Journal of Algebra 342 (2011) 212–234

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Toric varieties and spherical embeddings over an arbitrary field

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ARTICLE INFO

Article history:
Received 15 April 2011
Available online 13 July 2011
Communicated by Peter Littelmann

Keywords:
Toric variety
Spherical embedding

ABSTRACT

We are interested in two classes of varieties with group action, namely toric varieties and spherical embeddings. They are classified by combinatorial objects, called fans in the toric setting, and colored fans in the spherical setting. We characterize those combinatorial objects corresponding to varieties defined over an arbitrary field k. Then we provide some situations where toric varieties over k are classified by Galois-stable fans, and spherical embeddings over k by Galois-stable colored fans. Moreover, we construct an example of a smooth toric variety under a 3-dimensional nonsplit torus over k whose fan is Galois-stable but which admits no k-form. In the spherical setting, we offer an example of a spherical homogeneous space X_0 over $\mathbb{R}$ of rank 2 under the action of $SU(2,1)$ and a smooth embedding of X_0 whose fan is Galois-stable but which admits no $\mathbb{R}$-form.

Introduction

In the early 70's, Demazure [Dem70] gave a full classification of smooth toric varieties under a split torus in terms of combinatorial objects which he named fans. This classification was then extended to all toric varieties under a split torus during the next decade (see for example [Dan78]).

In the first part of this paper, we address the classification problem for a nonsplit torus T over a field k. Let K be a Galois extension of k which splits T. Then the Galois group $\mathrm{Gal}(K|k)$ acts on fans corresponding to toric varieties under $T_K = T \times_k K$, and one can speak of Galois-stable fans. The classification Theorem 1.22 says, roughly speaking, that toric varieties under a nonsplit torus T are classified by Galois-stable fans satisfying an additional condition, named (ii). For a quasi-projective fan (see Proposition 1.9) condition (ii) holds. If the torus T is of dimension 2 then every fan is

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.05.031

quasi-projective, and then condition (ii) holds. If T is split by a quadratic extension, condition (ii) is also automatically satisfied, by a result of Włodarczyk [Wło93] which asserts that any two points in a toric variety (under a split torus) lie on a common affine open subset. In these situations, toric varieties under T are thus classified by Galois-stable fans (Theorem 1.25). Condition (ii) is nonetheless necessary, as we construct an example of a three-dimensional torus T over a field k, split by an extension K of k of degree 3, and a smooth toric variety under the split torus T_K with Galois-stable fan but which is not defined over k (Theorem 1.31). This provides an example of a toric variety (under a split torus) containing three points which do not lie on a common affine open subset.

Recently, Elizondo, Lima-Filho, Sottile and Teitler also studied toric varieties under nonsplit tori [ELST10]. They also obtain the statement of Theorem 1.25, but do not address the problem of descent in general. Using Galois cohomology, they are able to classify toric k-forms of $\mathbb{P}^n_K$ if the extension K of k is cyclic and, toric smooth surfaces in general.

In the second part of this paper, we address the classification problem for embeddings of spherical homogeneous spaces. A homogeneous space (X_0, x_0) under a connected reductive group G over k is called spherical if there is a Borel subgroup B of $G_{\bar{k}}$ with Bx_0 open in $X_0(\bar{k})$. An embedding of X_0 is a normal G-variety over k containing X_0 as an open orbit. The main difference with the toric case is the base point, introduced in order to kill automorphisms.

The classification of spherical embeddings was obtained by Luna and Vust [LV83] when k is algebraically closed of characteristic 0, and extended by Knop [Kno91] to all characteristics. The classifying objects, called colored fans, are also of combinatorial nature.

In Section 2.2, we show that the Galois group $\mathrm{Gal}(\bar{k}|k)$ acts on those colored fans, so that we can speak of Galois-stable colored fans. The main classification theorem is Theorem 2.26; like Theorem 1.22 it asserts that the embeddings of X_0 are classified by Galois-stable fans satisfying an additional assumption, named (ii). We provide some situations where this condition (ii) is automatically satisfied (including the split case, which is not a part of the Luna–Vust theory), and an example of a homogeneous space X_0 over $\mathbb{R}$, under the action of $SU(2,1)$, and an embedding X of $X_{0,\mathbb{C}}$ with Galois-stable colored fan but which is not defined over $\mathbb{R}$. This gives an example of a smooth spherical variety containing two points which do not lie on a common affine subset.

A motivation for studying embeddings of spherical homogeneous spaces is to construct equivariant smooth compactifications of them. In the toric case, this construction is due to Colliot-Thélène, Harari and Skorobogatov [CTHS05].

I'd like to thank M. Brion for his precious advice about that work and his careful reading. I'd also like to thank E.J. Elizondo, P. Lima-Filho, F. Sottile and Z. Teitler for communicating their work to me.

1. Classification of toric varieties over an arbitrary field

Let k be a field and $\bar{k}$ a fixed algebraic closure. We denote by T a torus defined over k. By a variety over k we mean a separated geometrically integral scheme of finite type over k. We define toric varieties under the action of T in the following way:

Definition 1.1. A toric variety over k under the action of T is a normal T-variety X such that the group $T(\bar{k})$ has an open orbit in $X(\bar{k})$ in which it acts with trivial isotropy subgroup scheme. A morphism between toric varieties under the action of T is a T-equivariant morphism defined over k.

It follows from the definition that a toric variety X under the action of T contains a principal homogeneous space under T as a T-stable open subset. We will denote it by X_0.

Definition 1.2. We will say that X is split if X_0 is isomorphic to T, that is to say, if X_0 has a k-point.

Remark 1.3. If X is a split toric variety, then the automorphism group of X is the group $T(k)$.

In the rest of this section, we classify the toric varieties under the action of T (up to isomorphism). Assuming first that the torus T is split, we recall how the classification works in terms of combinatorial data named fans (Section 1.1). In Section 1.2, we derive the general case from the split case.

We show (Theorem 1.22) that toric varieties under the action of T are, roughly speaking, classified by Galois-stable fans satisfying an additional assumption. In Section 1.3, we provide some situations where this additional assumption is always satisfied, and an example where it is not.

1.1. The split case

In this section, we assume that the torus T is split, and give the classification of toric varieties under the action of T. This classification was obtained by Demazure in the case of smooth toric varieties, and by many other people in the general case. See [Dan78] for more details and proofs.

Proposition 1.4. *Every toric variety under the action of T is split.*

Proof. By Hilbert's 90 theorem, every principal homogeneous space under T has a k-point. □

In order to state the main theorem of this section, we need more notations and definitions.

Notation 1.5. We denote by M the character lattice of T, and by N the lattice of one parameter subgroups, which is dual to M. We call V the $\mathbb{Q}$-vector space $N \otimes_{\mathbb{Z}} \mathbb{Q}$, and $V^* = M \otimes_{\mathbb{Z}} \mathbb{Q}$ its dual.

By a cone in V, we mean the $\mathbb{Q}^+$-linear span of finitely many elements of V. We say that a cone is strictly convex if it contains no line.

Notation 1.6. If $\mathcal{C}$ is a cone in V, we denote by $\mathcal{C}^\vee \subseteq V^*$ its dual cone, and by $\mathrm{Int}(\mathcal{C})$ its relative interior.

Recall the classical:

Definition 1.7. A fan in V is a finite collection $\mathcal{E}$ of strictly convex cones in V satisfying:

- Every face of $\mathcal{C} \in \mathcal{E}$ belongs to $\mathcal{E}$.
- The intersection of two cones in $\mathcal{E}$ is a face of each.

Let us now recall how a fan can be associated to a toric variety under the action of T. Fix a toric variety X under T. If $x \in X(\bar{k})$, the orbit of x under $T(\bar{k})$ is actually defined over k. For simplicity, we will denote by Tx this orbit. If ω is a T-orbit, the open subset

$$X_\omega := \left\{x \in X(\bar{k}),\ \omega \subseteq \overline{Tx}\right\}$$

is affine and defined over k. Fix $x \in X_0(k)$. One can show that the subset

$$\left\{\lambda \in N,\ \lim_{t \to 0} \lambda(t)x \text{ exists in } X \text{ and belongs to} X_\omega\right\}$$

of N is a finitely generated monoid whose $\mathbb{Q}^+$-linear span $\mathcal{C}_\omega$ is a strictly convex cone in V. Observe that the cone $\mathcal{C}_\omega$ does not depend on the point $x \in X_0(k)$.

Theorem 1.8. *By mapping a toric variety X to the collection*

$$\{\mathcal{C}_\omega,\ \omega \subseteq X \text{ is a } T\text{-orbit}\},$$

one gets a bijection between (isomorphism classes of) toric varieties under T and fans in V. We will denote by $\mathcal{E}_X$ the fan associated to the toric variety X, and by $X_{\mathcal{E}}$ the toric variety associated to the fan $\mathcal{E}$.

To construct a toric variety under T out of a fan $\mathcal{E}$ in V, one proceeds as follows. If $\mathcal{E}$ contains only one maximal cone $\mathcal{C}$, then the variety $X_{\mathcal{E}}$ is

$$X_{\mathcal{E}} = \operatorname{Spec}\bigl(k\bigl[\mathcal{C}^{\vee} \cap M\bigr]\bigr).$$

Observe that $X_{\mathcal{E}}$ is affine. In the general case, one glues the toric varieties $(X_{\mathcal{C}})_{\mathcal{C}\in\mathcal{E}}$ along their intersections: $X_{\mathcal{C}} \cap X_{\mathcal{C}'} = X_{\mathcal{C}\cap\mathcal{C}'}$.

