New fixed point theorems in Banach algebras under weak topology features and applications to nonlinear integral equations

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Abstract

We introduce a class of Banach algebras satisfying certain sequential condition (P) and we prove fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators. Later on, we give some examples of applications of these types of results to the existence of solutions of nonlinear integral equations in Banach algebras.
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1. Introduction

The study of functional integral equations and differential equations is the main object of research on nonlinear functional analysis. These equations occur in physical, biological and economic problems.

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Some of these equations can be formulated into nonlinear operators equations:

\[ x = AxBx + Cx \]  

(1.1)

in suitable Banach algebras.

In recent years, many authors have focused on the resolution of Eq. (1.1) and obtained a lot of valuable results (see for example [3–6,9,11–14,16] and the references therein). These studies were mainly based on the convexity of the bounded domain, the celebrate Schauder fixed point theorem [16] and properties of operators \( A, B \) and \( C \) (cf. completely continuous, k-set contractive, condensing and the potential tool of the axiomatic measures of non-compactness). Because the weak topology is the practice setting and natural to investigate the problems of existence of solutions of different types of nonlinear integral equations and nonlinear differential equations in Banach algebras, it turns out that the results mentioned above cannot be easily applied. One of difficulties arising when treating such situations, is that a bounded linear functional \( \varphi \) acting on a Banach algebra does not necessarily satisfy the following inequality:

\[ |\varphi(x.y)| \leq c|\varphi(x)||\varphi(y)|, \quad \text{with } c \geq 0 \text{ and } x, y \in X. \]

In the present paper, for this interest, we introduce a class of Banach algebras satisfying certain sequential conditions called here the condition \((P)\) (see Definition 3.1).

The main goal of this paper is to prove some new fixed point theorems in a nonempty closed convex subset of any Banach algebras or Banach algebras satisfying the condition \((P)\) under weak topology setting. Our main conditions are formulated in term of weak sequential continuity to the three nonlinear operators \( A, B \) and \( C \) involved in Eq. (1.1). Besides, no weak continuity conditions are required for this work.

Our main results are applied to investigate the existence of solutions for the two following nonlinear functional integral equations in Banach algebra \( C(J, X) \)

\[
\begin{align*}
x(t) &= a(t) + (T_1 x)(t) \left[ \left( q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) \, ds \right) u \right], \quad 0 < \lambda < 1, \\
x(t) &= a(t)x(t) + (T_2 x)(t) \left[ \left( q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) \, ds \right) u \right], \quad 0 < \lambda < 1,
\end{align*}
\]

where \( J \) is the interval \([0, 1]\) and \( X \) is any Banach algebra. The functions \( a, q, \sigma \) are continuous on \( J \); \( T_1, T_2, p(\ldots) \) are nonlinear functions and \( u \) is a nonvanishing vector.

The organization of this paper is as follows: in the next section, we give some definitions that will be needed in the sequel. Section 3 is devoted to the existence results for Eq. (1.1). So, we present some new fixed point theorems in Banach algebras. The main results of this section are Theorem 3.3 and Corollary 3.1. We end this section by discussing briefly the existence of positive solutions. In Section 4, we will apply the results obtained in precedent section to investigate the two FIE (4.1) and (4.2). The main result of this section is Theorems 4.1 and 4.2. Finally, we close this section by giving an application to resolve a particular example of functional differential equations.
2. Preliminaries

**Definition 2.1.** An algebra $\mathcal{E}$ is a vector space endowed with an internal composition law noted by $(.)$ i.e.,

$$
\begin{align*}
(\cdot) : \mathcal{E} \times \mathcal{E} &\to \mathcal{E}, \\
(x, y) &\to x.y
\end{align*}
$$

which is associative and bilinear.

A normed algebra is an algebra endowed with a norm satisfying the following property

$$
\text{for all } x, y \in \mathcal{E}; \quad \|x.y\| \leq \|x\| \|y\|.
$$

A complete normed algebra is called a Banach algebra.

**Definition 2.2.** Let $X$ be a Banach space. An operator $A : X \to X$ is said to be weakly compact if $A(B)$ is relatively weakly compact for every bounded subset $B \subset X$.

**Definition 2.3.** Let $X$ be a Banach space. An operator $A : X \to X$ is said to be sequentially weakly continuous on $X$ if for every sequence $\{x_n\}$ with $x_n \rightharpoonup x$, we have $Ax_n \rightharpoonup Ax$; here $\rightharpoonup$ denotes weak convergence.

**Definition 2.4.** Let $X$ be a Banach space. An operator $A : X \to X$ is said to be strongly continuous on $X$, if for every sequence $\{x_n\}$ with $x_n \to x$, we have $Ax_n \to Ax$; here $\to$ denotes convergence in $X$.

**Definition 2.5.** Let $X$ be a Banach space. A mapping $G : X \to X$ is called $D$-Lipschitzian if there exists a continuous and nondecreasing function $\phi_G : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$
\|Gx - Gy\| \leq \phi_G(\|x - y\|)
$$

for all $x, y \in X$, with $\phi_G(0) = 0$. Sometimes we call the function $\phi_G$ a $D$-function of $G$ on $X$. If $\phi_G(r) = kr$ for some $k > 0$, then $G$ is called a Lipschitzian function on $X$ with the Lipschitz constant $k$. Further if $k < 1$, then $G$ is called a contraction on $X$ with the contraction $k$.

**Remark 2.1.** Every Lipschitzian mapping is $D$-Lipschitzian, but the converse may not be true. If $\phi_G$ is not necessarily nondecreasing and satisfies $\phi_G(r) < r$, for $r > 0$, the mapping $G$ is called a nonlinear contraction with a contraction function $\phi_G$.

The following fixed points results stated in [2] will be used throughout the next section. The proof follows from O. Arino. S. Gautier and J.P. Penot theorem [1]

**Theorem 2.1.** (See [2, Theorem 2.5].) Let $X$ be a Banach space, $S$ be a nonempty closed convex subset of $X$ and $N : S \to S$ be a sequentially weakly continuous map. If $N(S)$ is relatively weakly compact, then $N$ has a fixed point in $S$. 
3. Fixed point theorems

Now, we are ready to state our first fixed point theorems in Banach algebras to provide the existence results of Eq. (1.1). First we have

**Theorem 3.1.** Let $\mathcal{E}$ be a Banach algebra and $S$ be a nonempty closed convex subset of $\mathcal{E}$. Let $A, C : \mathcal{E} \to \mathcal{E}$ and $B : S \to \mathcal{E}$ be three operators such that

(i) $(\frac{I-C}{A})^{-1}$ exists on $B(S)$.

(ii) $(\frac{I-C}{A})^{-1}B$ is sequentially weakly continuous.

(iii) $(\frac{I-C}{A})^{-1}B(S)$ is relatively weakly compact.

(iv) $x = AxBy + Cx \Rightarrow x \in S$, for all $y \in S$.

Then Eq. (1.1) has at least one solution in $S$.

**Remark 3.1.** Note that the assumption (iv) of above theorem is introduced by Burton [8] instead of assuming that $AxBy + Cx \in S$, for all $x, y \in S$.

