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Nonlinear Equations Involving m-Accretive Operators

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Let X be a Banach space and T an m-accretive operator defined on a subset $D(T)$ of X and taking values in 2^X. For the class of spaces whose bounded closed and convex subsets have the fixed point property for nonexpansive self-mappings, it is shown here that two boundary conditions which imply existence of zeroes for T, appear to be equivalent. This fact is then used to prove that if there exists $x_0 \in D(T)$ and a bounded open neighborhood U of x_0, such that $|T(x_0)| < r \leqslant |T(x)|$ for all $x \in \partial U \cap D(T)$, then the open ball $B(0; r)$ is contained in the range of T.

Throughout this note we suppose X is a Banach space, and we use $B(x; r)$ to denote the open ball centered at $x \in X$ with radius $r > 0$ and ∂U to denote the boundary of a subset U of X. We also use the notation $|A| = \inf\{\|x\| : x \in A\}$, $A \subset X$ (see [1]).

An operator $T: D(T) \subset X \to 2^X$ is said to be accretive if for each $u, v \in D(T)$ and $r > 0$,

$$\|u - v\| \leqslant |u - v + r(T(u) - T(v))|.$$

If in addition the range of $I + rT$ is precisely X for all $r > 0$, then T is said to be *m-accretive*. For this class of operator, the resolvent $J_r = (I + rT)^{-1}$, $r > 0$, is a single-valued nonexpansive mapping whose domain is all X; also $R(T) = \{y : y \in T(x),\ x \in D(T)\}$ denotes the range of T.

The purpose of this work is to study the solvability of nonlinear equations of the type

$$z \in T(x), \tag{1}$$

where T is a multivalued m-accretive operator (without any continuity assumptions). We first show that, for the class of spaces X whose bounded closed and convex subsets have the fixed point property for nonexpansive self-mappings, a mapping $T: D(T) \subset X \to 2^X$ m-accretive has a zero iff there

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exists a bounded open neighborhood $U(x_0)$ (with $x_0 \in D(T)$) for which $t(x - x_0) \notin T(x)$ whenever $x \in \partial U \cap D(T)$ and $t < 0$. This theorem is then used to show that if for some $x \in D(T)$,

$$|T(x_0)| < r \leqslant |T(x)| \qquad \text{for all } x \in \partial U \cap D(T),$$

then $B(0; r) \subset R(T)$. This result represents a significant generalization of Theorem 2 of Kartsatos [3] who assumes (for single-valued mapping) that X and X^* are uniformly convex. We should also mention that our development is independent of existence theorems of differential equations.

Finally, we obtain a new result which implies the existence of solution of (1) for a certain type of single-valued operator satisfying some kind of sign condition at infinity.

We start showing an extension of an author's result ([6, Proposition 2]) for multivalued operators.

PROPOSITION. *Let X be a Banach space and D a subset of X (with $0 \in D$), and let $T: D \to 2^X$ be m-accretive. Then there exists a mapping $\Psi: (-\infty, 0) \to D$ defined by*

$$\Psi(t) = x_t$$

where $tx_t \in T(x_t)$. Moreover, this mapping Ψ has the following properties:

(i) *Ψ is continuous.*

(ii) *If $U \subset X$ is an open neighborhood of the origin for which $tx \notin T(x)$ for $x \in \partial U \cap D$ and $t < 0$, then $x_t \in U$ for $t < 0$.*

Proof. The existence of Ψ is an immediate consequence of the definition of m-accretivity.

(i) Let $t, s < 0$. Since T is accretive, then for each $r > 0$ we have

$$\begin{aligned} \|x_t - x_s\| &\leqslant |x_t - x_s + r(T(x_t) - T(x_s))| \\ &\leqslant \|x_t - x_s + r(tx_t - sx_s)\| \end{aligned} \tag{2}$$

and thus by choosing $r = -1/t$ we conclude that

$$\|\Psi(t) - \Psi(s)\| \leqslant \|x_s\| \, |t - s|/|t|.$$

(ii) Let $t < 0$ and $r = -1/t$. Then by (2),

$$\begin{aligned} |T(x_t)| &\leqslant |t| \, \|x_t\| \\ &\leqslant |t| \, |x_t - (T(x_t) - T(0))/t| \\ &\leqslant \|u_0\|, \end{aligned}$$

where $u_0 \in T(0)$. If $t \to -\infty$, then $x_t \to 0$. Hence there exists $t_0 < 0$ such that $x_{t_0} \in U$. Since by (i) the set of eigenvectors is connected, it follows by (ii) that $x_t \in U$ for all $t < 0$.

Now we prove the main theorem of this paper.