The following proposition enables us to detect the quasi-projectivity of a toric variety by looking at its fan:

Proposition 1.9. *Let $\mathcal{E}$ be a fan in V. The variety $X_{\mathcal{E}}$ is quasi-projective if and only if there exists a family of linear forms $(l_{\mathcal{C}})_{\mathcal{C}\in\mathcal{E}}$ on V satisfying the following conditions*:

- $\forall \mathcal{C}, \mathcal{C}' \in \mathcal{E}$, $l_{\mathcal{C}} = l_{\mathcal{C}'}$ *over* $\mathcal{C} \cap \mathcal{C}'$.
- $\forall \mathcal{C}, \mathcal{C}' \in \mathcal{E}$, $\forall x \in \operatorname{Int}(\mathcal{C})$, $l_{\mathcal{C}}(x) > l_{\mathcal{C}'}(x)$.

In this situation, we say that the fan $\mathcal{E}$ is quasi-projective.

Remark 1.10. Every two-dimensional fan is quasi-projective.

Remark 1.11. If the fan $\mathcal{E}$ has one maximal cone, then it is quasi-projective because $X_{\mathcal{E}}$ is affine. If $\mathcal{E}$ has two maximal cones, then it is also quasi-projective. Indeed, let $l \in V^*$ be positive on the first maximal cone $\mathcal{C}_1$, and negative on the second $\mathcal{C}_2$. Putting $l_{\mathcal{C}_1} = l$ and $l_{\mathcal{C}_2} = 0$ one gets the result.

1.2. Forms of a split toric variety

In this section, we go back to the general setting (T is not necessarily split). We fix a finite Galois extension K of k with Galois group Γ such that the torus T_K is split. The notations $M, N, \ldots$ will refer to the corresponding objects associated to T_K in Section 1.1. These objects are equipped with an action of the group Γ.

Fix a toric variety X under T_K. We address the following problem:

Question 1.12. Does X admit a k-form?

By a k-form of X, we mean a toric variety Y under the action of T, such that $Y_K \simeq X$ as T_K-varieties. Let F_X be the set of isomorphism classes of k-forms of X. We denote by Ω the open orbit of T_K in X, and define F_{Ω} similarly. By definition, F_{Ω} is the set of principal homogeneous spaces under T which become trivial under T_K. By mapping a k-form of X to the principal homogeneous space that it contains, one obtains a natural map

$$\delta_X : F_X \to F_{\Omega}.$$

Question 1.13. What can be said about the map δ_X?

Theorem 1.22 will give an answer to Questions 1.12 and 1.13. We will obtain a criterion involving the fan $\mathcal{E}_X$ for the set F_X to be nonempty, and show that if this criterion is satisfied, the map δ_X is a bijection. As usual in Galois descent issues, semi-linear actions on X respecting the ambient structure turn out to be very helpful.

Definition 1.14. An action of Γ on X is called toric semi-linear if for every $\sigma \in \Gamma$ the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \xrightarrow{\ \sigma\ } & \mathrm{Spec}(K) \end{array}, \qquad \begin{array}{ccc} T_K \times X & \longrightarrow & X \\ \downarrow{\scriptstyle (\sigma,\sigma)} & & \downarrow{\scriptstyle \sigma} \\ T_K \times X & \longrightarrow & X \end{array}$$

are commutative. The group $\mathrm{Aut}_K(X) = T(K)$ acts by conjugacy on the whole set of toric semi-linear actions of Γ on X. We denote by E_X the set of conjugacy classes of toric semi-linear actions of Γ on X.

If Y is a k-form of X and if one fixes a T_K-equivariant isomorphism $X \to Y_K$, then one can let the group Γ act on X. Replacing the isomorphism $X \to Y_K$ by another one, one obtains a $T(K)$-conjugated toric semi-linear action. This proves that there is a natural map

$$\alpha_X : F_X \to E_X.$$

The following proposition is part of the folklore and can be found for example in [ELST10].

Proposition 1.15. *The map α_X is injective. A toric semi-linear action on X is in the image of α_X if and only if the quotient X/Γ exists, or, in other words, if and only if one can cover X by Γ-stable quasi-projective subsets.*

Remark 1.16. If the variety X itself is quasi-projective, the map α_X is thus bijective.

Proposition 1.17. *The map α_Ω is bijective. Otherwise stated, a principal homogeneous space under T which becomes trivial under T_K is characterized by the toric semi-linear action it induces on Ω.*

Proof. Use Proposition 1.15 and observe that Ω is affine. □

Remark 1.18. The set F_Ω is naturally the Galois cohomology set $H^1(\Gamma, T(K))$.

The following proposition (also obtained in [ELST10]) enables us to see very easily whether the set E_X is empty or not, by looking at the fan $\mathcal{E}_X$.

Proposition 1.19. *The set E_X is nonempty if and only if the fan $\mathcal{E}_X$ is Γ-stable in the sense that, for every cone $\mathcal{C} \in \mathcal{E}_X$, and for every $\sigma \in \Gamma$, the cone $\sigma(\mathcal{C})$ still belongs to $\mathcal{E}_X$. In this case, the restriction map $E_X \to E_\Omega$ is a bijection.*

Remark 1.20. The open orbit Ω is easily seen to be Γ-stable for any toric semi-linear action of Γ on X. By mapping such an action to its restriction to Ω one gets what we call the restriction map $E_X \to E_\Omega$.

Proof of Proposition 1.19. Assume first that the set E_X is nonempty. The variety X is thus endowed with a toric semi-linear action of Γ. Let ω be an orbit of T_K on X, and σ be an element of Γ. Let x be a K-point in Ω (it exists because T_K is split). One has

$$\sigma(\mathcal{C}_\omega) \cap N = \Big\{ \sigma(\lambda),\ \lambda \in N,\ \lim_{t \to 0} \lambda(t)x \text{ exists in } X \text{ and belongs to } X_\omega \Big\}.$$

Thus,

$$\sigma(\mathcal{C}_\omega) \cap N = \Big\{ \lambda \in N,\ \lim_{t \to 0} \lambda(t)\sigma(x) \text{ exists in } X \text{ and belongs to } X_{\sigma(\omega)} \Big\}.$$

But the K-point $\sigma(x)$ belongs to Ω, showing that $\sigma(\mathcal{C}_\omega) = \mathcal{C}_{\sigma(\omega)}$. This proves that $\mathcal{E}_X$ is stable under the action of Γ.

Assume now that $\mathcal{E}_X$ is Γ-stable. Let $\mathcal{C}$ be a cone in $\mathcal{E}$ and σ an element of Γ. The linear map σ on M gives a morphism of monoids

$$\sigma(\mathcal{C})^\vee \cap M \to \mathcal{C}^\vee \cap M$$

and then a semi-linear morphism of K-algebras

$$K\big[\sigma(\mathcal{C})^\vee \cap M\big] \to K\big[\mathcal{C}^\vee \cap M\big],$$

inducing a morphism of varieties

$$U_{\mathcal{C}} \to U_{\sigma(\mathcal{C})}$$

which respects the toric structures on both sides. These morphisms patch together, and enable us to construct the desired toric semi-linear action on X. This completes the proof of the first point.

Suppose from now on that the set E_X is nonempty, and fix a toric semi-linear action of Γ on X. Denote by $*$ a toric semi-linear action of Γ on Ω. Then, for all $\sigma \in \Gamma$, the morphism

$$\Omega \to \Omega,$$
$$x \mapsto \sigma^{-1} * \big(\sigma(x)\big)$$

is a toric automorphism of Ω, that is, the multiplication by an element of $T(K)$. But such a multiplication extends to X, proving that the semi-linear action $*$ of Γ extends to X. In other words, the restriction map is surjective. But it is also injective because Ω is open in X. This completes the proof of the proposition. □

Remark 1.21. By the previous arguments, one sees that if $\mathcal{E}$ is Γ-stable, if ω is an orbit of T_K in X, and σ an element of Γ, the notation $\sigma(\omega)$ makes sense, and does not depend on the chosen toric semi-linear action of Γ on X. Moreover, one sees that for every toric semi-linear action of Γ on X, and for every cone $\mathcal{C} \in \mathcal{E}$, $\sigma(X_{\mathcal{C}}) = X_{\sigma(\mathcal{C})}$.

We are now able to answer Questions 1.12 and 1.13:

Theorem 1.22. *The set F_X is nonempty if and only if the two following conditions are satisfied*:

(i) *The fan $\mathcal{E}_X$ is Γ-stable.*
(ii) *For every cone $\mathcal{C} \in \mathcal{E}_X$, the fan consisting of the cones $(\sigma(\mathcal{C}))_{\sigma \in \Gamma}$ and their faces is quasi-projective.*

In that case, the map δ_X is bijective. Otherwise stated, for every principal homogeneous space X_0 under T, there is a unique k-form of X containing X_0, up to isomorphism.

Proof. Assume first that conditions (i) and (ii) are fulfilled. By Proposition 1.19, the set E_X is nonempty. Fix a toric semi-linear action of Γ on X. By condition (ii) and Proposition 1.9, for every cone $\mathcal{C} \in \mathcal{E}$, the open subset $\bigcup_{\sigma \in \Gamma} X_{\sigma(\mathcal{C})}$ is quasi-projective. But these open subsets are Γ-stable (by Remark 1.21) and cover X. By Proposition 1.15 the quotient X/Γ exists. Performing this argument for every toric semi-linear action of Γ on X, one proves that the map δ_X is bijective. In particular, the set F_X is nonempty.

Assume now that the set F_X is nonempty. We want to prove that conditions (i) and (ii) are fulfilled. By Proposition 1.15, the set E_X is nonempty, so that condition (i) holds. Fix a T-orbit ω on X. By Remark 1.21, the open subset $U = \bigcup_{\sigma\in\Gamma} X_{\sigma(\omega)}$ is Γ-stable. Moreover, closed T-orbits in U form a unique orbit under Γ. There exists therefore an affine open subset V in U intersecting every closed T-orbit. By Proposition 1.23 below, one concludes that U is quasi-projective, or, using Proposition 1.9, that the fan consisting of the cones $(\sigma(\mathcal{C}_\omega))_{\sigma\in\Gamma}$ and their faces is quasi-projective. This being true for every T-orbit ω, we are done. □

Let G be a linear algebraic group over k. Sumihiro proved in [Sum74] that a normal G-variety containing only one closed orbit is quasi-projective. By the same arguments one gets the next proposition, whose proof we give for the convenience of the reader.