**Proof of Theorem 3.1.** From assumption (i), it follows that for each $y$ in $S$, there is a unique $x_y \in \mathcal{E}$ such that

$$
(\frac{I-C}{A})x_y = By
$$

or, equivalently

$$
Ax_yBy + Cx_y = x_y.
$$

Since the hypothesis (iv) holds, then $x_y \in S$. Therefore, we can define

$$
\mathcal{N} : S \to S,
y \mapsto \mathcal{N}y = (\frac{I-C}{A})^{-1}By.
$$

By using the hypotheses (ii), (iii) and Theorem 2.1, we conclude that $\mathcal{N}$ has a fixed point $y$ in $S$. Hence, $y$ verifies Eq. (1.1) i.e.,

$$
AyBy + Cy = y.
$$

We also have

**Proposition 3.1.** Let $\mathcal{E}$ be a Banach algebra and $S$ be a nonempty closed convex subset of $\mathcal{E}$. Let $A, C : \mathcal{E} \to \mathcal{E}$ and $B : S \to \mathcal{E}$ be three operators such that

(i) $A$ and $C$ are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions $\phi_A$ and $\phi_C$ respectively,

(ii) $A$ is regular on $\mathcal{E}$, i.e., $A$ maps $\mathcal{E}$ into the set of all invertible elements of $\mathcal{E}$,

(iii) $B$ is a bounded function with bound $M$.

Then $(\frac{I-C}{A})^{-1}$ exists on $B(S)$ as soon as $M\phi_A(r) + \phi_C(r) < r$, for $r > 0$. 

Proof. Let \( y \) be fixed in \( S \) and define the mapping
\[
\varphi_y : \mathcal{E} \to \mathcal{E},
\]
\[
x \to \varphi_y(x) = AxBy +Cx.
\]

Let \( x_1, x_2 \in \mathcal{E} \), the use of the assumption (i) leads to
\[
\| \varphi_y(x_1) - \varphi_y(x_2) \| \leq \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\
\leq \|Ax_1 - Ax_2\| \|By\| + \|Cx_1 - Cx_2\| \\
\leq M\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|).
\]

Now, an application of a fixed point theorem of Boyd and Wong [7] yields that there is a unique element \( xy \in \mathcal{E} \) such that
\[
\varphi_y(xy) = xy.
\]

Hence, \( xy \) verifies Eq. (3.2) and so, by virtue of the hypothesis (ii), \( xy \) verifies Eq. (3.1). Therefore, the mapping \((\frac{I-C}{A})^{-1}\) is well defined on \( B(S) \) and \((\frac{I-C}{A})^{-1}By = xy\) and the desired result is deduced.

In what follows, we will combine Theorem 3.1 and Proposition 3.1 to obtain the following fixed point theorems in Banach algebras.

**Theorem 3.2.** Let \( \mathcal{E} \) be a Banach algebra and \( S \) be a nonempty closed convex subset of \( \mathcal{E} \). Let \( A, C : \mathcal{E} \to \mathcal{E} \) and \( B : S \to \mathcal{E} \) be three operators such that

(i) \( A \) and \( C \) are \( D \)-Lipschitzians with the \( D \)-functions \( \phi_A \) and \( \phi_C \) respectively,
(ii) \( A \) is regular on \( \mathcal{E} \),
(iii) \( B \) is strongly continuous,
(iv) \( B(S) \) is bounded with bound \( M \),
(v) \((\frac{I-C}{A})^{-1}\) is weakly compact on \( B(S) \),
(vi) \( x = AxBy + Cx \Rightarrow x \in S \), for all \( y \in S \).

Then Eq. (1.1) has at least one solution in \( S \) as soon as \( M\phi_A(r) + \phi_C(r) < r \), for all \( r > 0 \).

**Remark 3.2.** Here, we suppose \( B \) is strongly continuous and \( B(S) \) is bounded but not totally bounded. Thus, Theorem 1.1 in [12] follows as a consequence of Theorem 3.2.

**Proof of Theorem 3.2.** From Proposition 3.1, it follows that \((\frac{I-C}{A})^{-1}\) exists on \( B(S) \). By virtue of assumption (vi), we obtain
\[
(\frac{I-C}{A})^{-1}B(S) \subset S.
\]
Moreover, the use of hypotheses (iv) and (v) leads that \((I - C_A)^{-1} B(S)\) is relatively weakly compact. Now, we show that \((I - C_A)^{-1} B(S)\) is sequentially weakly continuous. To see this, let \(\{u_n\}\) be any sequence in \(S\) such that \(u_n \rightharpoonup u\) in \(S\). By virtue of assumption (iii), we have

\[
Bu_n \to Bu.
\]

Since \((I - C_A)^{-1}\) is a continuous mapping on \(B(S)\) (see [14, Theorem 2.4]), we deduce that

\[
(I - C_A)^{-1} B u_n \to (I - C_A)^{-1} Bu.
\]

This shows that \((I - C_A)^{-1} B(S)\) is sequentially weakly continuous. Finally, an application of Theorem 3.1 yields that Eq. (1.1) has a solution in \(S\).

**Theorem 3.3.** Let \(S\) be a nonempty closed convex subset of a Banach algebra \(E\). Let \(A, C : E \to E\) and \(B : S \to E\) be three operators such that

(i) \(A\) and \(C\) are \(D\)-Lipschitzians with the \(D\)-functions \(\phi_A\) and \(\phi_C\) respectively,
(ii) \(B\) is sequentially weakly continuous and \(B(S)\) is relatively weakly compact,
(iii) \(A\) is regular on \(E\),
(iv) \((I - C_A)^{-1}\) is sequentially weakly continuous on \(B(S)\),
(v) \(x = AxBy + Cy \Rightarrow x \in S\), for all \(y \in S\).

Then Eq. (1.1) has at least one solution in \(S\) as soon as \(M\phi_A(r) + \phi_C(r) < r\), for all \(r > 0\).

**Remark 3.3.** Recently, some fixed point theorems involving three operators in Banach algebras were established for completely continuous maps. Because every totally bounded subset of \(X\) is relatively weakly compact, Theorem 2.1 in [12] follows as a sequence of Theorem 3.3. Further, in general, the continuity condition is not easy to verify. In Theorem 3.3, the continuity is not required.