THEOREM 1. *Let X be a Banach space for which each nonempty bounded closed convex subset has the fixed point property for nonexpansive self-mappings, and let* $T: D(T) \subset X \to 2^X$ *be m-accretive. Then the following are equivalent*:

(a) $0 \in R(T)$;

(b) $E = \{x \in D(T): tx \in T(x)$ *for some* $t < 0\}$ *is bounded*;

(c) *there exists* x_0 *in* $D(T)$ *and a bounded open neighborhood* U *of* x_0, *such that*

$$t(x - x_0) \notin T(x) \qquad \textit{for } x \in \partial U \cap D(T) \textit{ and } t < 0. \tag{3}$$

Proof. It is easily seen that (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c), we first show that (b) $\Rightarrow$ (a). Choose $n \in \mathbb{N}$ and $x_n \in E$ for which $(-1/n)\, x_n \in T(x_n)$. Let $L = \limsup_{n \to \infty} \|x_n\|$. Then the set $K = \{x \in X: \limsup_{n \to \infty} \|x - x_n\| \leqslant L\}$ is a nonempty, bounded, closed, convex subset of X. Since $|TJ_\lambda x_n| \leqslant |Tx_n|$ (see, e.g., [4, proof of the lemma]) and $|Tx_n| \to 0$ as $n \to \infty$, K is also invariant under J_λ. Hence, by assumption J_λ has a fixed point which is a zero for T. (For equivalent versions of (b) $\Rightarrow$ (a), see [4, 8]).

To prove (c) $\Rightarrow$ (a), one may assume without loss of generality that $x_0 = 0$. Since by the Proposition the set E is bounded, then by the previous implication the proof is complete.

Next we show two interesting corollaries of Theorem 1. First, we observe that if X^* denotes the dual space of X, then the normalized duality mapping $J: X \to 2^{X^*}$ is defined by

$$J(x) = \{j \in X^*: (x, j) = \|x\|^2, \|j\| = \|x\|\}$$

and for each x and $y \in X$ we let

$$[y, x]_+ = \sup\{(y, j); j \in J(x)\}.$$

It is easily seen that $[\alpha x, x]_+ = \alpha \|x\|^2$ for all $\alpha \in \mathbb{R}$.

Our first corollary is precisely Theorem 2.1 of [10].

COROLLARY 1. *Let X be a Banach space whose nonempty closed convex* *and bounded subsets have the fixed point property for nonexpansive self-mappings, let $T: D(T) \to 2^X$ be m-accretive, and $z \in X$ for which*

$$\liminf_{\|x\| \to \infty} \frac{[y - z, x]_+}{\| y - z\|^q} \geqslant M(z) > -\infty, \qquad q \in [1, 2) \tag{4}$$

and $y \in T(x)$ is fulfilled. Then $z \in R(T)$.

Proof. Since the mapping $U = T - z$ is clearly m-accretive, it would be sufficient to show that the set E (defined in Theorem 1) is bounded. Suppose there is a sequence $\{x_n\} \subset E$ such that $\|x_n\| \to +\infty$ as $n \to \infty$, and $t_n x_n \in U(x_n)$ for $t_n < 0$. Then by (4) we have

$$\liminf_{n \to \infty} \frac{[t_n x_n, x_n]_+}{\|t_n x_n\|^q} = -\liminf_{n \to \infty} \frac{\|x_n\|^{2-q}}{|t_n|^{q-1}}.$$

Since the sequence $\{t_n\}$ is bounded, we conclude

$$\liminf_{n \to \infty} \frac{[t_n x_n, x_n]_+}{\|t_n x_n\|^q} = -\infty$$

which is a contradiction. It follows that E is bounded and, thus, Theorem 1 implies that U has a zero in $D(T)$. Therefore $z \in R(T)$

COROLLARY 2. *Let X, T and $D(T)$ as in Corollary* 1 *and suppose that for some $\delta > 0$ the set*

$$F = \{x \in D(T) \colon \| y\| \leqslant \delta \text{ for some } y \in T(x)\}$$

is nonempty and bounded. Then T has a zero.

Proof. Choose $z \in F$. Suppose $x \in D(T)$ for which $t(x - z) \in T(x)$ where $t < 0$. Then by the accretiveness of T,

$$\begin{aligned} \|x - z\| &\leqslant |x - z + r(T(x) - T(z))| \\ &\leqslant \|x - z + r(t(x - z) - y)\| \end{aligned}$$

for some $y \in T(z)$ with $\| y\| \leqslant \delta$. By selecting $r = -1/t$, we obtain $\|x - z\| \leqslant \| y\|/|t|$ and therefore $x \in F$. Since F is bounded, Theorem 1 completes the proof.