Proposition 1.23. *Let X be a normal G-variety over k. Assume that there is an affine subset of X which meets every closed orbit of G on X. Then X is quasi-projective.*

Proof. Quasi-projectivity is of geometric nature, so one can assume that k is algebraically closed in what follows. We will use the following quasi-projectivity criterion given in [Sum74] (Lemma 7):

Lemma 1.24. *If there exist a line bundle $\mathcal{L}$ on X and global sections $s_1, \dots, s_n$ generating $\mathcal{L}$ at every point and such that $X_{s_1}, \dots, X_{s_n}$ are affine, then X is quasi-projective.*

Let U be an affine subset of X meeting every closed orbit of G and $D = X \setminus U$. Then D is a Weil divisor. Denote by $\mathcal{L}$ the coherent sheaf $\mathcal{O}_X(D)$, and by $i : X^0 \to X$ the inclusion of the regular locus of X in X. The natural morphism $\varphi : \mathcal{L} \to i_*(\mathcal{L}_{|X^0})$ is an isomorphism, because X is normal. Moreover, the sheaf $\mathcal{L}_{|X^0}$ is invertible on the smooth G-variety X^0, so that one can let a finite covering G' of G act on $\mathcal{L}_{|X^0}$, and thus on $\mathcal{L}$, using the isomorphism φ. This enables us to see that the set

$$A = X \setminus \{x \in X,\ \mathcal{L} \text{ is invertible in a neighborhood of } x\}$$

is a G-stable closed subset of X. Moreover, $\mathcal{L}_{|U}$ is trivial, so that A is contained in D, proving that A is empty, or, otherwise stated, the sheaf $\mathcal{L}$ is invertible: the divisor D is Cartier. Let now s be the canonical global section of the sheaf $\mathcal{L}$. The common zero locus of the translates $(g'.s)_{g'\in G'}$ is also a G-closed subset of D, and is therefore empty. We can thus find a finite number of elements $g'_1, \dots, g'_n$ of G' such that the global sections $g'_1.s, \dots, g'_n.s$ generate $\mathcal{L}$ at every point. Moreover, the open subsets $X_{g'_i.s}$ are G-translates of U and are therefore affine. This completes the proof. □

1.3. Applications

In this section we give first some conditions on T for every toric variety under T_K to have a k-form. Then we construct an example of a 3-dimensional torus T split by a degree 3 extension K of k and a toric variety X under T with F_X empty.

Theorem 1.25. *Assume that $\dim_k(T) = 2$, or that the torus T is split by a quadratic extension. Then, for every toric variety X under the action of T_K whose associated fan is Γ-stable, the map δ_X is bijective.*

Proof. If $\dim_k(T) = 2$, then $\dim_K T_K = 2$, and Remark 1.10 shows that the variety X is quasi-projective. By Remark 1.16, the map δ_X is bijective in that case. Assume now that the torus T is split by a quadratic extension K of k. Then the Galois group Γ has two elements, and thus for every cone $\mathcal{C} \in \Sigma$, the fan consisting of the cones $(\sigma(\mathcal{C}))_{\sigma\in\Gamma}$ and their faces has only one or two maximal cones. By Remark 1.11, this fan is automatically quasi-projective, so that Theorem 1.22 applies. □

Remark 1.26. The tori T split by a fixed quadratic extension K of k being exactly the products of the three following tori: $\mathbb{G}_m$, $R_{K|k}(\mathbb{G}_m)$, $R^1_{K|k}(\mathbb{G}_m)$. Here $R_{K|k}(\mathbb{G}_m)$ is the Weil restriction of the torus $\mathbb{G}_m$, and $R^1_{K|k}(\mathbb{G}_m)$ is the kernel of the norm map

$$N_{K|k} : R_{K|k}(\mathbb{G}_m) \to \mathbb{G}_m.$$

We now provide an example of a split toric variety over K which admits no k-form. We assume that the Galois extension K over k is cyclic of degree 3. Let σ be a generator of Γ. Fix a basis (u, v, w) of the lattice $N = \mathbb{Z}^3$, and let σ act on N by the following matrix in the basis (u, v, w)

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One defines in this way an action of Γ on N. Let T be the torus over k corresponding to this action. This is a three-dimensional torus split by K.

Lemma 1.27. *Let $\mathcal{C} = \mathrm{Cone}(5u + v - 5w, -5u - 5v + 14w, 4u - v)$. Then $\mathcal{C} \cap \sigma(\mathcal{C}) = \{0\}$.*

Proof. One has $\sigma(\mathcal{C}) = \mathrm{Cone}(-u + 4v - 5w, 5u + 14w, u + 5v)$. Let $x \in \mathcal{C} \cap \sigma(\mathcal{C})$. There exist $a, b, c, d, e, f \geqslant 0$ such that

$$\begin{aligned} x &= (5a - 5b + 4c)u + (a - 5b - c)v + (-5a + 14b)w \\ &= (-d + 5e + f)u + (4d + 5f)v + (-5d + 14e)w. \end{aligned}$$

Consequently

$$\begin{cases} 5a - 5b + 4c = -d + 5e + f, \\ a - 5b - c = 4d + 5f, \\ -5a + 14b = -5d + 14e, \end{cases}$$

and then 14(second line) + 5(third line) gives

$$-11a - 14c = 31d + 70e + 70f.$$

The left-hand side is nonpositive, and the right-hand side is nonnegative, so that $d = e = f = 0$, and then $x = 0$. □

Definition 1.28. Let $\mathcal{E}$ be the fan consisting of $\mathcal{C}, \sigma(\mathcal{C}), \sigma^2(\mathcal{C})$ and their faces.

Lemma 1.29. *The fan $\mathcal{E}$ is smooth.*

Proof. It is enough to check that $\mathcal{C}$ is a smooth cone, and this holds:

$$\begin{vmatrix} 5 & -5 & 4 \\ 1 & -5 & -1 \\ -5 & 14 & 0 \end{vmatrix} = 1. \qquad \square$$

Proposition 1.30. *The fan $\mathcal{E}$ is not quasi-projective.*

Proof. Observe first that $-w \in \mathcal{C} - \sigma(\mathcal{C})$. Indeed, $45u - 25w = 5((5u + v - 5w) + (4u - v)) \in \mathcal{C}$ and $45u + 126w = 9(5u + 14w) \in \sigma(\mathcal{C})$, so that $-151w = (45u - 25w) - (45u + 126w) \in \mathcal{C} - \sigma(\mathcal{C})$, and finally, $-w \in \mathcal{C} - \sigma(\mathcal{C})$. Applying σ and σ^2, one obtains

$$-w \in \big(\mathcal{C} - \sigma(\mathcal{C})\big) \cap \big(\sigma(\mathcal{C}) - \sigma^2(\mathcal{C})\big) \cap \big(\sigma^2(\mathcal{C}) - \mathcal{C}\big).$$

Now suppose that the fan $\mathcal{E}$ is quasi-projective, and use the notations of Definition 1.9. The linear form $l_{\mathcal{C}} - l_{\sigma(\mathcal{C})}$ is strictly positive on $(\mathcal{C} - \sigma(\mathcal{C})) \setminus \{0\}$, and then: $(l_{\mathcal{C}} - l_{\sigma(\mathcal{C})})(-w) > 0$. But similarly, $l_{\sigma(\mathcal{C})} - l_{\sigma^2(\mathcal{C})}$ is strictly positive on $(\sigma(\mathcal{C}) - \sigma^2(\mathcal{C})) \setminus \{0\}$ (resp. $(\sigma^2(\mathcal{C}) - \mathcal{C}) \setminus \{0\}$) and thus $(l_{\sigma(\mathcal{C})} - l_{\sigma^2(\mathcal{C})})(-w) > 0$ (resp. $(l_{\sigma^2(\mathcal{C})} - l_{\mathcal{C}})(-w) > 0$). This gives a contradiction, proving that the fan $\mathcal{E}$ is not quasi-projective. □

In view of Theorem 1.22, one has the following:

Theorem 1.31. *The toric variety $X_{\mathcal{E}}$ under T_K is smooth and does not admit any k-form.*

In this example, T is of minimal dimension and Γ of minimal order, in view of Theorem 1.25.

Remark 1.32. The variety $X_{\mathcal{E}}$ gives an example of a toric variety containing three points which do not lie on an open affine subset. In [Wło93], it is shown that any two points in a toric variety lie on a common affine subset.