**Proof of Theorem 3.3.** Similarly to the proof of preceding Theorem 3.2, we obtain \((I - C_A)^{-1}\) exists on \(B(S)\) and

\[
(I - C_A)^{-1} B(S) \subset S.
\]

Since \((I - C_A)^{-1}\) and \(B\) are sequentially weakly continuous, so, by composition we have \((I - C_A)^{-1} B\) is sequentially weakly continuous. Finally, we claim that \((I - C_A)^{-1} B(S)\) is relatively weakly compact. To see this, let \(\{u_n\}\) be any sequence in \(S\) and let

\[
v_n = (I - C_A)^{-1} Bu_n.
\]
Since $B(S)$ is relatively weakly compact, there is a renamed subsequence \( \{Bu_n\} \) weakly converging to an element \( w \). This fact, together with hypothesis (iv) gives that

\[
v_n = \left( \frac{I - CA}{A} \right)^{-1} Bu_n \rightharpoonup \left( \frac{I - CA}{A} \right)^{-1} w.
\]

We infer that \( \left( \frac{I - CA}{A} \right)^{-1} B(S) \) is sequentially relatively weakly compact. An application of the Eberlein–Šmulian theorem [10] yields that \( \left( \frac{I - CA}{A} \right)^{-1} B(S) \) is relatively weakly compact, which leads our claim. The result is concluded immediately from Theorem 3.1. \( \square \)

Because the product of two sequentially weakly continuous functions is not necessarily sequentially weakly continuous, we will introduce:

**Definition 3.1.** We will say that the Banach algebra \( \mathcal{E} \) satisfies condition \((\mathcal{P})\) if

\[
(\mathcal{P}) \quad \begin{cases} 
\text{For any sequences } \{x_n\} \text{ and } \{y_n\} \text{ in } \mathcal{E} \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y, \\
\text{then } x_n y_n \rightharpoonup xy; \text{ here } \rightharpoonup \text{ denotes weak convergence.}
\end{cases}
\]

Note that, every finite dimensional Banach algebra satisfies condition \((\mathcal{P})\). Even, if \( X \) satisfies condition \((\mathcal{P})\) then \( C(K, X) \) is also Banach algebra satisfying condition \((\mathcal{P})\), where \( K \) is a compact Hausdorff space. The proof is based on Dobrokov’s theorem:

**Theorem 3.4.** (See [15, Dobrakov, p. 36].) Let \( K \) be a compact Hausdorff space and \( X \) be a Banach space. Let \( (f_n)_n \) be a bounded sequence in \( C(K, X) \), and \( f \in C(K, X) \).

Then \( (f_n)_n \) is weakly convergent to \( f \) if and only if \( (f_n(t))_n \) is weakly convergent to \( f(t) \) for each \( t \in K \).

**Theorem 3.5.** Let \( \mathcal{E} \) be a Banach algebra satisfying condition \((\mathcal{P})\). Let \( S \) be a nonempty closed convex subset of \( \mathcal{E} \). Let \( A, C : \mathcal{E} \to \mathcal{E} \) and \( B : S \to \mathcal{E} \) be three operators such that

(i) \( A \) and \( C \) are \( \mathcal{D} \)-Lipschitzians with the \( \mathcal{D} \)-functions \( \phi_A \) and \( \phi_C \) respectively,
(ii) \( A \) is regular on \( \mathcal{E} \),
(iii) \( A, B \) and \( C \) are sequentially weakly continuous on \( S \),
(iv) \( B(S) \) is bounded with bound \( M \),
(v) \( \left( \frac{I - CA}{A} \right)^{-1} \) is weakly compact on \( B(S) \),
(vi) \( x = Ax By + Cx \Rightarrow x \in S \), for all \( y \in S \).

Then Eq. (1.1) has at least one solution in \( S \) as soon as \( M\phi_A(r) + \phi_C(r) < r \), for all \( r > 0 \).

**Proof.** Similarly to the proof of Theorem 3.2, we obtain \( \left( \frac{I - CA}{A} \right)^{-1} \) exists on \( B(S) \),

\[
\left( \frac{I - CA}{A} \right)^{-1} B(S) \subset S
\]
and \((\frac{I-C}{A})^{-1} B(S)\) is relatively weakly compact. In view of Theorem 3.1, it suffices to establish that \((\frac{I-C}{A})^{-1} B\) is sequentially weakly continuous. To see this, let \(\{u_n\}\) be a weakly convergent sequence of \(S\) to a point \(u\) in \(S\). Now, define the sequence \(\{v_n\}\) of the subset \(S\) by

\[ v_n = \left(\frac{I-C}{A}\right)^{-1} B u_n. \]

Since \((\frac{I-C}{A})^{-1} B(S)\) is relatively weakly compact, so, there is a renamed subsequence such that

\[ v_n = \left(\frac{I-C}{A}\right)^{-1} B u_n \rightharpoonup v. \]

But, on the other hand, the subsequence \(\{v_n\}\) verifies

\[ v_n - C v_n = A v_n B u_n. \]

Therefore, from assumption (iii) and in view of condition \((P)\), we deduce that \(v\) verifies the following equation

\[ v - C v = A v B u, \]

or, equivalently

\[ v = \left(\frac{I-C}{A}\right)^{-1} B u. \]

Next we claim that the whole sequence \(\{u_n\}\) verifies

\[ \left(\frac{I-C}{A}\right)^{-1} B u_n = v_n \rightharpoonup v. \]

Indeed, suppose that this is not the case, so, there is \(V^w\) a weakly neighborhood of \(v\) satisfying for all \(n \in \mathbb{N}\), there exists an \(N \geq n\) such that \(v_N \notin V^w\). Hence, there is a renamed subsequence \(\{v_n\}\) verifying the property

\[ \text{for all } n \in \mathbb{N}, \quad v_n \notin V^w. \quad (3.3) \]

However

\[ \text{for all } n \in \mathbb{N}, \quad v_n \in \left(\frac{I-C}{A}\right)^{-1} B(S). \]

Again, there is a renamed subsequence such that

\[ v_n \rightharpoonup v'. \]
According to the preceding, we have

\[ v' = \left( \frac{I - C}{A} \right)^{-1} Bu, \]

and, consequently

\[ v = v', \]

which is a contradiction with the property (3.3). This yields that \( \left( \frac{I - C}{A} \right)^{-1} B \) is sequentially weakly continuous. \( \Box \)

An interesting corollary of Theorem 3.5 is

**Corollary 3.1.** Let \( E \) be a Banach algebra satisfying condition \((P)\) and let \( S \) be a nonempty closed convex subset of \( E \). Let \( A, C : E \to E \) and \( B : S \to E \) be three operators such that

(i) \( A \) and \( C \) are \( \mathcal{D} \)-Lipschitzians with the \( \mathcal{D} \)-functions \( \phi_A \) and \( \phi_C \) respectively,

(ii) \( A \) is regular on \( E \),

(iii) \( A, B \) and \( C \) are sequentially weakly continuous on \( S \),

(iv) \( A(S), B(S) \) and \( C(S) \) are relatively weakly compacts,

(v) \( x = AxBy + Cy \Rightarrow x \in S, \) for all \( y \in S \).

Then Eq. (1.1) has at least one solution in \( S \) as soon as \( M\phi_A(r) + \phi_C(r) < r \), for all \( r > 0 \).

