M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_2 M_2 b + c_2$$
$$+ M_2 c_1 \int_0^t \mu(s)\, ds + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^m d_k\Bigg\}.$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = \frac{1}{1-c_1}\left\{M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 M_2 b + c_2 + \sum_{k=1}^m d_k\right\},$$

$$\mu(t) \leq v(t), \qquad t \in J,$$

and

$$v'(t) = \frac{1}{1-c_1}[M_2 c_1 \mu(t) + p(t)\psi(\mu(t))], \qquad t \in J.$$

Using the nondecreasing character of ψ we get

$$\begin{aligned} v'(t) &\leq \frac{1}{1-c_1}[M_2 c_1 v(t) + p(t)\psi(v(t))] \\ &\leq \hat{m}(t)[v(t) + \psi(v(t)], \qquad t \in J. \end{aligned}$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\tau + \psi(\tau)} \leq \int_0^b \hat{m}(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}.$$

This inequality implies that there exists a constant K such that $v(t) \leq K,\ t \in J$, and hence $\mu(t) \leq K,\ t \in J$. Since for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\Omega \leq K' = \max\{\|\phi\|, K\},$$

where K' depends on b and on the functions p and ψ. This shows that $\Phi(N)$ is bounded.

Set $X := \Omega$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a solution of (1.1)–(1.3). ∎

4. SECOND ORDER IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

In this section we study the initial value problem (1.4)–(1.7) by using the theory of strongly continuous cosine and sine families.

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in E),

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$, and

(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \qquad y \in E,\ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books by Goldstein [7] and Fattorini [6] and to the papers by Travis and Webb [17, 18].

DEFINITION 4.1. A function $y \in C([-r, b], E)$ is said to be a mild solution of (1.4)–(1.7) if $y(t) = \phi(t)$ on $[-r, 0]$, $y'(0) = \eta$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$, $\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-))$, $k = 1, \dots, m$, and there exists a $v \in L^1(J, E)$ such that $v(t) \in F(t, y_t)$ a.e. on J, and

$$y(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s)g(s, y_s)\, ds + \int_0^t S(t-s)v(s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad t \in J.$$

a.e. on $J - \{t_1, \dots, t_m\}$.

We need the following auxiliary result. Its proof is very simple, so we omit it.

LEMMA 4.2. *$y \in \Omega^1$ is a mild solution of* (1.4)–(1.7), *if and only if $y \in \Omega$ is a solution of the impulsive integral equation*

$$y(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s)g(s, y_s)\, ds + \int_0^t S(t-s)v(s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad t \in J.$$

Assume the, following.

(A1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators from E into itself.

(A2) There exists constants c_1 and c_2 such that

$$|f(t, u)| \leq c_1\|u\| + c_2, \qquad t \in J, \quad u \in C(J_0, E).$$

(A3) There exist constants $d_k, \bar{d}_k$ such that $|I_k(y)| \leq d_k, |\overline{I}_k(y)| \leq \overline{d}_k$, $k = 1, \dots, m$, for each $y \in E$.

(A4) $F: J \times C(J_0, E) \longrightarrow BCC(E)$; $(t, u) \longmapsto F(t, u)$ is measurable with respect to t for each $u \in C(J_0, E)$ and u.s.c. with respect to u for each $t \in J$; and for each fixed $u \in C(J_0, E)$ the set

$$S_{F,u} = \left\{g \in L^1(J, E) : \ g(t) \in F(t, u) \text{ for a.e. } t \in J\right\}$$

is nonempty.

(A5) $\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$\int_0^b \hat{m}(s)\, ds < \int_c^\infty \frac{d\tau}{\tau + \psi(\tau)},$$

where

$$c = M\|\phi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + Mc_2 b + \sum_{k=1}^{m}[d_k + (b - t_k)\overline{d}_k],$$

$$M = \sup\{|C(t)| : t \in J\},$$

and

$$\hat{m}(t) = \max\{Mc_1, Mp(t)\}.$$

(A6) The function g is completely continuous and for any bounded set $D \subseteq C(J_1, E)$ the set $\{t \longrightarrow g(t, y_t) : y \in D\}$ is equicontinuous in $C(J, E)$.

(A7) $C(t), t \in J$, is completely continuous.

Now, we are in a position to state and prove our main theorem in this section.

THEOREM 4.3. *Assume that the hypotheses* (A1)–(A7) *hold. Then the IVP* (1.4)–(1.7) *has at least one mild solution on* $[-r, b]$.