Remark 1.33. Using techniques from [CTHS05], one can "compactify" the fan $\mathcal{E}$ in a Γ-equivariant way, and thus produce an example of a smooth complete toric variety under the action of T_K which has no k-form. We first produce a complete simplicial fan Γ-stable fan $\mathcal{E}_0$ containing $\mathcal{E}$. For simplicity, we note

$$r_1 = -5u - 5v + 14w, \qquad s_1 = 4u - v, \qquad t_1 = 5u + v - 5w,$$
$$r_2 = \sigma(r_1), \qquad s_2 = \sigma(s_1), \qquad t_2 = \sigma(t_1), \qquad r_3 = \sigma^2(r_1), \qquad s_3 = \sigma^2(s_1), \qquad t_3 = \sigma^2(t_1).$$

The fan $\mathcal{E}_0$ has maximal cones

$$\mathrm{Cone}(r_1, t_3, s_1), \qquad \mathrm{Cone}(t_3, s_1, t_1), \qquad \mathcal{C} = \mathrm{Cone}(r_1, s_1, t_1),$$
$$\mathrm{Cone}(r_1, r_2, t_1), \qquad \mathrm{Cone}(r_1, r_2, r_3), \qquad \mathrm{Cone}(t_1, t_2, t_3)$$

and their images under Γ. Because the fan $\mathcal{E}_0$ is complete, it gives a triangulation of the unit sphere in $\mathbb{R}^3$. By projecting this triangulation from the South Pole to the tangent plane of the North Pole, one gets Fig. 1. The maximal cone $\mathrm{Cone}(t_1, t_2, t_3)$ of $\mathcal{E}_0$ is missing in this picture because it is sent to infinity by the projection. In order to smoothen the fan $\mathcal{E}_0$ in a Γ-equivariant way, we use the method described in [CTHS05]. We first subdivide $\mathcal{E}_0$ using the vectors $r = -5v + 28w$, w, $t = u - 4v - 10w$ and $-w$. We thus obtain a complete, simplicial Γ-stable fan $\mathcal{E}_0'$ satisfying Property $(*)$ defined in Proposition 2 of [CTHS05]. This fan $\mathcal{E}_0'$ has maximal cones

$$\mathrm{Cone}(r_1, r, t_1), \qquad \mathrm{Cone}(r, r_2, t_1), \qquad \mathrm{Cone}(r_1, r, w), \qquad \mathrm{Cone}(r, r_2, w), \qquad \mathcal{C} = \mathrm{Cone}(r_1, s_1, t_1),$$
$$\mathrm{Cone}(r_1, s_1, t_3) \qquad \mathrm{Cone}(t_3, s_1, t) \qquad \mathrm{Cone}(t, t_1, s_1) \qquad \mathrm{Cone}(t_3, t, -w) \qquad \mathrm{Cone}(t, t_1, -w)$$

and their images under Γ. We now can apply to $\mathcal{E}_0'$ the algorithm explained in the proof of Proposition 3 of [CTHS05] because it satisfies Property $(*)$. We thus get a complete, smooth Γ-stable fan containing $\mathcal{E}$.

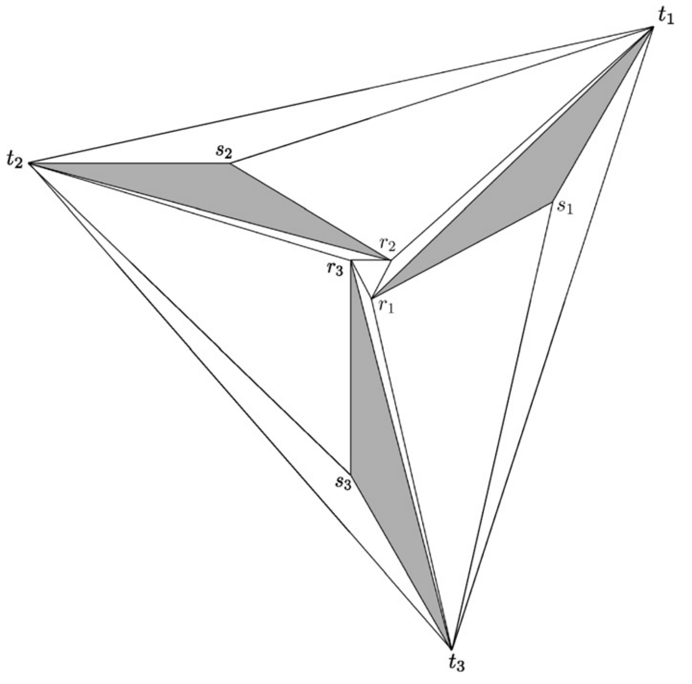

Fig. 1. The fans $\mathcal{E}$ (grey) and $\mathcal{E}_0$.

2. Spherical embeddings over an arbitrary field

Let k be a field, and $\bar{k}$ a fixed algebraic closure. Throughout this section, we denote by Γ the absolute Galois group of k. Let G be a connected reductive algebraic group over k. We note $G_{\bar{k}} = G \times_k \bar{k}$.

Definition 2.1. A spherical homogeneous space under G over k is a pointed G-variety (X_0, x_0) over k such that:

- The group $G(\bar{k})$ acts transitively on $X_0(\bar{k})$.
- $x_0 \in X_0(k)$.
- There exists a Borel subgroup B of $G_{\bar{k}}$ such that the orbit of x_0 under $B(\bar{k})$ is open in $X_0(\bar{k})$.

We say that X_0 is split over k if there exists a split Borel subgroup B of G such that Bx_0 is open in $X_{0,\bar{k}}$. In this case, the group G itself is split.

Remark 2.2. For simplicity, we will often denote by X_0 a spherical homogeneous space, omitting the base point.

Definition 2.3. Let X_0 be a spherical homogeneous space under G. An embedding of X_0 is a pointed normal G-variety (X, x) together with a G-equivariant immersion $i : X_0 \to X$ preserving base points. A morphism between two embeddings is defined to be a G-equivariant morphism defined over k preserving base points.

Remark 2.4. For simplicity again, when speaking about an embedding of X_0, we will often omit the base point and the immersion.

Remark 2.5. If there exists a morphism between two embeddings, then it is unique.

In this section, we classify the embeddings of a fixed spherical homogeneous space X_0 under G up to isomorphism. We first recall in Section 2.1 the Luna–Vust classification theory of spherical embeddings, assuming that the field k is algebraically closed. Fundamental objects named colored fans are the cornerstone of that theory. In Section 2.2 we let the Galois group Γ act on these colored fans, and prove in Section 2.3 that the embeddings of X_0 are classified by Γ-stable colored fans satisfying an additional condition. In Section 2.4 we provide several situations where this condition is fulfilled (including the split case), and an example where it is not.

2.1. Recollections on spherical embeddings

We assume that k is algebraically closed. Fix a spherical homogeneous space X_0 under G, and a Borel subgroup B such that Bx_0 is open in X_0. We now list some facts about X_0 and give the full classification of its embeddings (see [Kno91] for proofs and more details). We will introduce along the way notations that will be systematically used later on.

Notation 2.6. We will denote by:

- $\mathcal{K} = k(X_0)$ the function field of X_0. The group G acts on $\mathcal{K}$.
- $\mathcal{V}$ the set of k-valuations on $\mathcal{K}$ with values in $\mathbb{Q}$. The group G acts on $\mathcal{V}$. We will denote by $\mathcal{V}^G$ the set of G-invariant valuations, and by $\mathcal{V}^B$ the set of B-invariant valuations.
- Ω the orbit of x_0 under the action of B. This is an affine variety.
- $\mathcal{D}$ the set of prime divisors in $X_0 \setminus \Omega$. These are finitely many B-stable divisors.
- $\mathcal{X}$ the set of weights of B-eigenfunctions in $\mathcal{K}$. This is a sublattice of the character lattice of B. The rank of $\mathcal{X}$ is called the rank of X_0 and denoted by $rk(X_0)$.
- $V = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{X}, \mathbb{Q})$. This is a $\mathbb{Q}$-vector space of dimension $rk(X_0)$.
- ρ the map $\mathcal{V} \to V$, $\nu \mapsto (\chi \mapsto \nu(f_\chi))$, $f_\chi \in \mathcal{K}$ being a B-eigenfunction of weight χ (such an f_χ is uniquely determined up to a scalar). The restriction of ρ to $\mathcal{V}^G$ is injective, and the image $\rho(\mathcal{V}^G)$ is a finitely generated cone in V whose interior is nonempty.

The Luna–Vust theory of spherical embeddings gives a full classification of embeddings of X_0 in terms of combinatorial data living in the ones we have just defined.

Definition 2.7. A **colored cone** inside V with colors in $\mathcal{D}$ is a couple $(\mathcal{C}, \mathcal{F})$ with $\mathcal{F} \subseteq \mathcal{D}$, and satisfying:

- $\mathcal{C}$ is a cone generated by $\rho(\mathcal{F})$ and finitely many elements in $\rho(\mathcal{V}^G)$.
- The relative interior of $\mathcal{C}$ in V meets $\rho(\mathcal{V}^G)$.
- The cone $\mathcal{C}$ is strictly convex and $0 \notin \rho(\mathcal{F})$.

A colored cone $(\mathcal{C}', \mathcal{F}')$ is called a **face** of $(\mathcal{C}, \mathcal{F})$ if $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}' = \mathcal{F} \cap \rho^{-1}(\mathcal{C}')$. A **colored fan** is a finite set $\mathcal{E}$ of colored cones satisfying:

- $(0, \emptyset) \in \mathcal{E}$.
- Every face of $(\mathcal{C}, \mathcal{F}) \in \mathcal{E}$ belongs to $\mathcal{E}$.
- There exists at most one colored cone $(\mathcal{C}, \mathcal{F}) \in \mathcal{E}$ containing a given $\nu \in \rho(\mathcal{V}^G)$ in its relative interior.

Let us now recall how a colored fan can be associated to an embedding X of X_0. The open subset Ω being affine, its complement $X \setminus \Omega$ is pure of codimension one, and thus a union of B-stable prime divisors. Let ω be a G-orbit on X. We denote by $\mathcal{D}_\omega$ the set of prime divisors $D \subseteq X \setminus \Omega$

with $\omega \subseteq D$, and by $\mathcal{F}_\omega$ the subset of $D \in \mathcal{D}$ with $\omega \subseteq \overline{D}$. The elements of $\mathcal{F}_\omega$ are called the **colors** associated to the orbit ω. Let $\mathcal{C}_\omega$ be the cone in V generated by the elements

$$\rho(\nu_D), \quad D \in \mathcal{D}_\omega$$

where ν_D is the normalized valuation of $\mathcal{K}$ associated to the divisor D. We are now able to state the main theorem of this section:

Theorem 2.8. *By mapping X to*

$$\big\{(\mathcal{C}_\omega, \mathcal{F}_\omega),\ \omega \subseteq X \text{ is a } G\text{-orbit}\big\}$$

one gets a bijection between embeddings of X_0 and colored fans in V, with colors in the set $\mathcal{D}$. We will denote by $\mathcal{E}_X$ the colored fan associated to the embedding X of X_0, and by $X_{\mathcal{E}}$ the embedding of X_0 associated to the colored fan $\mathcal{E}$.