**Proof.** In view of Theorem 3.5, it is enough to prove that \( \left( \frac{I - C}{A} \right)^{-1} B(S) \) is relatively weakly compact. To do this, let \( \{u_n\} \) be any sequence in \( S \) and let

\[ v_n = \left( \frac{I - C}{A} \right)^{-1} Bu_n. \quad (3.4) \]

Since \( B(S) \) is relatively weakly compact, there is a renamed subsequence \( \{Bu_n\} \) weakly converging to an element \( w \). On the other hand, by Eq. (3.4), we obtain

\[ v_n = Av_nBu_n + Cv_n. \quad (3.5) \]

Since \( \{v_n\} \) is a sequence in \( S \), so, by assumption (iv), there is a renamed subsequence such that \( Av_n \to x \) and \( Cv_n \to y \). Hence, in view of condition \((P)\) and the last equation (3.5), we obtain

\[ v_n \to xw + y. \]

This shows that \( \left( \frac{I - C}{A} \right)^{-1} B(S) \) is sequentially relatively weakly compact. An application result of the Eberlein–Šmulian theorem [10] yields that \( \left( \frac{I - C}{A} \right)^{-1} B(S) \) is relatively weakly compact. \( \Box \)

Now, we shall discuss briefly on the existence of positive solutions. Let \( E_1 \) and \( E_2 \) be two Banach algebras, with positive closed cones \( E_1^+ \) and \( E_2^+ \), respectively. An operator \( G \) from \( E_1 \) into \( E_2 \) is said to be positive if it carries the positive cone \( E_1^+ \) into \( E_2^+ \) (i.e., \( G(E_1^+) \subseteq E_2^+ \)).
Theorem 3.6. Let \( E \) be a Banach algebra satisfying condition (\( P \)) and \( S \) be a nonempty closed convex subset of \( E \) such that \( S^+ = S \cap E^+ \neq \emptyset \). Let \( A, C : E \to E \) and \( B : S \to E \) be three operators such that

(i) \( A \) and \( C \) are \( D \)-Lipschitzians with the \( D \)-functions \( \phi_A \) and \( \phi_C \) respectively,
(ii) \( A \) is regular on \( E \),
(iii) \( A, B \) and \( C \) are sequentially weakly continuous on \( S^+ \),
(iv) \( A(S^+), B(S^+) \) and \( C(S^+) \) are relatively weakly compacts,
(v) \( x = AxBy + Cx \Rightarrow x \in S^+, \) for all \( y \in S^+ \).

Then Eq. (1.1) has at least one solution in \( S^+ \) as soon as \( M^+ \phi_A(r) + \phi_C(r) < r \), for all \( r > 0 \), where \( M^+ = \|B(S^+)\| \).

Proof. Obviously \( S^+ = S \cap E^+ \) is a closed convex subset of \( E \). From Proposition 3.1, it follows that \( (\frac{I - C}{A})^{-1} \) exists on \( B(S^+) \). By virtue of assumption (v), we have

\[
\left( \frac{I - C}{A} \right)^{-1} B(S^+) \subset S^+.
\]

Then we can define the mapping:

\[
\mathcal{N} : S^+ \to S^+ \quad \text{by} \quad y \to \mathcal{N}y = \left( \frac{I - C}{A} \right)^{-1} By.
\]

Now, an application of Corollary 3.1 yields that \( \mathcal{N} \) has a fixed point in \( S^+ \). As a result, by the definition of \( \mathcal{N} \), Eq. (1.1) has a solution in \( S^+ \). \( \Box \)

4. Functional integral equations

In this section we illustrate the applicability of our Corollary 3.1 and Theorem 3.3 by considering the following examples of nonlinear functional integral equations.

4.1. Example

Let \( (X, \|\|) \) be a Banach algebra satisfying condition (\( P \)). Let \( J = [0, 1] \) the closed and bounded interval in \( \mathbb{R} \), the set of all real numbers. Let \( \mathcal{E} = C(J,X) \) the Banach algebra of all continuous functions from \([0, 1]\) to \( X \), endowed with the sup-norm \( \|\|_\infty \), defined by \( \|f\|_\infty = \sup\{\|f(t)\| : t \in [0, 1]\}, \) for each \( f \in C(J,X) \). We consider the nonlinear functional integral equation (in short, FIE):

\[
x(t) = a(t) + \left( T_1 x \right)(t) \left[ \left( q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s)) \, ds \right) u \right], \quad 0 < \lambda < 1, \quad (4.1)
\]

for all \( t \in J \), where \( u \neq 0 \) is a fixed vector of \( X \) and the functions \( a, q, \sigma, p, T_1 \) are given, while \( x = x(t) \) is an unknown function.
We shall obtain the solution of FIE (4.1) under some suitable conditions. Suppose that the functions involved in Eq. (4.1) verify the following conditions:

(H_1) a : J \rightarrow X is a continuous function.
(H_2) \sigma : J \rightarrow J is a continuous and nondecreasing function.
(H_3) q : J \rightarrow \mathbb{R} is a continuous function.
(H_4) The operator \( T_1 : C(J, X) \rightarrow C(J, X) \) is such that
(a) \( T_1 \) is Lipschitzian with a Lipschitzian constant \( \alpha \),
(b) \( T_1 \) is regular on \( C(J, X) \),
(c) \( T_1 \) is sequentially weakly continuous on \( C(J, X) \),
(d) \( T_1 \) is weakly compact.
(H_5) There exists \( r_0 > 0 \) such that
(a) \( |p(t, s, x, y)| \leq r_0 - \|q\|_\infty \) for each \( t, s \in J; x, y \in X \) such that \( \|x\| \leq r_0 \) and \( \|y\| \leq r_0 \),
(b) \( \|T_1 x\|_\infty \leq \left(1 - \frac{\|u\|_\infty}{r_0}\right) \frac{1}{\|u\|} \) for each \( x \in C(J, X) \),
(c) \( \alpha r_0 \|u\| < 1 \).

**Theorem 4.1.** Under assumptions (H_1)–(H_6), Eq. (4.1) has at least one solution \( x = x(t) \) which belongs to the space \( C(J, X) \).

**Remark 4.1.** When \( X \) is infinite dimensional, the subset \( A_{r_0} = \{ x \in X; \|x\| \leq r_0 \} \) is not compact. So, the restriction \( p \) on \( J \times J \times A_{r_0} \times A_{r_0} \) is not uniformly continuous. Thus, we note that the operator \( B \) in Eq. (4.1) is not necessarily continuous on \( S \).

**Proof of Theorem 4.1.** First, we begin by showing that \( C(J, X) \) verifies condition \( (\mathcal{P}) \). To see this, let \( \{x_n\}, \{y_n\} \) any sequences in \( C(J, X) \) such that \( x_n \rightharpoonup x \) and \( y_n \rightharpoonup y \). So, for each \( t \in J \), we have \( x_n(t) \rightharpoonup x(t) \) and \( y_n(t) \rightharpoonup y(t) \) (cf. Theorem 3.4). Since \( X \) verify condition \( (\mathcal{P}) \), then

\[
x_n(t)y_n(t) \rightharpoonup x(t)y(t),
\]

because \( (x_ny_n)_n \) is a bounded sequence [18], this, further, implies that

\[
x_ny_n \rightharpoonup xy \quad (\text{cf. Theorem 3.4}),
\]

which shows that the space \( C(J, X) \) verifies condition \( (\mathcal{P}) \).

