Wave propagation in transversely isotropic cylinders

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Abstract

Various approaches have been used for modelling problems dealing with interaction of acoustic/elastic waves with transversely isotropic cylinders. The authors developed the first mathematical model for the scattering of acoustic waves from transversely isotropic cylinders [Honarvar, F., Sinclair, A.N., 1996. Acoustic wave scattering from transversely isotropic cylinders. Journal of the Acoustical Society of America 100, 57–63]. In the current paper, this model is used for derivation of the frequency equations of longitudinal and flexural wave propagation in free transversely isotropic cylinders. Consistency of this model with the physics of the problem is demonstrated and a systematic solution to the corresponding equations is developed. Numerical results obtained for a number of transversely isotopic cylinders are used for verification of the mathematical model.

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1. Introduction

Most cylindrical components such as wires, rods, pipes, tubes, and fibers are manufactured by processes which induce transversely isotropic elastic properties in them. Many fiber-reinforced materials also have transversely isotropic properties. Modeling the propagation/scattering of waves in/from these components is important in various applications including ultrasonic nondestructive evaluation techniques. In this paper, we consider solving the problems of interaction of mechanical waves with transversely isotropic cylinders.

The propagation of stress waves in transversely isotropic media was first studied by Morse (1954) and Buchwald (1961) who independently developed the frequency equation for transversely isotropic cylinders and plates. Buchwald’s approach was based on potential functions while Morse used the series method for solving the governing equations. In 1965, Mirsky (1965) studied the problem of propagation of longitudinal waves in transversely isotropic cylinders based on an extension of Buchwald’s work. Several other researchers

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including Tsai et al. (1990, 1991), Nagy (1995), Ahmad (2001), Berliner and Solecki (1996) also studied the propagation of waves in transversely isotropic cylinders with the following contributions to this field: Tsai et al. (1990) and Tsai (1991) investigated the cylindrically guided waves in transversely isotropic shafts and thick hollow cylinders. Nagy (1995) formulated the problem of the propagation of longitudinal waves in fluid-loaded homogeneous transversely isotropic cylinders. Ahmad (2001) extended Nagy’s results to include flexural modes of free and immersed transversely isotropic cylinders by the potential functions method. Limited numerical results for propagation of waves in free cylinders were presented by Ahmad. Berliner and Solecki (1996) extended Mirsky’s work to the problem of propagation of flexural waves in transversely isotropic cylindrical shells.

Honarvar and Sinclair (1996) developed a mathematical model based on potential functions for the scattering of acoustic waves from transversely isotropic cylinders. Kim and Ih (2003) used this model to investigate the scattering of acoustic waves from transversely isotropic shells. Pan et al. (2004) also recently used this model to study the acoustic waves in isotropic cylinders. In the model developed by the authors (Honarvar and Sinclair, 1996), the expressions for displacement potentials were somehow guessed. In the current paper, a systematic solution to the governing differential equations is developed where there is no need to guess the form of the solutions. To some extent, this model will also be compared with other existing models. Numerical results for both longitudinal and flexural modes in transversely isotropic cylinders are presented. For results pertaining to longitudinal waves, this is the first time they are being obtained from the proposed model. For the case of flexural waves, the authors believe that no similar results have been reported in the literature.

2. Acoustic waves in transversely isotropic cylinders

Various approaches have been used for modeling wave propagation and wave scattering problems which involve transversely isotropic cylinders. These approaches have differences such that there have been discussions about their relative suitability (Honarvar and Sinclair, 1998) and ease of solution (Rahman and Ahmad, 1998).

In general, mathematical approaches used in solving wave equations in transversely isotropic cylinders can be divided into two models. The first model was used by Zhang et al. (1995), Honarvar and Sinclair (1996), and also by Pan et al. (2004). In the most general form, the wave field inside an elastic medium can be decomposed into three components, i.e. one compression (P) and two shear (SH and SV) wave components. In an anisotropic material, these waves are not necessarily independent. Following Morse and Feshbach (1953); Honarvar and Sinclair (1996) used the three scalar potential functions $\phi$, $\chi$ and $\psi$ to express the displacement field as follows,

$$
\nabla \phi + \nabla \times (\gamma e_\tau) + a \nabla \times \nabla \times (\psi e_\tau).
$$

The second approach, which was introduced by Buchwald (1961), has been used, with small changes, by Mirsky (1965), Niklasson and Datta (1998), Ahmad and Rahman (2000), and Pan et al. (2003). The expressions used for the displacement field by these authors are as follows,