Remark 2.9. We will say that the embedding X has no color if, for every orbit ω of G on X, the set $\mathcal{F}_\omega$ is empty.

We are not going to explain how an embedding of X_0 can be construct out of a colored fan. The curious reader can find the recipe in [Kno91].

In the rest of the section, we list some properties of spherical embeddings that will be used later. Firstly, as in the toric case, one is able to say whether an embedding of X_0 is quasi-projective by using the associated colored fan (see [Bri89]).

Proposition 2.10. *Let $\mathcal{E}$ be a colored fan. The embedding $X_{\mathcal{E}}$ is quasi-projective if and only if there exists a collection $(l_{\mathcal{C},\mathcal{F}})_{(\mathcal{C},\mathcal{F})\in\mathcal{E}}$ of linear forms on V satisfying*:

- $\forall(\mathcal{C},\mathcal{F}) \in \mathcal{E}$, $\forall(\mathcal{C}',\mathcal{F}') \in \mathcal{E}$, $l_{\mathcal{C},\mathcal{F}} = l_{\mathcal{C}',\mathcal{F}'}$ *over* $\mathcal{C} \cap \mathcal{C}'$.
- $\forall(\mathcal{C},\mathcal{F}) \in \mathcal{E}$, $\forall x \in \operatorname{Int}(\mathcal{C}) \cap \rho(\mathcal{V}^G)$, $\forall(\mathcal{C}',\mathcal{F}') \in \mathcal{E} \setminus \{(\mathcal{C},\mathcal{F})\}$, $l_{\mathcal{C},\mathcal{F}}(x) > l_{\mathcal{C}',\mathcal{F}'}(x)$.

In this situation, we say that the fan $\mathcal{E}$ is quasi-projective.

We now give some results of local nature on spherical embeddings. Knowing the local structure of toric varieties (as explained in Section 1.1) was a crucial point in order to prove Theorem 1.22. A toric variety X under T is covered by open affine T-stable subset. This fact is not true on a spherical embedding X of X_0, but we still have a nice atlas of affine charts at our disposal. If ω is a G-orbit on X, we denote

$$X_{\omega,G} := \{y \in X,\ \omega \subseteq \overline{Gy}\}.$$

This is a G-stable open subset of X containing ω as its unique closed orbit. It is quasi-projective by a result of Sumihiro [Sum74]. We define

$$X_{\omega,B} := X_{\omega,G} \setminus \bigcup_{D \in \mathcal{D}\setminus\mathcal{F}_\omega} \overline{D}.$$

This is a B-stable affine open subset of X. Moreover, the G-translates of $X_{\omega,B}$ cover $X_{\omega,G}$. One has

$$k[X_{\omega,B}] = \big\{f \in k[\Omega],\ \forall D \in \mathcal{D}_\omega\ \nu_D(f) \geqslant 0\big\}.$$

Let

$$P := \bigcap_{D \in \mathcal{D} \setminus \mathcal{F}_\omega} \mathrm{Stab}(D).$$

This is a parabolic subgroup of G containing B, and the open subset $X_{\omega,B}$ is P-stable. The following theorem describes the action of P in $X_{\omega,B}$. See [BLV86] for a proof.

Theorem 2.11. *Assume that the field k is of characteristic 0. There exist a Levi subgroup L of P and a closed L-stable subvariety S of $X_{\omega,B}$ containing x_0, such that the natural map*

$$R_u(P) \times S \to X_{\omega,B}, \quad (g,x) \mapsto gx$$

is a P-equivariant isomorphism.

Finally, let us introduce the class of horospherical homogeneous spaces:

Definition 2.12. A homogeneous space X_0 is said to be horospherical if the isotropy group of x_0 contains a maximal unipotent subgroup of G.

Any horospherical homogeneous space is spherical. Moreover, thanks to the following proposition, we are able to recognize horospherical homogeneous spaces among spherical ones in a very simple way (see [Kno91]).

Proposition 2.13. *A spherical homogeneous space X_0 is horospherical if and only if $\rho(\mathcal{V}^G) = V$.*

2.2. Galois actions

We return to an arbitrary field k with algebraic closure $\bar{k}$. Recall that Γ is the absolute Galois group of k, and G is a connected reductive algebraic group over k. Fix a spherical homogeneous space X_0 under the action of G, and a Borel subgroup B of $G_{\bar{k}}$ such that Bx_0 is open in $X_0(\bar{k})$. In the previous section, we introduced some data attached to $X_{0,\bar{k}}$. In this section, we let the group Γ act on these data.

2.2.1. Action on $\mathcal{K}$

The group Γ acts on $\mathcal{K}$ by the following formula

$$\forall \sigma \in \Gamma,\ \forall f \in \mathcal{K},\ \forall x \in X_0(\bar{k}), \quad \sigma(f)(x) = \sigma\big(f\big(\sigma^{-1}(x)\big)\big).$$

Proposition 2.14. *One has*

$$\forall \sigma \in \Gamma,\ \forall g \in G(\bar{k}),\ \forall f \in \mathcal{K}, \quad \sigma(gf) = \sigma(g)\sigma(f).$$

Proof. A straightforward computation. □

2.2.2. Action on $\mathcal{V}^B$

If $\sigma \in \Gamma$, then the group $\sigma(B)$ is a Borel subgroup of G, so that (see [Bor91, Chap. IV, Theorem 11.1]) there exists $g_\sigma \in G(\bar{k})$ satisfying

$$\sigma(B) = g_\sigma B g_\sigma^{-1}.$$

Moreover, g_σ is unique up to right multiplication by an element of B. We let the group Γ act on $\mathcal{V}$ by

$$\forall \sigma \in \Gamma,\ \forall \nu \in \mathcal{V},\ \forall f \in \mathcal{K}, \quad \sigma(\nu)(f) = \nu\big(\sigma^{-1}(g_\sigma f)\big).$$

Proposition 2.15. *We define in this way an action of Γ on $\mathcal{V}^B$ which does not depend on the particular choice of the $(g_\sigma)_{\sigma\in\Gamma}$.*

Proof. Fix $\nu \in \mathcal{V}^B$, $\sigma \in \Gamma$ and $f \in \mathcal{K}$. Let $b \in B$, and $b' \in B$ such that: $g_\sigma b^{-1} = \sigma(b')g_\sigma$. Then one has

$$\big(b\sigma(\nu)\big)(f) = \nu\big(\sigma^{-1}\big(g_\sigma b^{-1} f\big)\big) = \nu\big(b'\sigma^{-1}(g_\sigma f)\big) = \sigma(\nu)(f)$$

because $\nu \in \mathcal{V}^B$. Thus we have proved that for all $\sigma \in \Gamma$ and $\nu \in \mathcal{V}^B$, $\sigma(\nu) \in \mathcal{V}^B$. Now let us check that one defines an action of Γ on $\mathcal{V}^B$. For this we will need the following lemma:

Lemma 2.16.

$$\forall(\sigma,\tau) \in \Gamma^2,\ \exists b_{\sigma,\tau} \in B \quad \textit{with}\ \sigma(g_\tau)g_\sigma = g_{\sigma\tau} b_{\sigma,\tau}.$$

Proof. Express $\sigma(\tau(B))$ in two different ways. □

Fix $(\sigma,\tau) \in \Gamma^2$, $\nu \in \mathcal{V}^B$ and $f \in \mathcal{K}$. Then

$$\begin{aligned}(\sigma\tau)(\nu)(f) &= \nu\big(\tau^{-1}\big(\sigma^{-1}(g_{\sigma\tau} f)\big)\big) = \nu\big(\tau^{-1}\big(\sigma^{-1}\big(\sigma(g_\tau)g_\sigma b_{\sigma,\tau}^{-1}\big) f\big)\big) \\ &= \nu\big(\tau^{-1}\big(g_\tau \sigma^{-1}\big(g_\sigma b_{\sigma,\tau}^{-1} f\big)\big)\big).\end{aligned}$$

This shows that

$$(\sigma\tau)(\nu) = b_{\sigma,\tau}\big(\sigma\big(\tau(\nu)\big)\big) = \sigma\big(\tau(\nu)\big)$$

because $\sigma(\tau(\nu)) \in \mathcal{V}^B$. The fact that this action does not depend on the choice of the g_σ readily follows from the fact that we are working with B-invariant valuations. □

2.2.3. Action on $\mathcal{X}$

Let us denote by $\mathcal{X}(B)$ the character lattice of the group B. For $\sigma \in \Gamma$ and $\chi \in \mathcal{X}(B)$ we define

$$\sigma(\chi) : B \to \mathbb{G}_{m,\bar{k}}, \qquad b \mapsto \sigma\big(\chi\big(g_{\sigma^{-1}}^{-1}\sigma^{-1}(b)g_{\sigma^{-1}}\big)\big).$$

As in the proof of Proposition 2.15, one checks that this defines an action of Γ on $\mathcal{X}(B)$ which does not depend on the choice of the $(g_\sigma)_{\sigma\in\Gamma}$. The following proposition shows that this action restricts to an action on $\mathcal{X}$:

Proposition 2.17. *Let $\chi \in \mathcal{X}$ and $f_\chi \in \mathcal{K}$ be a B-eigenfunction of weight χ. Fix $\sigma \in \Gamma$. Then $\sigma(g_{\sigma^{-1}} f)$ is a B-eigenfunction of weight $\sigma(\chi)$.*

Proof. A straightforward computation. □

2.2.4. Action on V

Being dual to $\mathcal{X}$, the vector space $V = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{X}, \mathbb{Q})$ inherits a linear action of Γ defined by

$$\forall \varphi \in V,\ \forall \sigma \in \Gamma,\ \forall \chi \in \mathcal{X}, \quad \sigma(\varphi)(\chi) = \varphi\big(\sigma^{-1}(\chi)\big).$$

Proposition 2.18. *The map*

$$\rho : \mathcal{V}^B \to V$$

is Γ-equivariant.