Let us define the subset \( S \) of \( C(J, X) \) by

\[
S := \{ y \in C(J, X), \|y\|_\infty \leq r_0 \} = B_{r_0}.
\]

Obviously \( S \) is nonempty, convex and closed. Let us consider three operators \( A, B \) and \( C \) defined on \( C(J, X) \) by

\[
(Ax)(t) = (T_1x)(t),
\]
\[ (Bx)(t) = \left[ q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) \, ds \right] u, \quad 0 < \lambda < 1, \]
\[ (Cx)(t) = a(t). \]

We shall prove that the operators \( A, B \) and \( C \) satisfy all the conditions of Corollary 3.1.
(i) From assumption \((H_4)(a)\), it follows that \( A \) is Lipschitzian with a Lipschitzian constant \( \alpha \). Clearly \( C \) is Lipschitzian with a Lipschitzian constant 0.
(ii) From assumption \((H_4)(b)\), it follows that \( A \) is regular on \( C(J, X) \).
(iii) Since \( C \) is constant, so, \( C \) is sequentially weakly continuous on \( S \). From assumption \((H_4)(c)\), \( A \) is sequentially weakly continuous on \( S \). Now, we show that \( B \) is sequentially weakly continuous on \( S \). Firstly, we verify that if \( x \in S \), then \( Bx \in C(J, X) \). Let \( \{t_n\} \) be any sequence in \( J \) converging to a point \( t \) in \( J \). Then

\[
\| (Bx)(t_n) - (Bx)(t) \| = \left\| \left[ \int_0^{\sigma(t_n)} p(t_n, s, x(s), x(\lambda s)) \, ds \right. \right. \\
- \left. \left. \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) \, ds \right] u \right\| \\
\leq \left[ \int_0^{\sigma(t_n)} \left| p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s)) \right| \, ds \right] \| u \| \\
+ \left( r_0 - \| q \|_{\infty} \right) \| \sigma(t_n) - \sigma(t) \| \| u \|.
\]

Since \( t_n \to t \), so, \((t_n, s, x(s), x(\lambda s)) \to (t, s, x(s), x(\lambda s))\), for all \( s \in J \). Taking into account the hypothesis \((H_5)\), we obtain

\[ p(t_n, s, x(s), x(\lambda s)) \to p(t, s, x(s), x(\lambda s)) \text{ in } \mathbb{R}. \]

Moreover, the use of assumption \((H_6)\) leads to

\[ \left| p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s)) \right| \leq 2(r_0 - \| q \|_{\infty}) \]

for all \( t, s \in J, \lambda \in (0, 1) \). Consider

\[
\begin{cases}
\varphi : J \to \mathbb{R}, \\
s \to \varphi(s) = 2(r_0 - \| q \|_{\infty}).
\end{cases}
\]
Clearly $\varphi \in L^1(J)$. Therefore, from the dominated convergence theorem and assumption $(H_2)$, we obtain

$$(Bx)(t_n) \to (Bx)(t) \text{ in } X.$$ 

It follows that

$$Bx \in C(J, X).$$

Next, we prove $B$ is sequentially weakly continuous on $S$. Let $\{x_n\}$ be any sequence in $S$ weakly converging to a point $x$ in $S$. So, from assumptions $(H_5)$–$(H_6)$ and the dominated convergence theorem, we get

$$\lim_{n \to \infty} \int_0^1 p(t, s, x_n(s), x_n(\lambda s)) \, ds = \int_0^1 p(t, s, x(s), x(\lambda s)) \, ds,$$

which implies

$$\lim_{n \to \infty} \left( q(t) + \int_0^1 p(t, s, x_n(s), x_n(\lambda s)) \, ds \right) \cdot u = \left( q(t) + \int_0^1 p(t, s, x(s), x(\lambda s)) \, ds \right) \cdot u.$$

Hence,

$$(Bx_n)(t) \to (Bx)(t) \text{ in } X.$$ 

Since $(Bx_n)_n$ is bounded by $r_0\|u\|$, then

$$Bx_n \rightharpoonup Bx \quad (\text{cf. Theorem 3.4}).$$

We conclude that $B$ is sequentially weakly continuous on $S$.

(iv) We will prove that $A(S)$, $B(S)$ and $C(S)$ are relatively weakly compact. Since $S$ is bounded by $r_0$ and taking into account the hypothesis $(H_4)(d)$, it follows that $A(S)$ is relatively weakly compact. Now, we show $B(S)$ is relatively weakly compact.

(Step I) By definition,

$$B(S) := \{ B(x), \|x\|_\infty \leq r_0 \}.$$ 

For all $t \in J$, we have

$$B(S)(t) = \{ (Bx)(t), \|x\|_\infty \leq r_0 \}.$$ 

We claim that $B(S)(t)$ is sequentially weakly relatively compact in $X$. To see this, let $\{x_n\}$ be any sequence in $S$, we have $$(Bx_n)(t) = r_n(t) \cdot u,$$ where $r_n(t) = q(t) + \int_0^1 p(t, s, x_n(s), x_n(\lambda s)) \, ds$. Since $|r_n(t)| \leq r_0$ and $(r_n(t))$ is a real sequence, so, there is a renamed subsequence such that

$$r_n(t) \to r(t) \text{ in } \mathbb{R},$$
which implies
\[ r_n(t).u \rightarrow r(t).u \quad \text{in } X, \]
and, consequently
\[ (Bx_n)(t) \rightarrow (q(t) + r(t)).u \quad \text{in } X. \]

We conclude that \( B(S)(t) \) is sequentially relatively compact in \( X \), then \( B(S)(t) \) is sequentially relatively weakly compact in \( X \).

(Step II) We prove that \( B(S) \) is weakly equicontinuous on \( J \). If we take \( \epsilon > 0 \); \( x \in S; x^* \in X^* \); \( t, t' \in J \) such that \( t \leq t' \) and \( t' - t \leq \epsilon \). Then
\[
\|x^*((Bx)(t) - (Bx)(t'))\| \leq \left[ \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s)) \, ds - \int_0^{\sigma(t')} p(t',s,x(s),x(\lambda s)) \, ds \right] \|x^*(u)\|
\]
\[
\leq [w(p,\epsilon) + (r_0 - \|q\|_{\infty})w(\sigma, \epsilon)] \|x^*(u)\|,
\]
where
\[
w(p, \epsilon) = \sup \left\{ \left| p(t,s,x,y) - p(t',s,x,y) \right| : t, t', s \in J; |t - t'| \leq \epsilon; x,y \in B_{r_0} \right\},
\]
\[
w(\sigma, \epsilon) = \sup \left\{ |\sigma(t) - \sigma(t')| : t, t' \in J; |t - t'| \leq \epsilon \right\}.
\]

Taking into account the hypothesis \((H_5)\) and in view of the uniform continuity of the function \( \sigma \) on the set \( J \), it follows that \( w(p, \epsilon) \rightarrow 0 \) and \( w(\sigma, \epsilon) \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). An application of the Arzelà–Ascoli theorem [17], we conclude that \( B(S) \) is sequentially weakly relatively compact in \( X \). Again an application result of Eberlein–Šmulian theorem [10] yields that \( B(S) \) is relatively weakly compact. As \( C(S) = \{a\} \), hence \( C(S) \) is relatively weakly compact.