Mirsky (1965) : $U = \nabla_\perp \phi + \nabla \times (\psi e_\tau) + \lambda \phi e_\tau$,

Niklasson and Datta (1998) : $U = \nabla_\perp \phi + \nabla \times (\gamma e_\tau) + \frac{\partial \psi}{\partial z} e_\tau$,

Ahmad and Rahman (2000) : $U = \nabla \phi + \nabla \times (\gamma e_\tau) + \frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z} e_\tau$,

where,

$$
\nabla_\perp = \frac{\partial}{\partial \rho} e_\rho + \frac{1}{\rho} \frac{\partial}{\partial \theta} e_\theta.
$$

For the purpose of brevity, we will hereafter refer to Eq. (1) as “Model A” and to Eqs. (2)–(4) as “Model B.” Model B has been introduced here so that when discussing the properties of Model A, one could compare these properties with those of the other models. The focus of this paper is on Model A and a detailed comparison between the two models is not intended. Model B has been used by many authors for solving both
propagation and scattering problems. As mentioned earlier, Model A has also been successfully used for modeling the scattering of waves from transversely isotropic cylinders but it has not been considered for solving wave propagation problems. In what follows, we will first investigate some useful properties of Model A and will then derive a systematic solution to the equations of wave propagation in transversely isotropic cylinders using this mathematical model.

3. Consistency with the physics of the problem

In the most general form, a wave field consists of a rotation-free P-wave and two dilatation-free (equivoluminal) S-waves. The polarization directions of these three waves are mutually perpendicular. The scalar potentials used in Model A, Eq. (1), satisfy these criteria.

Model A can also be examined against the Helmholtz second theorem. According to this theorem, any vector field can be expressed as a combination of two distinct fields; one dilatation-free and the other rotation-free (Morse and Feshbach, 1953). This condition is satisfied in Model A, where the term containing potential $\phi$ corresponds to a rotation-free field (curl of the gradient is zero), and the other two terms containing $\psi$ and $\chi$ are equivoluminal (divergence of the curl is zero). This point is illustrated by first taking the divergence and then the curl of the displacement field in Model A: the rotation-free and dilatation-free parts easily separate from each other,

$$\nabla \cdot \mathbf{U} = \nabla \cdot (\nabla \phi + \nabla \times (\chi \mathbf{e}_z) + a \nabla \times \nabla \times (\psi \mathbf{e}_z)) = \nabla \cdot \nabla \phi,$$

$$\nabla \times \mathbf{U} = \nabla \times (\nabla \phi + \nabla \times (\chi \mathbf{e}_z) + a \nabla \times \nabla \times (\psi \mathbf{e}_z)) = \nabla \times (\nabla \times (\chi \mathbf{e}_z) + a \nabla \times \nabla \times (\psi \mathbf{e}_z)).$$

(6)

(7)

However, Model B does not satisfy these same criteria.

4. Mathematical solution

In this section, a systematic approach for solving equations of Model A is presented. Using Model A, the displacement vector is written in terms of three scalar potential functions $\phi, \chi$ and $\psi$ as given in Eq. (1) where $a$ is the radius of the cylinder which is a constant with dimensions of length.

In a cylindrical coordinate system, Fig. 1, the equations of motion for a transversely isotropic material are as follows,

$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) \left\{c_{11} \nabla^2 \phi + (c_{13} + 2c_{44} - c_{11}) \frac{\partial^2 \phi}{\partial z^2} - \rho_c \frac{\partial^2 \phi}{\partial t^2}\right\} + a \frac{\partial}{\partial z} \left[ \left(c_{13} - c_{13} - c_{44}\right) \nabla^2 \psi + (c_{13} + 2c_{44} - c_{11}) \frac{\partial^2 \psi}{\partial z^2} - \rho_c \frac{\partial^2 \psi}{\partial t^2} \right] = 0,$$

$$\frac{\partial}{\partial z} \left[ \left(c_{13} + 2c_{44}\right) \nabla^2 \phi + (c_{33} - c_{13} - 2c_{44}) \frac{\partial^2 \phi}{\partial z^2} - \rho_c \frac{\partial^2 \phi}{\partial t^2} \right]$$

$$+ a \left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) \left[c_{44} \nabla^2 \psi + (c_{33} - c_{13} - 2c_{44}) \frac{\partial^2 \psi}{\partial z^2} - \rho_c \frac{\partial^2 \psi}{\partial t^2}\right] = 0,$$