Proof. Let $\nu \in \mathcal{V}^B$, $\sigma \in \Gamma$ and $\chi \in \mathcal{X}$. Denote by $f_\chi \in \mathcal{K}$ a B-eigenfunction of weight χ. By Proposition 2.17, $\sigma^{-1}(g_\sigma f)$ is a B-eigenfunction of weight $\sigma^{-1}(\chi)$. Then

$$\rho\big(\sigma(\nu)\big)(\chi) = \sigma(\nu)(f_\chi) = \nu\big(\sigma^{-1}(g_\sigma f)\big) = \rho(\nu)\big(\sigma^{-1}(\chi)\big),$$

completing the proof. □

2.2.5. Action on $\mathcal{D}$

Let $\sigma \in \Gamma$. Observe that $g_\sigma^{-1}\sigma(\Omega)$ is an open B-orbit in $X_{0,\bar{k}}$. Thus $g_\sigma^{-1}\sigma(\Omega) = \Omega$, and the map

$$\mathcal{D} \to \mathcal{D}, \qquad D \mapsto \sigma \cdot D = g_\sigma^{-1}\sigma(D)$$

is a bijection. The elements of $\mathcal{D}$ being B-invariant divisors, this map does not depend on the choice of g_σ. As in the proof of Proposition 2.15, one checks that this defines an action of Γ on $\mathcal{D}$. Moreover, one has the following:

Proposition 2.19. *The natural map*

$$\mathcal{D} \to \mathcal{V}^B, \qquad D \mapsto \nu_D$$

is Γ-equivariant.

Proof. Indeed, if a valuation ν on $\mathcal{K}$ has a center Y on $X_{0,\bar{k}}$ and if φ is an automorphism of $X_{0,\bar{k}}$, then the valuation

$$\varphi(\nu) : \mathcal{K} \to \mathbb{Q}, \qquad f \mapsto \nu\big(\varphi^{-1}(f)\big)$$

is centered on $\varphi(Y)$. □

2.3. k-forms of embeddings

We assume in this section that k is a perfect field. We keep the notations from Section 2.2. If X is an embedding of $X_{0,\bar{k}}$, we call a k-**form** of X an embedding of X_0 isomorphic to X after extending scalars to $\bar{k}$. We obtain in this section a criterion for an embedding X of $X_{0,\bar{k}}$ to admit a k-form, in terms of its associated colored fan.

Proposition 2.20. *If a k-form of X exists, then it is unique up to isomorphism.*

Proof. Take Y and Z two k-forms of X. There exists an isomorphism of embeddings

$$f : Y_{\bar{k}} \to Z_{\bar{k}}$$

because they are both isomorphic to X. But a morphism between two embeddings, if exists, is unique. This shows that f is unchanged under twisting by Γ. In other words, f is defined over k. □

The following proposition gives a characterization of embeddings of $X_{0,\bar{k}}$ admitting a k-form:

Proposition 2.21. *The embedding X admits a k-form if and only if it satisfies the two following conditions*:

(i) *The semi-linear action of Γ on $X_{0,\bar{k}}$ extends to X.*
(ii) *X is covered by Γ-stable affine open sets.*

Remark 2.22. Condition (ii) is automatically satisfied if X is quasi-projective or covered by Γ-stable quasi-projective subsets. By a result of Sumihiro [Sum74], this is the case if there is only one closed orbit of $G_{\bar{k}}$ on X, or, in other words, if the embedding is simple.

Proof of Proposition 2.21. These two conditions are satisfied exactly when X admits a k-form as a variety. Because the semi-linear action of Γ on $X_{0,\bar{k}}$ is $G_{\bar{k}}$-semi-linear in the sense that

$$\forall \sigma \in \Gamma,\ \forall g \in G(\bar{k}),\ \forall x \in X_0(\bar{k}), \quad \sigma(gx) = \sigma(g)\sigma(x),$$

the k-form is naturally an embedding of X_0. □

Condition (i) in this proposition can be made very explicit in terms of the colored fan associated to X.

Theorem 2.23. *The $G_{\bar{k}}$-semi-linear action of Γ on $X_{0,\bar{k}}$ extends to X if and only if the colored fan of X is Γ-stable. In that case, for every $G_{\bar{k}}$-orbit ω on X and every $\sigma \in \Gamma$ one has*

$$\sigma(\mathcal{C}_\omega) = \mathcal{C}_{\sigma(\omega)} \quad \textit{and} \quad \sigma(\mathcal{F}_\omega) = \mathcal{F}_{\sigma(\omega)}.$$

Remark 2.24. We say that a colored fan $\mathcal{E}$ is Γ-**stable** if for every colored cone $(\mathcal{C}, \mathcal{F}) \in \mathcal{E}$, the colored cone $(\sigma(\mathcal{C}), \sigma(\mathcal{F}))$ still belongs to $\mathcal{E}$.

Proof of Theorem 2.23. Assume (i) and let ω be a $G_{\bar{k}}$-orbit on X. Fix $\sigma \in \Gamma$. Observe that $\sigma(\omega)$ is also a $G_{\bar{k}}$-orbit on X. By mapping a divisor D to $\sigma \cdot D$, one gets bijections

$$\mathcal{D}_\omega \to \mathcal{D}_{\sigma(\omega)} \quad \text{and} \quad \mathcal{F}_\omega \to \mathcal{F}_{\sigma(\omega)}.$$

Thus,

$$\big(\sigma(\mathcal{C}_\omega), \sigma(\mathcal{F}_\omega)\big) = (\mathcal{C}_{\sigma(\omega)}, \mathcal{F}_{\sigma(\omega)})$$

because the map ρ is Γ-equivariant (Proposition 2.18).

Assume now that the colored fan of X is Γ-stable. Let ω be a $G_{\bar{k}}$-orbit on X. Fix $\sigma \in \Gamma$, and denote by ω' the $G_{\bar{k}}$-orbit on X satisfying

$$\big(\sigma(\mathcal{C}_\omega), \sigma(\mathcal{F}_\omega)\big) = (\mathcal{C}_{\omega'}, \mathcal{F}_{\omega'}).$$

Lemma 2.25. *The automorphism σ of $\mathcal{V}^B$ sends the set $\{\nu_D, D \in \mathcal{D}_\omega\}$ onto the set $\{\nu_D, D \in \mathcal{D}_{\omega'}\}$.*

Proof. Let $D \in \mathcal{D}_\omega$. If D is not $G_{\bar{k}}$-stable, then $D \in \mathcal{F}_\omega$. In this case $\sigma \cdot D \in \mathcal{F}_{\omega'}$, and thus Proposition 2.19 shows that

$$\sigma(\nu_D) \in \{\nu_{D'},\ D' \in \mathcal{D}_{\omega'}\}.$$

Assume now that D is $G_{\bar{k}}$-stable. We know from Lemma 2.4 [Kno91] that $\mathbb{Q}^+\rho(\nu_D)$ is an extremal ray of $\mathcal{C}_\omega$ which contains no element of $\rho(\mathcal{F}_\omega)$. The map ρ being equivariant, this proves that $\mathbb{Q}^+\rho(\sigma(\nu_D))$ is an extremal ray of $\mathcal{C}_{\omega'}$ which contains no element of $\rho(\mathcal{F}_{\omega'})$. Using Lemma 2.4 [Kno91] again and the injectivity of ρ on $\mathcal{V}^G$, we get that $\sigma(\nu_D) = \nu_{D'}$, for some $G_{\bar{k}}$-stable divisor D' in $\mathcal{D}_{\omega'}$. So far we have proved that σ sends the set $\{\nu_D, D \in \mathcal{D}_\omega\}$ into the set $\{\nu_D, D \in \mathcal{D}_{\omega'}\}$. Using this result for σ^{-1} and ω' instead of ω, one obtains the lemma. □

Using the description of $\bar{k}[X_{\omega,B}]$ and $\bar{k}[X_{\omega',B}]$ given in Section 2.1 and the previous lemma, we get that the morphism

$$\Omega \to \Omega, \qquad x \mapsto g_\sigma^{-1}\sigma(x)$$

extends to a morphism

$$X_{\omega,B} \to X_{\omega',B}.$$

Let U be the largest open subset of X on which this morphism extends. The action of Γ on $X_{0,\bar{k}}$ being $G_{\bar{k}}$-semi-linear, U is $G_{\bar{k}}$-stable. But X is covered by the $G_{\bar{k}}$-translates of $X_{\omega,B}$, ω being a $G_{\bar{k}}$-orbit on X. We conclude that $U = X$, completing the proof of the theorem. □

Assuming that the condition (i) of Proposition 2.21 holds, we now make (ii) explicit.