(v) Finally, it remains to prove the hypothesis (v) of Corollary 3.1. To see this, let \( x \in C(J, X) \) and \( y \in S \) such that
\[
x = AxBy + Cx,
\]
or, equivalently for all \( t \in J \),
\[
x(t) = a(t) + (T_1 x)(t)(By)(t).
\]
But, for all \( t \in J \), we have

\[
\|x(t)\| \leq \|x(t) - a(t)\| + \|a(t)\|.
\]

Then

\[
\|x(t)\| \leq \|(T_1x)(t)\| r_0 \|u\| + \|a\|_\infty \\
\leq \left(1 - \frac{\|a\|_\infty}{r_0}\right) r_0 + \|a\|_\infty \\
= r_0.
\]

From the last inequality and taking the supremum over \( t \), we obtain

\[
\|x\|_\infty \leq r_0,
\]

and, consequently \( x \in S \).

We conclude that the operators \( A, B \) and \( C \) satisfy all the requirements of Corollary 3.1. Thus, an application of it yields that the FIE (4.1) has a solution in the space \( C(J, X) \).

\[\square\]

4.2. Example

To illustrate Theorem 3.3, We consider the nonlinear functional integral equation (in short, FIE) in \( C(J, X) \).

\[
x(t) = a(t)x(t) + (T_2x)(t) \left[ \left( q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) \, ds \right) u \right], \quad 0 < \lambda < 1, \quad (4.2)
\]

for all \( t \in J \), where \( u \neq 0 \) is a fixed vector of \( X \) and the functions \( a, q, \sigma, p, T_2 \) are given, while \( x \) in \( C(J, X) \) is an unknown function.

We shall obtain the solution of (FIE (4.2)) under some suitable conditions on the functions involved in (4.2). Suppose that the functions \( a, q, \sigma, p \) and the operator \( T_2 \) verify the following conditions:

\begin{enumerate}
  \item[(H_1)] \( a : J \to X \) is a continuous function with \( \|a\|_\infty < 1 \).
  \item[(H_2)] \( \sigma : J \to J \) is a continuous and nondecreasing function.
  \item[(H_3)] \( q : J \to \mathbb{R} \) is a continuous function.
  \item[(H_4)] The operator \( T_2 : C(J, X) \to C(J, X) \) is such that
    \begin{enumerate}
      \item[(a)] \( T_2 \) is Lipschitzian with a Lipschitzian constant \( \alpha \),
      \item[(b)] \( T_2 \) is regular on \( C(J, X) \),
      \item[(c)] \( \left( \frac{T_2}{\alpha} \right)^{-1} \) is well defined on \( C(J, X) \),
      \item[(d)] \( \left( \frac{T_2}{\alpha} \right)^{-1} \) is sequentially weakly continuous on \( C(J, X) \).
    \end{enumerate}
  \item[(H_5)] The function \( p : J \times J \times X \times X \to \mathbb{R} \) is continuous such that for arbitrary fixed \( s \in J \) and \( x, y \in X \), the partial function \( t \to p(t, s, x, y) \) is continuous uniformly for \( (s, x, y) \in J \times X \times X \).
  \item[(H_6)] There exists \( r_0 > 0 \) such that
(a) \(|p(t, s, x, y)| \leq r_0 - \|q\|_{\infty}\) for each \(t, s \in J; x, y \in X\) such that \(\|x\| \leq r_0\) and \(\|y\| \leq r_0\),
(b) \(\|T_2 x\|_{\infty} \leq (1 - \frac{\|a\|_{\infty}}{r_0}) \frac{1}{\|u\|}\) for each \(x \in \mathcal{C}(J, X)\),
(c) \(\alpha r_0 \|u\| < 1\).

**Theorem 4.2.** Under assumptions \((H_1)-(H_6)\), Eq. (4.2) has at least one solution \(x = x(t)\) which belongs to the space \(\mathcal{C}(J, X)\).

**Remark 4.2.** Note that the operator \(C\) in Eq. (4.2) does not satisfy condition (iv) of Corollary 3.1. In fact, if we take \(X = \mathbb{R}\) and \(a \equiv \frac{1}{2}\), then \((Cx)(t) = \frac{1}{2}x(t)\). Thus

\[
C(S) = \left\{ \frac{1}{2}x / \|x\|_{\infty} \leq r_0 \right\},
\]

\[
= B_{r_0}^2.
\]

Because \(C(J, \mathbb{R})\) is with infinite dimensional, \(C(S)\) is not relatively compact. Furthermore, \(\mathbb{R}\) is with finite dimensional, so, \(C(S)\) is not relatively weakly compact [17].

**Proof of Theorem 4.2.** Let us consider three operators \(A, B\) and \(C\) defined on \(C(J, X)\) by

\[
(Ax)(t) = (T_2 x)(t),
\]

\[
(Bx)(t) = \left[ q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) \, ds \right] u, \quad 0 < \lambda < 1,
\]

\[
(Cx)(t) = a(t)x(t).
\]

We shall prove that the operators \(A, B\) and \(C\) satisfy all the conditions of Theorem 3.3.

(i) From assumption \((H_4)(a)\), \(A\) is Lipschitzian with a Lipschitzian constant \(\alpha\). Next, we show that \(C\) is Lipschitzian on \(C(J, X)\). To see this, fix arbitrarily \(x, y \in C(J, X)\). Then, if we take an arbitrary \(t \in J\), we get

\[
\| (Cx)(t) - (Cy)(t) \| = \| a(t)x(t) - a(t)y(t) \|
\]

\[
\leq \| a \|_{\infty} \| x(t) - y(t) \|.
\]

From the last inequality and taking the supremum over \(t\), we obtain

\[
\|Cx - Cy\|_{\infty} \leq \|a\|_{\infty} \|x - y\|_{\infty}.
\]

This proves that \(C\) is Lipschitzian with a Lipschitzian constant \(\|a\|_{\infty}\).

(ii) Arguing as in the proof of Theorem 4.1, we obtain \(B\) is sequentially weakly continuous on \(S\) and \(B(S)\) is relatively weakly compact.

(iii) From assumption \((H_4)(b)\), \(A\) is regular on \(C(J, X)\).

(iv) We show that \((I - C)A^{-1}\) is sequentially weakly continuous on \(B(S)\). To see this, let \(x, y \in C(J, X)\) such that

\[
\left( \frac{I - C}{A} \right)(x) = y,
\]
or, equivalently

\[
\frac{(1 - a)x}{T_2x} = y.
\]

Since \(\|a\|_\infty < 1\), so, \((1 - a)^{-1}\) exists on \(C(J, X)\), then

\[
\left(\frac{I}{T_2}\right)(x) = (1 - a)^{-1}y.
\]

This implies, from assumption \((H_3)(c)\), that

\[
x = \left(\frac{I}{T_2}\right)^{-1}((1 - a)^{-1}y).
\]

Thus

\[
\left(\frac{I - C}{A}\right)^{-1}(x) = \left(\frac{I}{T_2}\right)^{-1}((1 - a)^{-1}x)
\]

for all \(x \in C(J, X)\). Now, let \(\{x_n\}\) be a weakly convergent sequence of \(B(S)\) to a point \(x\) in \(B(S)\), then

\[
(1 - a)^{-1}x_n \rightharpoonup (1 - a)^{-1}x,
\]

and so, it follows from assumption \((H_4)(d)\) that

\[
\left(\frac{I}{T_2}\right)^{-1}((1 - a)^{-1}x_n) \rightharpoonup \left(\frac{I}{T_2}\right)^{-1}((1 - a)^{-1}x),
\]

we conclude that

\[
\left(\frac{I - C}{A}\right)^{-1}(x_n) \rightharpoonup \left(\frac{I - C}{A}\right)^{-1}(x).
\]

(v) Finally, a similar reasoning as, in the last point of Theorem 4.1, proves that the condition (v) of Theorem 3.3 is fulfilled.