Proof. Transform the problem into a fixed-point problem. In this setting, define the multivalued map $N : \Omega \longrightarrow 2^{\Omega}$ by

$$N(y) := \left\{ h \in C(J_1, E) : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \in J_0 \\ C(t)\phi(0) + S(t)[\eta - g(0, \phi)] & \\ \quad + \int_0^t C(t-s)g(s, y_s)\,ds & \\ \quad + \int_0^t S(t-s)v(s)\,ds & \\ \quad + \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y(t_k))], & \text{if } t \in J \end{array} \right. \right\},$$

where

$$v \in S_{F,y} = \left\{ v \in L^1(J, E) : v(t) \in F(t, y_t) \text{ for a.e. } t \in J \right\}.$$

Remark 4.4. It is clear that the fixed points of N are solutions to (1.1), (1.2).

We shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1. $N(y)$ is convex for each $y \in \Omega$.

This step is obvious since $S_{F,y}$ is convex (because F has convex values).

Step 2. N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in N(y)$, $y \in B_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$, one has $\|h\|_\Omega \leq \ell$.

If $h \in N(y)$, then there exists a $v \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s)f(s, y_s)\, ds + \int_0^t S(t-s)v(s)\, ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y(t_k))].$$

By (A2), (A3), and (A5) we have for each $t \in J$,

$$|h(t)| \leq Mq + bM(|\eta| + c_1 q + c_2) + Mb(c_1 q + c_2) + M \sup_{y\in[0,q]} \psi(y)\Big(\int_0^t p(s)\, ds\Big) + \sum_{k=1}^m [d_k + (T - t_k)\overline{d}_k].$$

Then for each $h \in N(B_q)$ we have

$$\|h\|_\Omega \leq Mq + Mb(|\eta| + c_1 q + c_2) + Mb(c_1 q + c_2) + M \sup_{y\in[0,q]} \psi(y)\Big(\int_0^b p(s)ds\Big) + \sum_{k=1}^m [d_k + (b - t_k)\overline{d}_k] := \ell.$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and B_q be a bounded set of Ω as in Step 2. For each $y \in B_q$ and $h \in N(y)$, there exists $v \in S_{F,y}$ such that

$$h(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s)g(s, y_s)\, ds + \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y(t_k))].$$

Thus

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| &\leq |C(\tau_2) - C(\tau_1)| + (|\eta| + c_1\|\phi\| + c_2)|S(\tau_2) - S(\tau_1)| \\ &\quad + \Big|\int_0^{\tau_2} [C(\tau_2 - s) - C(\tau_1 - s)]g(s, y_s)\, ds\Big| \\ &\quad + \Big|\int_{\tau_1}^{\tau_2} C(\tau_1 - s)g(s, y_s)\, ds\Big| \\ &\quad + \Big|\int_0^{t_2} [S(\tau_2 - s) - S(\tau_1 - s)]v(s)\, ds\Big| \\ &\quad + \Big|\int_{\tau_1}^{\tau_2} S(\tau_1 - s)v(s)\, ds\Big| + \sum_{0<t_k<\tau_2-\tau_1} [d_k + (T - t_k)\overline{d}_k] \\ &\leq |C(t_2) - C(\tau_1)| + (|\eta| + c_1 q + c_2)|S(\tau_2) - S(\tau_1)| \end{aligned}$$

$$\times \int_0^{\tau_2} |C(\tau_2 - s) - C(\tau_1 - s)]|(c_1 q + c_2)\, ds$$
$$+ \int_{\tau_1}^{\tau_2} |C(\tau_1 - s)|(c_1 q + c_2)\, ds$$
$$+ \int_0^{\tau_2} |S(\tau_2 - s) - S(\tau_1 - s)| M$$
$$\times \sup_{y \in [0,q]} \psi(y) \Big(\int_0^b p(s)\, ds \Big)\, ds$$
$$+ \int_{\tau_1}^{\tau_2} |S(t_1 - s)| M \sup_{y \in [0,q]} \psi(y) \Big(\int_0^b p(s)\, ds \Big)\, ds$$
$$+ \sum_{0 < t_k < \tau_2 - \tau_1} [d_k + (T - t_k)\overline{d}_k].$$

As $\tau_2 \longrightarrow \tau_1$, the right-hand side of the above inequality tends to zero.

The equicontinuities for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq t\tau_2$ are obvious.