Theorem 2.26. *Let X be an embedding of $X_{0,\bar{k}}$ with Γ-stable colored fan $\mathcal{E}_X$. Then X admits a k-form if and only if for every colored cone $(\mathcal{C},\mathcal{F}) \in \mathcal{E}_X$, the colored fan consisting of the cones $(\sigma(\mathcal{C}),\sigma(\mathcal{F}))_{\sigma\in\Gamma}$ and their faces is quasi-projective.*

Proof. The condition given in the proposition is equivalent to the following: for every $G_{\bar{k}}$-orbit ω on X, the open subset

$$\bigcup_{\sigma\in\Gamma} X_{\sigma(\omega),G_{\bar{k}}}$$

is quasi-projective. This set being Γ-stable, if this condition is fulfilled then X admits a k-form (see Remark 2.22). To prove the converse statement, one can clearly replace X by

$$\bigcup_{\sigma\in\Gamma} X_{\sigma(\omega),G_{\bar{k}}},$$

and thus suppose that maximal cones in $\mathcal{E}_X$ form a single orbit under the action of Γ. But maximal cones correspond to closed orbits, so using Theorem 2.23, one deduces that closed orbits are permuted by Γ. Because X admits a k-form, there exists an affine open subset U of X meeting every closed $G_{\bar{k}}$-orbit on X. We conclude by Proposition 1.23. □

2.4. Applications

2.4.1. Some situations where (i) ⇒ (ii)

We keep notations from Section 2.2, and denote by X an embedding of $X_{0,\bar{k}}$. In Proposition 2.21 we introduced two conditions called (i) and (ii) on X which were reformulated in terms of the colored fan $\mathcal{E}_X$ in Theorems 2.23 and 2.26. In this section, we prove that (i) ⇒ (ii) under some additional assumptions on G, X_0 or X.

Proposition 2.27. *In each of the following situations, one has* (i) ⇒ (ii):

1. X_0 *is split.*
2. X_0 *is of rank* 1.
3. X_0 *is horospherical and of rank* 2.
4. X_0 *is horospherical, and* G *is split by a quadratic extension* K *of* k.
5. X *has no colors, and* X_0 *is of rank* 2.
6. X *has no colors, and* G *is split by a quadratic extension* K *of* k.

Proof. We suppose that condition (i) is satisfied, and prove that (ii) holds. Let us first consider situation 1. We fix a split Borel subgroup B of G with Bx_0 open in $X_{0,\bar{k}}$. One can choose $g_\sigma = 1$ for every $\sigma \in \Gamma$. Looking at the very definition of the action of Γ on $\mathcal{X}$ and using the fact that B is split, one deduces that Γ acts trivially on $\mathcal{X}$. Let ω be a $G_{\bar{k}}$-orbit on X. For every $\sigma \in \Gamma$ one has

$$\big(\sigma(\mathcal{C}_\omega), \sigma(\mathcal{F}_\omega)\big) = (\mathcal{C}_\omega, \mathcal{F}_\omega).$$

But the colored fan consisting of $(\mathcal{C}_\omega, \mathcal{F}_\omega)$ and its faces is quasi-projective, so condition (ii) is satisfied.

We now turn to situation 2. One can check using Proposition 2.10 and the fact that V is of dimension 1 that X is automatically quasi-projective. Condition (ii) is thus fulfilled.

In the remaining situations, observe that for every closed orbit ω of $G_{\bar{k}}$ on X the cone $\mathcal{C}_\omega$ is contained in $\rho(\mathcal{V}^G)$ (this is obvious if X has no colors; if X_0 is horospherical use Proposition 2.13), so that the collection of cones

$$\{\mathcal{C}_\omega,\ \omega \text{ is an orbit of } G_{\bar{k}} \text{ on } X\}$$

is a fan. Moreover, this fan is quasi-projective if and only if the colored fan $\mathcal{E}_X$ is.

Every 2-dimensional fan is quasi-projective, so condition (ii) is satisfied in situations 3 and 5.

If G is split by a quadratic extension K of k, then the Galois group Γ acts through a quotient of order 2 on V, and thus for every orbit ω of $G_{\bar{k}}$ on X, the fan consisting of the cones $(\sigma(\mathcal{C}_\omega))_{\sigma\in\Gamma}$ and their faces has only one or two maximal cones. By Remark 1.11, this fan is automatically quasi-projective. We thus see that condition (ii) is also satisfied in situations 4 and 6. □

Remark 2.28. Situations 1, 2, 3 and 4 don't depend on X. This means that in these situations, the embeddings of X_0 are classified by Γ-stable colored fans. In the split case, the colored fan $\mathcal{E}_X$ is Γ-stable if and only if for every $G_{\bar{k}}$-orbit ω on X, the set of colors $\mathcal{F}_\omega$ is Γ-stable.

2.4.2. A spherical embedding with no k-form

In this section $k = \mathbb{R}$. Thus $\bar{k} = \mathbb{C}$ and $\Gamma = \mathbb{Z}/2\mathbb{Z}$. We let σ be the nontrivial element of Γ. We construct here a connected reductive group G over $\mathbb{R}$, a spherical homogeneous space X_0 of rank 2 under the action of G and an embedding X of $X_{0,\mathbb{C}}$ whose colored fan is Γ-stable, but which admits no $\mathbb{R}$-form.

2.4.3. The group G

We denote by E a 3-dimensional $\mathbb{C}$-vector space with an $\mathbb{R}$-structure $E_{\mathbb{R}}$. Let ε be a bilinear form on E defined over $\mathbb{R}$ and of signature $(1,2)$. We denote by q its associated quadratic form. If g is an element of $GL(E)$, we denote by g^* the inverse of its adjoint. We consider the semi-linear action of Γ on $SL(E)$ given by

$$\forall g \in SL(E), \quad \sigma(g) = \bar{g}^*.$$

Here the conjugation is relative to the $\mathbb{R}$-form $SL(E_{\mathbb{R}})$ of $SL(E)$. One checks that Γ acts by automorphisms of the group $SL(E)$, so that this action corresponds to an $\mathbb{R}$-form G of $SL(E)$, which is isomorphic to $SU(2,1)$. Fix an isotropic line l in E defined over $\mathbb{R}$, and denote by B the stabilizer of the complete flag

$$l \subset l^{\perp}$$

in E. Then B is a Γ-stable Borel subgroup of $SL(E)$. Thus G is quasi-split, and we can choose $g_\sigma = 1$ in what follows. The character lattice of B is given by

$$\mathcal{X}(B) = (\mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2 \oplus \mathbb{Z}\chi_3)/\mathbb{Z}(\chi_1 + \chi_2 + \chi_3),$$

where the group B acts through the character χ_1 on l, χ_2 on $l^{\perp}/l$ and χ_3 on $E/l^{\perp}$.

We denote by V the dual $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{X}(B), \mathbb{Q})$. We thus have

$$V = \{r_1\mu_1 + r_2\mu_2 + r_3\mu_3,\ (r_1, r_2, r_3) \in \mathbb{Q}^3 \text{ and } r_1 + r_2 + r_3 = 0\}$$

where $\mathbb{Z}\mu_1 \oplus \mathbb{Z}\mu_2 \oplus \mathbb{Z}\mu_3$ is the dual lattice of $\mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2 \oplus \mathbb{Z}\chi_3$.

2.4.4. The homogeneous space X_0

Consider the following affine variety over $\mathbb{C}$:

$$Y = \{(p, \mathcal{P}),\ p \in E,\ \mathcal{P} \subset E \text{ of dimension } 2,\ p \notin \mathcal{P}\}.$$

The group $SL(E)$ acts naturally and transitively on Y. Fix a point $p_0 \in E_{\mathbb{R}} \setminus \{l\}$ with $q(p_0) = 1$. We note $\mathcal{P}_0 = p_0^{\perp}$ and $x_0 = (p_0, \mathcal{P}_0)$.

Remark 2.29. The stabilizer of x_0 in $SL(E)$ is

$$\{g \in G(\mathbb{C}),\ g(p_0) = p_0 \text{ and } g(\mathcal{P}_0) = \mathcal{P}_0\},$$

and is therefore isomorphic to SL_2. The homogeneous space Y is thus isomorphic to SL_3/SL_2.

We let the group Γ act semi-linearly on Y by

$$\sigma(p, \mathcal{P}) = (p', \bar{p}^{\perp})$$

where $p' \in \overline{\mathcal{P}}^{\perp}$ satisfies $\varepsilon(\bar{p}, p') = 1$. We call X_0 the $\mathbb{R}$-form of this variety corresponding to this semi-linear action. There is no problem to perform the quotient because Y is an affine variety. Observe that $x_0 \in X_0(\mathbb{R})$.

Proposition 2.30. *The homogeneous space X_0 is spherical. More precisely, Bx_0 is open in $X_0(\mathbb{C})$.*

Proof. As one can check, the subset

$$\Omega := \{(p,\mathcal{P}) \in X_0(\mathbb{C}),\ p \notin l^\perp \text{ and } l \nsubseteq \mathcal{P}\}$$

of $X_0(\mathbb{C})$ is an orbit of B, and is open in $X_0(\mathbb{C})$. Moreover $x_0 \in \Omega$. □

2.4.5. Data attached to X_0

We define

$$D_1 := \{(p,\mathcal{P}) \in X_0(\mathbb{C}),\ p \in l^\perp\} \quad \text{and} \quad D_2 := \{(p,\mathcal{P}) \in X_0(\mathbb{C}),\ l \subseteq \mathcal{P}\}.$$

These are two B-stable divisors on $X_0(\mathbb{C})$ and:

Proposition 2.31. *The set $\mathcal{D}$ equals $\{D_1, D_2\}$, and σ exchanges D_1 and D_2.*

Proof. This is a direct consequence of the description of Ω given in the proof of Proposition 2.30 and of the definition of the action of Γ on $\mathcal{D}$. □

Before we continue, we need to specify a particular basis of $E_{\mathbb{R}}$.

Notation 2.32. We denote by e_1 the unique vector in l satisfying $\varepsilon(e_1, p_0) = 1$, by e_2 the unique vector in $l^\perp \cap \mathcal{P}_0$ with $q(e_2) = 1$ and by e_3 the vector $p_0 - e_1$. The vectors e_1, e_2, e_3 give a basis of E defined on $\mathbb{R}$ and we denote by e_1^*, e_2^*, e_3^* the dual basis.