We conclude that the operators \(A, B\) and \(C\) satisfy all the requirements of Theorem 3.3.

**Corollary 4.1.** Let \((X, \|\|)\) be a Banach algebra satisfying condition \((P)\), with positive closed cone \(X^+\). Suppose that the assumption \((H_1)\)–\((H_6)\) hold. Also, assume that

\(u\) belongs to \(X^+\), \(a(J) \subset X^+\), \(q(J) \subset \mathbb{R}_+\), \(p(J \times J \times X^+ \times X^+) \subset \mathbb{R}_+\) and \(\left(\frac{I}{T_2}\right)^{-1}\) is a positive operator from the cone positive \(C(J, X^+)\) of \(C(J, X)\) into itself.

Then Eq. (4.2) has at least one positive solution \(x\) in the cone \(C(J, X^+)\).
Proof. Let

\[ S^+ := \{ x \in S, \ x(t) \in X^+ \text{ for all } t \in J \}. \]

Obviously \( S^+ \) is nonempty, closed and convex. Similarly to the proof of Theorem 4.2, we show that

(i) \( A \) and \( C \) are Lipschitzians with a Lipschitzian constant \( \alpha \) and \( \|a\|_\infty \) respectively.
(ii) \( A \) is regular on \( \mathcal{C}(J, X) \).
(iii) \( A, B \) and \( C \) are sequentially weakly continuous on \( S^+ \).
(iv) Because \( S^+ \) is a subset of \( S \), so, we have \( A(S^+), B(S^+) \) and \( C(S^+) \) are relatively weakly compacts.
(v) Finally, we shall show that the hypothesis (v) of Theorem 3.6 is satisfied. In fact, fix an arbitrarily \( x \in \mathcal{C}(J, X) \) and \( y \in S^+ \) such that

\[ x = AxBy + Cx. \]

Arguing as in the proof of Theorem 4.2, we get \( x \in S \). Moreover, the last equation leads that

for all \( t \in J \), \( x(t) = a(t)x(t) + (T_2 x)(t)(By)(t) \),

thus,

\[ \text{for all } t \in J, \quad \frac{x(t)(1-a(t))}{(T_2 x)(t)} = (By)(t). \]

Since for all \( t \in J \), \( \|a(t)\| < 1 \), so, \( (1-a(t))^{-1} \) exists in \( X \), and

\[ (1-a(t))^{-1} = \sum_{n=0}^{+\infty} a^n(t). \]

Since \( a(t) \) belongs to the closed positive cone \( X^+ \), then \( (1-a(t))^{-1} \) is positive. Also, we verify that for all \( t \in J \), \( (By)(t) \) is a positive. Therefore, the map \( \psi \) defined on \( J \) by

\[ \psi(t) = (1-a(t))^{-1} \left[ q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s)) \ ds \right].u \]

belongs to the positive cone \( \mathcal{C}(J, X^+) \) of \( \mathcal{C}(J, X) \). Then \( B \) maps \( \mathcal{C}(J, X^+) \) into itself. Seeing that

\[ \left( \left( \frac{I}{T_2} \right) x \right)(t) = (1-a(t))^{-1} \left[ q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s)) \ ds \right].u = \psi(t), \]
then
\[ x = \left( \frac{I}{T_2} \right)^{-1}(\psi). \]

Thus, \( x \in C(J, X^+) \) and, consequently \( x \in S^+ \). \( \square \)

Next, we provide an example of the operator \( T_2 \) presented in Theorem 4.2.

**Example of the operator \( T_2 \) in \( C(J, \mathbb{R}) \).** Let \( \mathcal{E} = C(J, \mathbb{R}) = C(J) \) denotes the Banach algebra of all continuous real-valued functions on \( J \) with norm \( \|x\|_\infty = \sup_{t \in J} |x(t)| \). Clearly \( C(J) \) satisfies condition \((\mathcal{P})\). Let \( b : J \to \mathbb{R} \) is continuous and nonnegative, and define
\[
\begin{cases}
T_2 : C(J) \to C(J), \\
\quad x \to T_2x = \frac{1}{1 + b|x|}.
\end{cases}
\]

We obtain the following functional integral equation:
\[
x(t) = a(t)x(t) + \frac{1}{1 + b(t)|x(t)|} \left[ q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s)) \, ds \right], \quad 0 < \lambda < 1. \tag{4.3}
\]

We will prove all the conditions (a)–(d) of \((H_4)\) in Theorem 4.2 to Eq. (4.3):
(a) Fix \( x, y \in C(J) \). Then, for all \( t \in J \), we have
\[
\left| (T_2x)(t) - (T_2y)(t) \right| = \left| \frac{1}{1 + b(t)|x(t)|} - \frac{1}{1 + b(t)|y(t)|} \right| = \frac{b(t)||y(t)| - |x(t)||}{(1 + b(t)|x(t)|(1 + b(t)|y(t)|)} \leq \|b\|_\infty |x(t) - y(t)|.
\]

Taking the supremum over \( t \), we obtain
\[
\|T_2x - T_2y\|_\infty \leq \|b\|_\infty \|x - y\|_\infty.
\]

Which shows that \( T_2 \) is Lipschitzian with a Lipschitzian constant \( \|b\|_\infty \).
(b) Clearly \( T_2 \) is regular on \( C(J) \).
(c) We show that \( \left( \frac{1}{T_2} \right)^{-1} \) exists on \( C(J) \). To see this, let \( x, y \in C(J) \) such that
\[
\left( \frac{I}{T_2} \right)x = y,
\]
or, equivalently
\[
x(1 + b|x|) = y,
\]
which implies

$$|x|(1 + b|x|) = |y|,$$

hence

$$\left( \sqrt{b}|x| \right)^2 + |x| = |y|.$$

For each $t_0 \in J$ such that $b(t_0) = 0$, we have $x = y$. Then for each $t \in J$ such that $b(t) > 0$, we obtain

$$\left( \sqrt{b(t)}|x(t)| + \frac{1}{2\sqrt{b(t)}} \right)^2 = \frac{1}{4b(t)} + |y(t)|,$$

which further implies

$$\sqrt{b(t)}|x(t)| = -\frac{1}{2} + \sqrt{\frac{1}{4} + |y(t)|} + b(t)|x(t)|,$$

hence

$$b(t)|x(t)| = -\frac{1}{2} + \sqrt{\frac{1}{4} + |y(t)|} + b(t),$$

and, consequently

$$x(t) = \frac{y(t)}{1 + b(t)|x(t)|} = \frac{y(t)}{\frac{1}{2} + \sqrt{\frac{1}{4} + b(t)}}.$$

We remark that the equality is also verified for each $t$ such that $b(t) = 0$.