We now use Theorem 1 to prove a generalization of Theorem 2 of [3].

THEOREM 2. *Let X be Banach space for which each nonempty bounded closed convex subset has the fixed point property for nonexpansive self-*

mappings, let $T: D(T) \subset X \to 2^X$ be m-accretive with $x_0 \in D$, and suppose U is a bounded neighborhood of x_0 such that

$$|T(x_0)| < r \leqslant |T(x)| \qquad \text{for all } x \in \partial U \cap D(T). \tag{5}$$

Then $B(0; r) \subset R(T)$.

Before proving our Theorem 2, the following lemma will be used.

LEMMA. *Under the assumptions of Theorem* 1,

$$B(0; (r - |T(0)|)/2) \subset R(T).$$

Proof. Without loss of generality, we may assume that $x_0 = 0$ in (5). Let $T_z(x) = T(x) - z$ for $z \in B(0; (r - |T(0)|)/2)$. Since T_z is clearly m-accretive, it suffices to show that $tx \notin T_z(x)$ for $x \in \partial U \cap D(T)$ and $t < 0$. Suppose $tx \in T_z(x)$ where $x \in \partial U \cap D(T)$ and $t < 0$. Then the accretiveness of T_z implies

$$\begin{aligned} \|x\| &\leqslant |x + r(T_z(x) - T_z(0))| \\ &\leqslant \|x + rtx - ru\| \end{aligned}$$

for each $u \in T_z(0)$. By choosing $r = -1/t$, we conclude that $\|tx\| \leqslant |T_z(0)|$. On the other hand, since $x \in \partial U \cap D(T)$ and $\|z\| < (r - |T(0)|)/2$, condition (5) yields the fact

$$\begin{aligned} |T_z(0)| &\leqslant |T(0)| + \|z\| \\ &< \|tx + z\| - \|z\| \\ &\leqslant \|tx\|, \end{aligned}$$

which is a contradiction. Then part (c) of Theorem 1 is fulfilled, and hence $z \in R(T)$.

Proof of Theorem 2. We follow the argument of Kirk and Schöneberg in [4]. Let $z \in B(0; r)$ and define

$$\mathscr{E} = \{t \in [0, 1]: tz \in R(T)\}.$$

By the lemma, $\mathscr{E} \neq \varnothing$. Let $t_0 = \sup \mathscr{E}$. Select $t_n \in \mathscr{E}$ with $t_n \to t_0^-$ as $n \to \infty$ and let $z_n \in D(T)$ for which $t_n z \in T(z_n)$. Define T_n, by

$$T_n(x) = T(x + z_n) - t_n z, \qquad x \in D(T) - z_n.$$

Then $0 \in T_n(0)$, and if $x \in \partial U \cap D(T) - z_n$,

$$|T_n(x)| \geqslant r - t_n \|z\|.$$

Since T_n is m-accretive and satisfies (5), the lemma implies

$$B(0; (r - t_n \|z\|)/2) \subset R(T_n).$$

However, there exists $t \geqslant t_0$ and $n \in \mathbb{N}$ so that

$$(t - t_n) \|z\| < (r - t_n \|z\|)/2,$$

and therefore we may select $w_n \in D(T) - z_n$ such that $(t - t_n) z \in T_n(w_n)$, implying that $tz \in T(w_n + z_n)$. Hence $t \in \mathscr{E}$ and thus $t_0 = 1 \in \mathscr{E}$.

By a slight modification of the set F mentioned in Corollary 2, we are able to improve this result as follows.

COROLLARY 3. *Let X, T and $D(T)$ as in Theorem 2 and suppose that for some $\delta > 0$ the set*

$$F = \{x \in D(T) : \| y\| < \delta \text{ for some } y \in T(x)\}$$

in nonempty and bounded. Then $B(0; \delta) \subset R(T)$.

Proof. Since F is nonempty and bounded, there exist $x_0 \in F$ and an open ball $B(x_0; \mu)$ for some $\mu > 0$ such that $F \subset B(x_0; \mu)$. Therefore

$$|T(x_0)| < \delta \leqslant |T(x)| \qquad \text{for } x \in \partial B(x_0; \mu) \cap D(T).$$

Then Theorem 2 implies that $B(0; \delta) \subset R(T)$.

We now prove Theorem 3 of Kirk and Schöneberg [4] for the class of spaces already mentioned in previous theorems.

COROLLARY 4. *Let X, T and $D(T)$ as in Theorem 2. Suppose for some $x_0 \in D(T)$,*

$$|T(x_0)| < r \leqslant \liminf_{\substack{\|x\| \to \infty \\ x \in D(T)}} |T(x)|. \tag{6}$$

Then $B(0; r) \subset R(T)$.