$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) \left[\frac{(c_{11} - c_{12})}{2} \nabla^2 \chi + \frac{(c_{11} - c_{12})}{2} \frac{\partial^2 \chi}{\partial z^2} - \rho_c \frac{\partial^2 \chi}{\partial t^2}\right] = 0.$$

(8)

(9)

(10)

Fig. 1. The coordinate system used for derivation of the equations.
The compression wave, represented by $\phi$, and the vertically polarized shear wave, represented by $\psi$, are coupled. The horizontally polarized shear wave, represented by $\chi$, is uncoupled. Following a common method used for solving similar equations (e.g. see (Graff, 1991) and (Mirskey, 1965)), the solution to Eq. (8) can be obtained by setting the term in curly brackets equal to zero, i.e.:

$$
\begin{align*}
&\left[ c_{11} \nabla^2 \phi + (c_{13} + 2c_{44} - c_{11}) \frac{\partial^2 \phi}{\partial z^2} - \rho_c \frac{\partial^2 \phi}{\partial t^2} \right] \\
&+ a \frac{\partial}{\partial z} \left[ (c_{11} - c_{13} - c_{44}) \nabla^2 \psi + (c_{13} + 2c_{44} - c_{11}) \frac{\partial^2 \psi}{\partial z^2} - \rho_c \frac{\partial^2 \psi}{\partial t^2} \right] = 0,
\end{align*}
$$

(11)

if,

$$
(c_{11} - c_{13} - c_{44}) \nabla^2 \psi + (c_{13} + 2c_{44} - c_{11}) \frac{\partial^2 \psi}{\partial z^2} - \rho_c \frac{\partial^2 \psi}{\partial t^2} = 0,
$$

(12)

and,

$$
(c_{11} - c_{13} - c_{44}) \nabla^2 \phi + (c_{13} + 2c_{44} - c_{11}) \frac{\partial^2 \phi}{\partial z^2} - \rho_c \frac{\partial^2 \phi}{\partial t^2} = 0,
$$

(13)

where,

$$
\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},
$$

(14)

then Eq. (11) is satisfied. Expanding Eq. (12) gives,

$$
c_{11} \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + (c_{13} + 2c_{44} - c_{11}) \frac{\partial^2 \phi}{\partial z^2} - \rho_c \frac{\partial^2 \phi}{\partial t^2} = 0.
$$

(15)

Letting,

$$
\phi(r, \theta, z, t) = \phi_r(r)\phi_\theta(\theta)\phi_z(z)\phi_t(t)
$$

or in short form $\phi(r, \theta, z, t) = \phi_r\phi_\theta\phi_z\phi_t$,

(16)

and substituting Eq. (16) into Eq. (15), one gets,

$$
c_{11} \left( \phi_r'' + \frac{1}{r} \phi_r' + \frac{1}{r^2} \frac{\phi_r''}{\phi_r} \right) + (c_{13} + 2c_{44} - c_{11}) \frac{\phi_z''}{\phi_z} = 0.
$$

(17)

After factorizing out $\phi_r\phi_z\phi_t$, we have,

$$
c_{11} (\phi_\theta\phi_\phi\phi_t) \left[ \phi_r'' + \frac{1}{r} \phi_r' + \left( \frac{c_{13} + 2c_{44}}{c_{11}} \right) \phi_z'' \phi_t - \frac{\rho_c}{c_{11}} \frac{\phi_z''}{\phi_z} \phi_t \right] = 0.
$$

(18)

Eq. (18) can also be written as,

$$
c_{11} (\phi_\theta\phi_\phi\phi_t) \left[ \phi_r'' + \frac{1}{r} \phi_r' + \left( \frac{c_{13} + 2c_{44}}{c_{11}} \phi_z'' \phi_t - \frac{\rho_c}{c_{11}} \phi_z'' \phi_t + \frac{\phi_z''}{r^2} \phi_t \right) \phi_r \right] = 0.
$$

(19)

For a non-trivial solution, we must have,

$$
\phi_r'' + \frac{1}{r} \phi_r' + \left( \frac{c_{13} + 2c_{44}}{c_{11}} \phi_z'' \phi_t - \frac{\rho_c}{c_{11}} \phi_z'' \phi_t + \frac{\phi_z''}{r^2} \phi_t \right) \phi_r = 0.
$$