With these notations, we have

$$x_0 = (e_1 + e_3, \langle e_2, e_3\rangle) \quad \text{and} \quad l^\perp = \langle e_1, e_2\rangle.$$

We define the two following functions on $X_0(\mathbb{C})$

$$f_1 : (p,\mathcal{P}) \mapsto e_3^*(p) \quad \text{and} \quad f_2 : (p,\mathcal{P}) \mapsto \frac{\varphi_{\mathcal{P}}(e_1)}{\varphi_{\mathcal{P}}(p)}$$

where $\varphi_{\mathcal{P}}$ is an equation of the 2-plane $\mathcal{P}$. They are respectively the equations of D_1 and D_2, and B-eigenfunctions of weight $-\chi_3$ and χ_1. We are now able to prove:

Proposition 2.33. *The lattice $\mathcal{X}$ is $\mathcal{X}(B)$ itself. We have*

$$\rho(\nu_{D_1}) = \mu_2 - \mu_3, \qquad \rho(\nu_{D_2}) = \mu_1 - \mu_2.$$

The automorphism σ of V is the reflection exchanging $\mu_2 - \mu_3$ and $\mu_1 - \mu_2$.

Proof. The first point follows from the fact that $-\chi_3$ and χ_1 generate $\mathcal{X}(B)$. Moreover,

$$\rho(\nu_{D_1})(-\chi_3) = \nu_{D_1}(f_1) = 1 \quad \text{and} \quad \rho(\nu_{D_1})(\chi_1) = \nu_{D_1}(f_2) = 0,$$

proving that $\rho(\nu_{D_1}) = \mu_2 - \mu_3$. We compute $\rho(\nu_{D_2})$ in the same way. For the remaining point, recall that the automorphism σ of $\mathcal{D}$ exchanges D_1 and D_2, and the map ρ is equivariant. □

Proposition 2.34. *We have*

$$\rho(\mathcal{V}^G) = \{r_1\mu_1 + r_2\mu_2 + r_3\mu_3 \in V,\ r_3 \geqslant r_1\}.$$

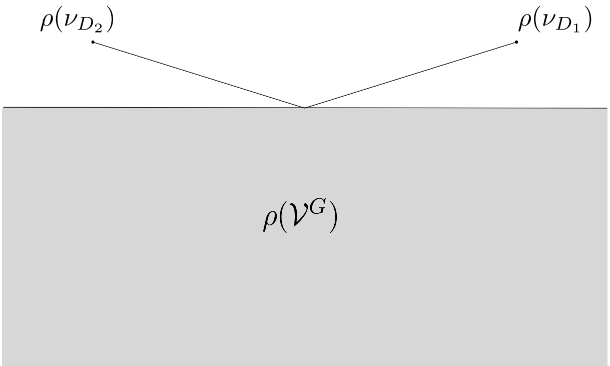

Fig. 2. The combinatorial data attached to X_0.

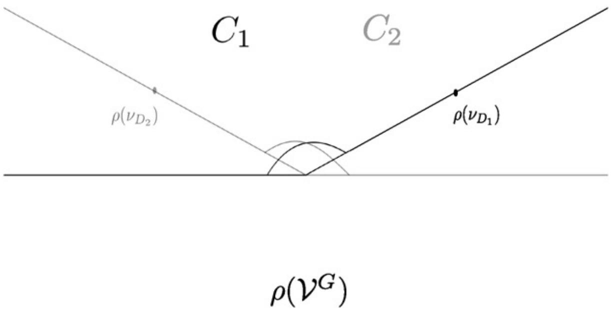

Fig. 3. The colored fan $\mathcal{E}$.

Proof. Denote by τ the permutation $(1,2,3)$ and also by $\tau \in SL(E)$ the automorphism sending e_i to $e_{\tau(i)}$. Then one checks that

$$\tau(f_1)f_2 + \tau^2(f_1)\tau^2(f_2) + f_1\tau(f_2) = 1$$

in the field $\mathcal{K}$. This proves that if $\nu \in \mathcal{V}$ is G-invariant, then

$$\langle \rho(\nu), \chi_1 - \chi_3 \rangle \leqslant 0.$$

So far we have proved "$\subseteq$". For the reverse inclusion, we use the following fact (see [Kno91]): the dimension of the linear part of the cone $\rho(\mathcal{V}^G)$ is equal to the codimension of $\mathrm{Stab}(x_0)$ in its normalizer. Using Remark 2.29 one sees that this normalizer is given by

$$\{g \in G(\mathbb{C}),\ g(p_0) \in \langle p_0 \rangle \text{ and } g(\mathcal{P}_0) = \mathcal{P}_0\}.$$

Thus the linear part of $\rho(\mathcal{V}^G)$ is of dimension 1, completing the proof of the proposition. See Fig. 2. □

2.4.6. The embedding X

We denote by $\mathcal{C}$ the cone in V spanned by $\mu_2 - \mu_3$ and $\mu_1 - 2\mu_2 + \mu_3$. Let $\mathcal{E}$ be the colored fan in V with colors in $\mathcal{D}$ whose maximal cones are $(\mathcal{C}, \{D_1\})$ and $(\sigma(\mathcal{C}), \{D_2\})$. The fan $\mathcal{E}$ is Γ-stable. It is depicted in Fig. 3.

Theorem 2.35. *The embedding $X = X_{\mathcal{E}}$ of $X_{0,\mathbb{C}}$ admits no $\mathbb{R}$-form. Moreover, X is smooth.*

Proof. The colored fan $\mathcal{E}$ has two maximal cones that are permuted by Γ. Moreover, these maximal cones meet, so the colored fan $\mathcal{E}$ cannot be quasi-projective. By Theorem 2.26 this proves the first point. Let us denote by ω the closed orbit of $G(\mathbb{C})$ on X corresponding to the colored cone $(\mathcal{C}, \{D_1\})$. Translating the open subset $X_{\omega,B}$ by elements of Γ and $G(\mathbb{C})$ one covers X, so in order to prove the second point, one only has to see that $X_{\omega,B}$ is smooth. Denote by P the stabilizer of D_2, or, in other words, the stabilizer of l in E. By Theorem 2.11, there exist a Levi subgroup L of P and a closed L-stable subvariety S of $X_{\omega,B}$ containing x_0 and such that the natural map

$$R_u(P) \times S \to X_{\omega,B}$$

is a P-equivariant isomorphism. By the following lemma, S is isomorphic to $\mathbb{C}^3$, and thus smooth. Let f_3 be the following function on $X_0(\mathbb{C})$:

$$f_3 : (p, \mathcal{P}) \mapsto e_2^*(p). \qquad \square$$

Lemma 2.36. *The functions f_2, $f_1 f_2$ and $f_3 f_2$ are algebraically independent in $\mathbb{C}[S]$, and*

$$\mathbb{C}[S] = \mathbb{C}[f_2, f_1 f_2, f_3 f_2].$$

Proof. Observe that:

$$\mathbb{C}[S] = \mathbb{C}[X_{\omega,B}]^{R_u(P)}$$

so that the algebra $\mathbb{C}[S]$ is the sub-L-module of $\mathbb{C}[Lx_0]$ generated by the $B \cap L$-eigenfunctions in $\mathbb{C}(Lx_0)$ of weight χ satisfying

$$\langle \mu_2 - \mu_3, \chi \rangle \geqslant 0, \qquad \langle \mu_1 - 2\mu_2 + \mu_3, \chi \rangle \geqslant 0.$$

In other words, χ belongs to the monoid generated by χ_1 and $\chi_1 - \chi_3$. The functions f_2 and $f_1 f_2$ are $B \cap L$-eigenfunctions of respective weights χ_1 and $\chi_1 - \chi_3$. Moreover, the L-module $\mathbb{C}[Lx_0]$ is multiplicity-free, because the homogeneous space Lx_0 is spherical. One deduces that

$$\mathbb{C}[S] = \bigoplus_{(m,n) \in \mathbb{N}^2} \langle L f_2^{m+n} f_1^n \rangle.$$

Using the fact that f_2 is a L-eigenfunction, the following lemma enables us to conclude. $\square$

Lemma 2.37. *Let $n \in \mathbb{N}$. A basis of the linear span of L-translates of f_1^n in $\mathbb{C}[Lx_0]$ is given by*

$$f_1^n, f_1^{n-1} f_3, \ldots, f_1 f_3^{n-1}, f_3^n.$$

Proof. We first prove the case $n = 1$. Observe that $l^\perp \subseteq V^*$ is generated by e_2^* and e_3^*. Then one easily checks that e_2^*, e_3^* is a basis of $\langle L e_3^* \rangle$. Moreover, if there exist $\lambda_1, \lambda_3 \in \mathbb{C}^2$ such that

$$\lambda_1 f_1 + \lambda_3 f_3 = 0$$

in $\mathbb{C}[S]$, then the same is true in $\mathbb{C}[X_{\omega,B}]$. The linear forms e_2^* and e_3^* are thus linearly dependent in V^*, which is a contradiction. Thus f_1 and f_3 are linearly independent in $\mathbb{C}[S]$, completing the proof in the case $n = 1$. The same argument proves that $f_1^n, f_1^{n-1} f_3, \ldots, f_1 f_3^{n-1}, f_3^n$ are linearly independent in $\mathbb{C}[S]$. By the case $n = 1$, they also generate $\langle L f_1^n \rangle$. $\square$

Remark 2.38. The embedding X gives an example of a spherical variety containing two points which do not lie on a common affine open subset. Indeed, there are two closed orbits of $G_{\bar{k}}$ on X. If the point y belongs to the first orbit and z to the second, then y and z cannot lie on a common affine open subset, because of Proposition 1.23. Moreover, in view of Theorem 2.27, this example has minimal rank.

Remark 2.39. With a little more work, one can compactify the previous example, and thus obtains a complete smooth embedding X of X_0 with Γ-stable fan, and with no $\mathbb{R}$-form.

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