Proof. Inequality (6) implies the existence of an $\varepsilon > 0$ so that $\inf\{|T(x)| : \|x\| \geqslant \varepsilon\} \geqslant r$. By choosing $\delta > 0$ (large enough) such that $B(0; \varepsilon) \subset B(x_0; \delta)$, we conclude that

$$|T(x_0)| < r \leqslant |T(x)| \qquad \text{for} \quad x \in \partial B(x_0; \delta) \cap D(T)$$

and thus, by Theorem 2, the proof is complete.

Theorem 2 will be used in the proof of our next result for single-valued mappings. We recall that a mapping $T: D \subset X \to X$ is said to be *ϕ-expansive* on D if for every $x, y \in D$

$$\|T(x) - T(y)\| \geqslant \phi(\|x - y\|),$$

where ϕ is a strictly increasing mapping from $\mathbb{R}^+$ into $\mathbb{R}^+$ with $\phi(0) = 0$.

THEOREM 3. *Let X be as in Theorem 2. Suppose $T: D \subset X \to X$ is m-accretive and ϕ-expansive mapping on D. Then $T(G)$ is open whenever $G \subset D$ is open. Moreover, if $B(x_0; r) \subset D$ for some $x_0 \in D$ and $r > 0$, then $B(T(x_0); \phi(r)) \subset T(B(x_0, r))$.*

Proof. Let $T(x) = \tilde{T}(x + x_0) - T(x_0)$, $x \in D - x_0$. Then

$$\|\tilde{T}(x)\| > \phi(r) > \|\tilde{T}(0)\| = 0$$

for all $x \in \partial B(0; r)$. Since $\tilde{T}$ is also m-accretive, Theorem 2 implies that $B(0; \phi(r)) \subset R(\tilde{T})$ and thus $B(T(x_0); \phi(r)) \subset T(B(x_0; r))$. The openness of $T(G)$ is an immediate consequence of the above.

Finally, we prove a new result which involves condition (3) introduced in Corollary 1 for $q = 1$. We first review some definitions. Let X be a Banach space and $A \subset X$. Following [5] we define the measure of noncompactness of A, $\gamma[A] = \inf\{d > 0: A$ can be covered by a finite number of sets of diameter $d\}$. A continuous mapping $T: D \to X$, $D \subset X$, is called a *condensing* (or densifying) mapping [2, 8] if $\gamma[T(A)] < \gamma[A]$ for all bounded set $A \subset D$ with $\gamma[A] > 0$. It follows immediately that any compact operator is condensing.

THEOREM 4. *Let X be a Banach space, z an element of X and $T: X \to X$ a mapping satisfying*

(i) *$I - T$ is condensing on X;*

(ii) $$\liminf_{\|x\| \to \infty} \frac{[T(x) - z, x]_+}{\|T(x) - z\|} \geqslant M(z) > -\infty.$$

Then $z \in R(T)$.

Proof. Let $U(x) = x - T(x) + z$. Since U is also condensing, it suffices to show that the set

$$E = \{x \in X: U(x) = \lambda x \text{ for some } \lambda > 1\}$$

is bounded. Suppose there exists a sequence $\{x_n\}$ in E for which $\|x_n\| \to +\infty$ as $n \to \infty$. Then

$$
\begin{aligned}
\liminf_{n\to\infty} \frac{[T(x_n)-z, x_n]_+}{\|Tx_n - z\|} &= \liminf_{n\to\infty} \frac{[(1-\lambda_n)x_n, x_n]_+}{\|(1-\lambda_n)x_n\|} \\
&= \liminf_{n\to\infty} \frac{(1-\lambda_n)\|x_n\|^2}{(\lambda_n - 1)\|x_n\|} \\
&= -\liminf_{n\to\infty} \|x_n\| \\
&= -\infty,
\end{aligned}
$$

which is a contradiction. Hence E is bounded. Since $E \subset B(0, R)$ for some $R > 0$, the operator U satisfies the Leray–Schauder condition on ∂B (i.e., $U(x) \neq \lambda x$ for $x \in \partial B$ and $\lambda > 1$). Therefore, Theorem 7 of [7] implies that U has a fixed point in X, and thus $z \in R(T)$.

The following corollary is an immediate consequence of Theorem 4, being an interesting generalization of Proposition 1.2 of Torrejón [10].

COROLLARY 5. *Let X be a Banach space of finite dimension, z an element of X and $T: X \to X$ a continuous mapping satisfying*

(i) $$\liminf_{\|x\|\to\infty} \frac{[T(x)-z, x]_+}{\|T(x)-z\|} \geqslant M(z) > -\infty.$$

Then $z \in R(T)$.

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