(20)

Eq. (20) is in the general form of Bessel’s differential equation, commonly presented in the form (Churchill, 1941),

$$
y'' + \frac{1}{x} y' + \left( s^2 - \frac{n^2}{x^2} \right) y = 0, \quad (s^2 \text{ and } n^2 \text{ are constants and } n \text{ is an integer}),
$$

(21)
Eq. (21) is the general form of Bessel’s differential equation. It can be observed that Eq. (20) has led to four ODE’s. By solving the three harmonic and one Bessel’s differential equations, the general solution to Eq. (12) is found to be of the form,

$$\phi(r, \theta, z, t) = \sum_{n=0}^{\infty} [B_{n1} J_n(sr) + B_{n2} Y_n(sr)][B_{n3} \cos(n\theta) + B_{n4} \sin(n\theta)](B_{n5} e^{ik_z z} + B_{n6} e^{-ik_z z})(B_{n7} e^{i\omega t} + B_{n8} e^{-i\omega t}),$$

(22)

where $J_n(sr)$ is the Bessel function of the first kind of order $n$ and $Y_n(sr)$ is the Bessel function of the second kind of order $n$.

Following a similar procedure, it can be shown that,

$$\psi(r, \theta, z, t) = \sum_{n=0}^{\infty} [C_{n1} J_n(sr) + C_{n2} Y_n(sr)][C_{n3} \cos(n\theta) + C_{n4} \sin(n\theta)](C_{n5} e^{ik_z z} + C_{n6} e^{-ik_z z})(C_{n7} e^{i\omega t} + C_{n8} e^{-i\omega t}),$$

and,

$$\chi(r, \theta, z, t) = \sum_{n=0}^{\infty} [D_{n1} J_n(sr) + D_{n2} Y_n(sr)][D_{n3} \cos(n\theta) + D_{n4} \sin(n\theta)](D_{n5} e^{ik_z z} + D_{n6} e^{-ik_z z})(D_{n7} e^{i\omega t} + D_{n8} e^{-i\omega t}).$$

(23)

(24)

According to the geometry and boundary conditions of the problem, appropriate combinations of the terms appearing in Eqs. (22)–(24) should be chosen. For example, in the case of the propagation of plane harmonic waves along a cylinder axis, i.e. along the z-direction, the potential functions should be of the following form:

$$\phi(r, \theta, z, t) = \sum_{n=0}^{\infty} B_{n} J_n(sr) \cos(n\theta)e^{(k_z z - i\omega t)},$$

(25)

$$\psi(r, \theta, z, t) = \sum_{n=0}^{\infty} C_{n} J_n(sr) \cos(n\theta)e^{(k_z z - i\omega t)},$$

(26)

$$\chi(r, \theta, z, t) = \sum_{n=0}^{\infty} D_{n} J_n(sr) \sin(n\theta)e^{(k_z z - i\omega t)}.$$

(27)

A portion of the general solution of $\phi$, i.e. all of the $Y_n(sr)$ terms, has been discarded because of its singular behavior at the origin. By substitution of Eqs. (25)–(27) into the governing equations, Eqs. (8)–(10) yield an eigenvalue problem where the eigenvalues are the phase velocities of wave modes and the eigenvectors are the corresponding polarization vectors, i.e.,

$$\begin{bmatrix}
-s^2((c_{11} s^2 - (\rho_c \omega^2 - (c_{11} + 2c_{44})k_z^2))) & -i k_z s^2((c_{11} - c_{11} - c_{44})) & 0 \\
-i k_z((c_{11} + 2c_{44})s^2 - (\rho_c \omega^2 - c_{11}k_z^2)) & a^2(c_{11} s^2 - (\rho_c \omega^2 - c_{11} - c_{44}k_z^2)) & 0 \\
0 & 0 & -s^2((c_{11} - c_{44})s^2 - 2(\rho_c \omega^2 - c_{44}k_z^2))
\end{bmatrix}
\begin{bmatrix}
B_{n1} \\
C_{n1} \\
D_{n1}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.$$ 

(28)

For a non-trivial solution, the coefficient determinant in Eq. (28) must be equal to zero. This yields the following characteristic equation,

$$s^4(s^2 + k_z^2)(c_{44}c_{11}s^4 - \xi s^2 + \zeta)
\left(\frac{c_{11} - c_{12}}{2}\right)s^2 - \rho_c \omega^2 + c_{44}k_z^2 = 0,$$