Consider $F$ the function defined by the expression

$$F : C(J) \rightarrow C(J), \quad x \mapsto F(x) = \frac{x}{\frac{1}{2} + \sqrt{\frac{1}{4} + b|x|}}.$$

It is easily to verify for all $x \in C(J)$

$$\left( \left( \frac{I}{T_2} \right) \circ F \right)(x) = \left( F \circ \left( \frac{I}{T_2} \right) \right)(x) = x.$$

We conclude that

$$\left( \frac{I}{T_2} \right)^{-1} = \frac{x}{\frac{1}{2} + \sqrt{\frac{1}{4} + b|x|}}.$$

(d) It is an easy exercise to show that $T_2$ and $\left( \frac{I}{T_2} \right)^{-1}$ is sequentially weakly continuous on $B(S)$. 
Remark 4.3. One can check easily that \((I_{T_2})^{-1}\) is a positive operator from the positive cone \(C(J, \mathbb{R}_+)\) of \(C(J, \mathbb{R})\) into itself.

4.3. Applications to functional differential equations

In this section, we prove the existence theorems for the nonlinear functional differential equation in Banach algebra by the applications of the abstract results of the previous section under generalized Lipschitz conditions.

We consider the following nonlinear functional differential equation (in short, FDE) in \(C(J)\)

\[
\left( \frac{x}{T_2x} - q_1 \right)'(t) = \int_0^t \frac{\partial p}{\partial t}(t,s,x(s),x(\lambda s))\,ds + p(t,t,x(t),x(\lambda t)),
\]

\(t \in J, 0 < \lambda < 1,\) 

satisfying the initial condition

\[
x(0) = \xi \in \mathbb{R}
\]

where the functions \(q_1, p\) and the operator \(T_2\) are given with \(q_1(0) = 0\), while \(x = x(t)\) is an unknown function. By a solution of the FDE (4.4)–(4.5), we mean an absolutely continuous function \(x : J \to \mathbb{R}\) that satisfies the two equations (4.4)–(4.5) on \(J\). The existence result for the FDE (4.4)–(4.5) is:

**Theorem 4.3.** We consider the following assumptions:

(H1) The \(q_1 : J \to \mathbb{R}\) is a continuous function.

(H2) The operator \(T_2 : C(J) \to C(J)\) is such that

(a) \(T_2\) is Lipschitzian with a Lipschitzian constant \(\alpha\),

(b) \(T_2\) is regular on \(C(J)\),

(c) \((I_{T_2})^{-1}\) is well defined on \(C(J)\),

(d) \((I_{T_2})^{-1}\) is sequentially weakly continuous on \(C(J)\),

(e) For all \(x \in C(J)\), we have \(\|T_2x\|_{\infty} \leq 1\).

(H3) The function \(p : J \times J \times \mathbb{R}^2 \to \mathbb{R}\) is continuous such that for arbitrary fixed \(s \in J\) and \(x, y \in \mathbb{R}\); the partial function \(t \mapsto p(t,s,x,y)\) is \(C^1\) on \(J\).

(H4) There exists \(r_0 > 0\) such that

(a) For all \(t, s \in J, y, z \in [-r_0, r_0]\) and \(x \in C(J)\), we have

\[
|p(t,s,y,z)| \leq r_0 - \|q_1\|_{\infty} - \frac{\|\xi\|}{|(T_2x)(0)|}.
\]

(b) \(\alpha r_0 < 1\).

Then the FDE (4.4)–(4.5) has at least one solution in \(C(J)\).
**Proof.** Note that the FDE (4.4)–(4.5) is equivalent to the integral functional equation:

\[
x(t) = (T_2x)(t) \left[ q_1(t) + \frac{\zeta}{(T_2x)(0)} + \int_0^t p(t, s, x(s), x(\lambda s)) \, ds \right], \quad t \in J, \quad 0 < \lambda < 1.
\]  

Eq. (4.6) represents a particular case of Eq. (4.2) with for all \( t \in J; \) \( \sigma(t) = t, a(t) = 0, u = 1 \) and \( q(t) = q_1(t) + \frac{\zeta}{(T_2x)(0)} \). Therefore, we have for all \( t \in J; \) \( (Ax)(t) = (T_2x)(t) \), \( (Bx)(t) = q(t) + \int_0^t p(t, s, x(s), x(\lambda s)) \, ds \) and \( C(x)(t) = 0 \).

Now, we shall prove that the operators \( A, B \) and \( C \) satisfy all the conditions of Theorem 3.3. Similarly to the proof of preceding Theorem 4.2, we obtain

(i) \( A \) and \( C \) are Lipschitzians with a Lipschitzian constant \( \alpha \) and 0 respectively.

(ii) \( B \) is sequentially weakly continuous on \( S \) and \( B(S) \) is relatively weakly compact where \( S = B_{r_0} := \{ x \in C(J), \| x \|_\infty \leq r_0 \} \).

(iii) \( A \) is regular on \( C(J) \).

(iv) \( (\frac{T - C}{A})^{-1} = (\frac{T_2}{T_2})^{-1} \) is sequentially weakly continuous on \( B(S) \).

It, thus, remains to prove (v) of Theorem 3.3. First, we show that \( M = \| B(S) \| \leq r_0 \). To see this, fix an arbitrarily \( x \in S \). Then, for \( t \in J \), we get

\[
(Bx)(t) \leq q_1(t) + \frac{\zeta}{(T_2x)(0)} + \int_0^t \left| p(t, s, x(s), x(\lambda s)) \right| \, ds
\]

\[
\leq q_1(t) + \frac{\zeta}{(T_2x)(0)} + \int_0^1 \left| p(t, s, x(s), x(\lambda s)) \right| \, ds
\]

\[
\leq \| q_1 \|_\infty + \frac{\zeta}{(T_2x)(0)} + r_0 - \| q_1 \|_\infty - \frac{\zeta}{(T_2x)(0)}
\]

\[
= r_0.
\]

Taking the supremum over \( t \), we obtain

\[
\| Bx \|_\infty \leq r_0.
\]

Thus

\[
M \leq r_0,
\]

and, consequently

\[
\alpha M + \beta = \alpha r_0 < 1.
\]

Next, fix an arbitrarily \( x \in C(J) \) and \( y \in S \) such that

\[
x = AxBy + Cx,
\]
or, equivalently

for all $t \in J$, \[ x(t) = (T_2x)(t)(By)(t), \]

then

\[ |x(t)| \leq \|T_2x\|_{\infty} \|By\|_{\infty}, \]

and thus, in view of assumption $(H_2)(e)$, yields that

\[ |x(t)| \leq \|By\|_{\infty}. \]

Since $y \in S$, this further implies

\[ |x(t)| \leq r_0, \]

taking the supremum over $t$, we obtain

\[ \|x\|_{\infty} \leq r_0. \]

As a result, $x$ is in $S$. This proves (v). Now, applying Theorem 3.3, we see that Eq. (4.6) has at least one solution in $C(J)$. \( \square \)

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