As a consequence of Step 2, Step 3, and (A6), (A7), together with the Ascoli–Arzela Theorem we can conclude that $N : \Omega \longrightarrow 2^{\Omega}$ is a compact multivalued map and, therefore, a condensing map.

Step 4. N has a closed graph.

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F, y_n}$ such that

$$h_n(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s) g(t, y_{ns})\, ds$$
$$+ \int_0^t S(t-s) v_n(s)\, ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y_n(t_k))], \qquad t \in J.$$

We must prove that there exists $v_* \in S_{F, y_*}$ such that

$$h_*(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s) g(t, y_{*s})\, ds$$
$$+ \int_0^t S(t-s) v_*(s)\, ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y_*(t_k))], \qquad t \in J.$$

Since f is continuous we have that

$$\Big\| \Big(h_n - C(t)\phi(0) - S(t)[\eta - g(0, \phi)] - \int_0^t C(t-s) g(t, y_{ns})\, ds$$
$$- \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y_n(t_k))] \Big)$$

$$-\Big(h_* - C(t)\phi(0) - S(t)[\eta - g(0,\phi)] - \int_0^t C(t-s)g(t, y_{*s})\,ds$$
$$- \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y_*(t_k))]\Big)\Big\|_\Omega \to 0,$$

as $n \longrightarrow \infty$.

Consider the linear continuous operator

$$\Gamma : L^1(J, E) \longrightarrow C(J, E),$$
$$v \longmapsto \Gamma(v)(t) = \int_0^t S(t-s)v(s)\,ds.$$

From Lemma 2.2, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - C(t)\phi(0) - S(t)[\eta - g(0,\phi)] - \int_0^t C(t-s)g(t, y_{ns})\,ds$$
$$- \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y_n(t_k))] \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y^*$, it follows from Lemma 2.2 that

$$h_*(t) - C(t)\phi(0) - S(t)[\eta - g(0,\phi)] - \int_0^t C(t-s)g(t, y_{*s})ds$$
$$- \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y_*(t_k))] = \int_0^t S(t-s)v_*(s)\,ds,$$

for some $v_* \in S_{F,y^*}$.

Therefore N is a completely continuous multivalued map, u.s.c. with convex closed values. In order to prove that N has a fixed point, we need one more step.

Step 5. The set

$$\mathcal{M} := \{y \in \Omega : \lambda y \in N(y), \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $v \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}C(t)\phi(0) + \lambda^{-1}S(t)[\eta - g(0,\phi)]$$
$$+ \lambda^{-1}\int_0^t C(t-s)g(s, y_s)\,ds + \lambda^{-1}\int_0^t S(t-s)v(s)\,ds$$
$$+ \lambda^{-1}\sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y(t_k))], \qquad t \in J.$$

This implies by (A2), (A3), and (A5) that for each $t \in J$ we have

$$|y(t)| \leq M\|\phi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + M\int_0^t (c_1\|y_s\| + c_2)\,ds$$
$$+ M\int_0^t p(s)\psi(\|y_s\|)\,ds + \sum_{k=1}^{m}[d_k + (b - t_k)\overline{d}_k].$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \qquad 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$ that

$$\mu(t) \leq M\|\phi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + Mc_1\int_0^t \mu(s)ds + Mc_2 b$$
$$+ M\int_0^t p(s)\psi(\mu(s))\,ds + \sum_{k=1}^{m}[d_k + (b - t_k)\overline{d}_k].$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = M\|\phi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + Mc_2 b + \sum_{k=1}^{m}[d_k + (b - t_k)\overline{d}_k],$$
$$\mu(t) \leq v(t), \qquad t \in J,$$

and

$$\begin{aligned} v'(t) &= Mc_1\mu(t) + Mp(t)\psi(\mu(t)) \\ &\leq Mc_1 v(t) + Mp(t)\psi(v(t)) \\ &\leq \hat{m}(t)[v(t) + \psi(v(t))], \qquad t \in J. \end{aligned}$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\tau + \psi(\tau)} \leq \int_0^b \hat{m}(s)\,ds < \int_{v(0)}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}.$$

This inequality implies that there exists a constant L such that $v(t) \leq L$, $t \in J$, and hence $\mu(t) \leq L$, $t \in J$. Since for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\Omega \leq L' = \max\{\|\phi\|, L\},$$

where L' depends on b and on the functions p and ψ. This shows that Ω is bounded.

Set $X := \Omega$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a solution of (1.4)–(1.7). ∎

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