(29)

where,

$$\xi = (c_{11} + c_{44})k_z^2 + c_{11}(\rho_c \omega^2 - c_{33}k_z^2) + c_{44}(\rho_c \omega^2 - c_{44}k_z^2),$$

$$\zeta = (\rho_c \omega^2 - c_{44}k_z^2)(\rho_c \omega^2 - c_{33}k_z^2).$$

(30)

(31)

In Eq. (29), $s = 0$ is a trivial solution, and the solution corresponding to $s^2 = -k_z^2$ corresponds to an imaginary wave vector in the z direction, i.e., a disturbance that will die out exponentially without propagating. It should
be mentioned that transfer of Buchwald’s equations (Model B) from Cartesian to cylindrical coordinate system also results in the first extraneous root (Buchwald, 1961).

There are three meaningful solutions (eigenvalues), \( s_1, s_2 \) and \( s_3 \), for Eq. (29),

\[
\begin{align*}
\frac{s_1^2}{2c_{11}c_{44}} &= \xi - \sqrt{\xi^2 - 4c_{11}c_{44}\xi} - \frac{1}{2}, \\
\frac{s_2^2}{2c_{11}c_{44}} &= \xi + \sqrt{\xi^2 - 4c_{11}c_{44}\xi} - \frac{1}{2}, \\
\frac{s_3^2}{c_{11} - c_{12}} &= 2\left(\rho_c\omega^2 - c_{44}k_z^2\right).
\end{align*}
\]

(32)

(33)

(34)

By the method of eigenfunction expansion, we may expand \( \phi, \psi \) and \( \chi \) as Bessel function series. This yields the following form for \( \phi \),

\[
\phi = \sum_{n=0}^{\infty} [B_{n1}J_n(s_1r) + B_{n2}J_n(s_2r)] \cos(n\theta)e^{ik_zz - j\omega t}.
\]

(35)

In a similar way for \( \psi \) we will have,

\[
\psi = \sum_{n=0}^{\infty} [C_{n1}J_n(s_1r) + C_{n2}J_n(s_2r)] \cos(n\theta)e^{ik_zz - j\omega t}.
\]

(36)

It should be noted that the coefficients appearing in the above equations for \( \phi \) and \( \psi \) are not independent of each other. Using Eq. (28), it can be shown that,

\[
\begin{align*}
B_{n2} &= -\frac{\alpha i k_z((c_{11} - c_{13} - c_{44})s_2^2 - (\rho_c\omega^2 - (c_{44}k_z^2)))}{(c_{11}s_1^2 - (\rho_c\omega^2 - (c_{13} + 2c_{44})k_z^2))} C_{n2} = q_2C_{n2}, \\
C_{n1} &= -\frac{\alpha i k_z((c_{11} - c_{13} - c_{44})s_1^2 - (\rho_c\omega^2 - (c_{13} + 2c_{44})k_z^2))}{(c_{11}s_1^2 - (\rho_c\omega^2 - (c_{13} + 2c_{44})k_z^2))} B_{n1} = q_1B_{n1},
\end{align*}
\]

(37)

(38)

To make it simpler, we rename \( B_{n1} \) and \( C_{n2} \), as \( B_n \) and \( C_n \), respectively. Therefore, we have,

\[
\phi(r, \theta, z, t) = \sum_{n=0}^{\infty} [B_nJ_n(s_1r) + q_2C_nJ_n(s_2r)] \cos(n\theta)e^{ik_zz - j\omega t},
\]

(39)

\[
\psi(r, \theta, z, t) = \sum_{n=0}^{\infty} [q_1B_nJ_n(s_1r) + C_nJ_n(s_2r)] \cos(n\theta)e^{ik_zz - j\omega t},
\]

(40)

and for \( \chi \) we can write,

\[
\chi(r, \theta, z, t) = \sum_{n=0}^{\infty} D_nJ_n(s_1r) \sin(n\theta)e^{ik_zz - j\omega t}.
\]

(41)

5. Discussion

Using the equations obtained in the previous section, propagation of flexural and longitudinal guided waves in free transversely isotropic cylinders is studied in this section. To verify the mathematical model, it is first applied to an isotropic aluminum cylinder, and then the results obtained for homogeneous transversely isotropic cylinders are presented.

5.1. Flexural modes

For a cylinder in vacuum, the traction-free boundary conditions hold. Therefore, at \( r = a \):

\[
\sigma_{rr} = \sigma_{rz} = \sigma_{r\theta} = 0.
\]

(42)
Expanded expressions for the stress and displacement at any point can be derived in terms of potential functions. Inserting the potential functions of Eqs. (39)–(41) into Eq. (42) results in the following system of linear algebraic equations:

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  A_n \\
  B_n \\
  C_n
\end{bmatrix}
= 0.
\]  

(43)

The expressions for \(a_{ij}\) are given in the Appendix. The solution to Eq. (43) is nontrivial only if the determinant of the coefficients vanishes, i.e.,

\[
\det(a_{ij}) = \begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = 0.
\]  

(44)

Eq. (44) represents the frequency equation for flexural waves propagating along a transversely isotropic cylinder in vacuum.

5.2. Longitudinal modes

For longitudinal waves traveling along the cylinder axis, the displacement field is independent of the \(\theta\)-coordinate and is of the form \((u_r, 0, u_z)\). This mode of wave propagation corresponds to \(n = 0\) in Eqs. (39)–(41) and results in \(\chi = 0\) (Rose, 1999). The cylinder is free from stresses at its surface, therefore, the boundary conditions can be written as,

\[
\sigma_{rr} = 0 \quad \text{and} \quad \sigma_{rz} = 0 \quad \text{at} \quad r = a.
\]  

(45)

Therefore,

\[
\det(a_{ij}) = \begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix} = 0.
\]  

(46)

Eq. (46) is the frequency equation for longitudinal waves propagating along a transversely isotropic cylinder in vacuum.

6. Numerical results

The frequency equations derived in Section 5 show the relationship between the frequencies and the wave numbers (phase velocities) of various modes of flexural and longitudinal guided waves in a free, transversely isotropic cylinder in terms of its elastic constants. The wave numbers of such modes can be calculated at any frequency by numerically searching for the zeros of the corresponding frequency equation. To compare our results with those of other researchers, we shall present the frequency relation in terms of the frequency-dependent wave number for flexural modes. For longitudinal modes, this relation is presented in terms of the frequency-dependent phase velocity. Since propagating modes are of interest here, only real parts of the frequency equation are used for plotting the dispersion curves.

To verify the mathematical model, it is first used for solving the frequency equation of an isotropic aluminum cylinder, and then it is applied to two transversely isotropic materials, viz. glass/epoxy fiber-reinforced composite and cobalt. Elastic properties of aluminum, glass/epoxy composite, and cobalt are given in Table 1.

6.1. Flexural modes

Previous applications of Model A deal with the scattering of waves from solid and hollow cylinders. This is the first time this model is being used for studying propagation of waves in cylinders. To verify Model A for the case of waves traveling along a cylinder, it is first applied to an isotropic aluminum cylinder in a vacuum. Fig. 2 shows the frequency spectrum for different flexural modes of aluminum calculated by Model A.
These curves are identical to those presented in Fig. 4 of Pao (1962), indicating the validity of Model A for isotropic materials. The corresponding curves for glass/epoxy and cobalt with transversely isotropic elastic properties are shown in Figs. 3 and 4, respectively. The general behavior of the frequency curves of these transversely isotropic materials is similar to those of aluminum; however, no similar results for such materials exist in the literature for comparison. Following a common practice, the vertical frequency axes in Figs. 2–4 are normalized with respect to $c_b$, where $c_b = \sqrt{E_a/\rho}$ and $E_a$ is the axial Young’s modulus defined as $E_a = c_{33} - 2c_{13}^2/(c_{11} + c_{12})$. The intersection of a mode curve with the frequency axis $(c_l/c_b)$ indicates a cut-off in the sense that it is a propagation limit, i.e. a resonance with infinite wavelength. In the case of

<table>
<thead>
<tr>
<th>Material</th>
<th>$c_{11}$ (N/m$^2$)</th>
<th>$c_{12}$</th>
<th>$c_{13}$</th>
<th>$c_{33}$</th>
<th>$c_{44}$</th>
<th>Density (kg/m$^3$)</th>
<th>$c_L$ (m/s)</th>
<th>$c_T$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>1.103</td>
<td>0.543</td>
<td>0.543</td>
<td>1.103</td>
<td>0.280</td>
<td>2800</td>
<td>6420</td>
<td>3040</td>
</tr>
<tr>
<td>Glass/epoxy</td>
<td>0.198</td>
<td>0.055</td>
<td>0.063</td>
<td>0.576</td>
<td>0.089</td>
<td>1900</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Cobalt (Pao, 1962)</td>
<td>2.95</td>
<td>1.59</td>
<td>1.11</td>
<td>3.35</td>
<td>0.71</td>
<td>8900</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Fig. 2. Calculated frequency curves for the flexural modes of a homogeneous isotropic aluminum cylinder.

Fig. 3. Calculated frequency curves for the flexural modes of a homogeneous transversely isotropic glass/epoxy fiber-reinforced composite cylinder.
the lowest mode for aluminum, the low-frequency limit for the phase velocity is zero and the high-frequency limit of phase velocity is the Rayleigh wave velocity.

6.2. Longitudinal modes

The dispersion curves for longitudinal modes of aluminum and glass/epoxy were calculated and plotted using Model A. The graphs obtained were identical to those calculated by other approaches (Nagy, 1995; Rose, 1999), indicating that Model A produces correct results for longitudinal waves propagating in isotropic and transversely isotropic cylinders. As an example, the dispersion curves of cobalt for $0 \leq ka \leq 50$ are plotted in Fig. 5.

7. Summary and conclusions

A mathematical model which was already used for solving scattering problems in transversely isotropic cylinders was applied to propagation of both flexural and longitudinal waves in transversely isotropic cylinders in vacuum. A systematic solution was developed and consistency of the model with the physics of the problem was demonstrated. Numerical results for both isotropic and transversely isotropic cylinders were presented.

In conjunction with experimental data, this mathematical model can be used for nondestructive testing of cylinders and online monitoring of the variations of mechanical properties of cylindrical products.
Appendix A

Elements of the matrices given in Eqs. (44) and (46) are as follows,

\[
\begin{align*}
    a_{11} &= \left[c_{11} + ikaq_1(c_{11} - c_{12})\right]
    \left[(n^2 - n - s_1^2a^2)J_n(s_1a) + (s_1a)J_{n+1}(s_1a)\right] \\
    &+ \left[c_{12} + ikaq_1(c_{12} - c_{11})\right]
    \left[n^2J_n(s_1a) - (s_1a)J_{n+1}(s_1a)\right] \\
    &+ (-c_{13}k^2a^2 - c_{12}n^2 + in^2kaq_1(c_{11} - c_{12}))J_n(s_1a), \\
    a_{12} &= \left[c_{11}q_2 + ika(c_{11} - c_{13})\right]
    \left[(n^2 - n - s_2^2a^2)J_n(s_2a) + (s_2a)J_{n+1}(s_2a)\right] \\
    &+ \left[c_{12}q_2 + ika(c_{12} - c_{13})\right]
    \left[n^2J_n(s_2a) - (s_2a)J_{n+1}(s_2a)\right] \\
    &+ (-c_{13}q_2k^2a^2 - c_{12}q_2n^2 + in^2ka(c_{11} - c_{12}))J_n(s_2a), \\
    a_{13} &= n(c_{11} - c_{12})\left[(n - 1)J_n(s_3a) - (s_3a)J_{n+1}(s_3a)\right], \\
    a_{21} &= c_{44}\left[q_1(s_1^2a^2 - k^2a^2) + 2ika\right]
    \left[nJ_n(s_1a) - (s_1a)J_{n+1}(s_1a)\right], \\
    a_{22} &= c_{44}\left[q_1(s_2^2a^2 - k^2a^2) + 2ikaq_2\right]
    \left[nJ_n(s_2a) - (s_2a)J_{n+1}(s_2a)\right], \\
    a_{23} &= c_{44}(inka)J_n(s_3a), \\
    a_{31} &= n(c_{11} - c_{12})(1 + ikaq_1)\left[(1 - n)J_n(s_1a) + (s_1a)J_{n+1}(s_1a)\right], \\
    a_{32} &= n(c_{11} - c_{12})(q_3 + ika)\left[(1 - n)J_n(s_2a) + (s_2a)J_{n+1}(s_2a)\right], \\
    a_{33} &= \left(\frac{c_{11} - c_{12}}{2}\right) \left\{(s_1^2a^2 - 2(n - 1))J_n(s_3a) - 2(s_3a)J_{n+1}(s_3a)\right\}.
\end{align*}
\]

References


