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Gabriel topologies on coherent quantales[1]

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Abstract

The set of Gabriel topologies on a coherent quantale ordered under inclusion is a frame (studied by Rosenthal, Simmons and others). The set of all those Gabriel topologies that are inaccessible by directed joins (we call such topologies compact) is a subframe of it. When the quantale under consideration is commutative the frame of compact topologies is coherent. Several notions of spectra in ring theory appear as instances of this construction. When the quantale is non-commutative and coherent and its finite elements are closed under (right) implication then the frame of compact topologies is locally compact and compact. We present an interpretation of the notion of compact Gabriel topology on a coherent quantale in terms of deductively closed sets of formulae for a system of propositional logic without the contraction and possibly the exchange rule (but admitting weakening). Our local compactness (and the subsequent spatiality) results for the frame of compact topologies correspond to a completeness theorem for such a system. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

A quantale is a complete lattice Q equipped with an associative binary operation &, which satisfies

$$a\&(\bigvee b_i) = \bigvee_i(a\&b_i) \text{ and } (\bigvee b_i)\&a = \bigvee_i(b_i\&a),$$

for all elements $a \in Q$ and sets of elements $\{b_i \in Q \mid i \in I\}$. This implies the existence of two operators $a \to_{\mathrm{r}} -$ and $a \to_{\mathrm{l}} -$, right adjoint to $a\& -$ and $-\&a$, respectively. All quantales considered in this work will be assumed to be *right-sided*, meaning that $a\&\top \leqslant a$ for all $a \in Q$, where $\top$ is the top element of the lattice. This implies that, for

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all $a, b \in Q$, $a \& b \leqslant a$ and that, if there is a unit element 1 for &, then $1 = \top$. An excellent introduction to the theory of quantales is given in [7] and is probably the only one where the subject is studied systematically.

We study here Gabriel topologies on quantales, a notion that generalizes directly the classical one in ring theory, studied extensively in the 1960s in connection with localizations of Abelian categories and culminating in works like [10]. The notion of Gabriel topology on a quantale makes sense only when the quantale is algebraic as a lattice, i.e. the finite elements generate. (The finite elements are also called compact in the literature; however, we reserve the term compact for other purposes.) The finite elements of the quantale play the role of the elements of the ring in the classical definition of a Gabriel topology on a category of modules. Further, in order to obtain any results about the set of topologies we need to assume that the quantale is coherent, i.e. that the top element is finite and that the finite elements are closed under &.

The set of Gabriel topologies on a coherent quantale is a frame [7, Proposition 4.5.8]. We give a proof of that result in the first section for the sake of completeness and since no explicit proof of this result appears in the literature. The proof indicated in [7], by adapting an argument in [9], requires an unnecessary (in this connection) complication: that of the correspondence of Gabriel topologies with a certain kind of nucleus. Our proof uses instead the traditional view of such topologies as filters.

The focus of this work is the set of Gabriel topologies that are inaccessible by directed joins (we call such topologies compact). We show that this set is a subframe of the frame of all topologies. When the quantale is commutative we show that the frame of compact topologies is coherent. In Section 2 we describe the topologies that constitute the finite elements of the coherent frame in hand. As it is the case with every coherent frame, the finite elements of it form a distributive lattice. In Section 3 we show that this distributive lattice has a certain universal property. In the non-commutative case, when the finite elements of the quantale are closed under implication, then the frame turns out to be a locally compact and compact one.

In Section 3 we present some applications to the construction of various types of ring spectra. The frame of compact Gabriel topologies (on the quantale of ideals) of a commutative ring is isomorphic to (the frame of opens of) the dual Zariski spectrum of the ring. Similar descriptions are given for the Brumfiel spectrum of a commutative ordered ring as well as for the real Zariski spectrum of a commutative ring, by applying our construction to the quantale of convex ideals and real ideals, respectively.

Finally, we present an interpretation of the notion of compact Gabriel topology on a coherent quantale in terms of deductively closed sets of sentences (theories) for a system of propositional logic without the contraction and possibly the exchange rule (but admitting weakening). The calculus of these theories forms to a great extent the basis for the semantical analysis of these logics. Through our interpretation we are in a position to give some heretofore unobserved results in this connection. Our local compactness (and the subsequent spatiality) results for the frame of compact topologies correspond to a completeness theorem for such a system.

These results extend certain parts of the author's Ph.D thesis [5], where we dealt with Gabriel topologies on categories of algebras for finitary theories, whose unary operations are central. In particular, the connection between Gabriel topologies on modules over a commutative ring and the Zariski spectrum was first obtained there.

1. Gabriel topologies and compact Gabriel topologies on quantales

The definition of a Gabriel topology on a quantale that follows has been proposed by Rosenthal in [7, Definition 4.5.5]:

Definition 1.1. Let Q be an algebraic, right sided quantale. A filter $S \subseteq Q$ is called a Gabriel topology on Q if it satisfies

(T1) If $b \in S$ and c is a finite element of Q then $c \to_{\mathrm{r}} b \in S$.

(T2) If $b \in S$ and, for all $c \leqslant b$, where c is finite, it is the case that $c \to_{\mathrm{r}} a \in S$, then $a \in S$.

Remarks. (a) We could postulate that S is non-empty and satisfies (T1) and (T2). Then we can deduce that S is a filter by well known arguments as they appear in [10]. The fact that they apply to our case is secured by the requirement that the quantale is right sided.

(b) When the quantale Q is that of the right ideals on a ring this notion of Gabriel topology coincides with the classical one of a Gabriel topology on a ring: Considering that the finite elements of the quantale are in this case the finitely generated ideals, we have that the above axioms imply the classical ones, since, for an element a and an ideal I of the ring, $\langle a \rangle \to_{\mathrm{r}} I = I : a$. They are also implied, due to the fact that

$$\langle a_1, \dots, a_n \rangle \to_{\mathrm{r}} I = I : \langle a_1, \dots, a_n \rangle \supseteq (I : a_1) \cap \dots \cap (I : a_n).$$

We will need the auxiliary notion of a pretopology on Q: A *pretopology* is a non-empty, upper closed $S \subseteq Q$, which satisfies (T1).

Topologies are obviously closed under intersection so, for every $X \subseteq Q$, there is a smallest topology containing X, in particular there is a smallest topology generated by a pretopology P. It can also be described by means of the following process, familiar from the ring case. Set:

$$S_0 = P,$$

$$S_{\alpha+1} = \{a \in Q \mid \exists b \in S_\alpha \text{ such that, } \forall c \leqslant b,\ c \text{ finite, } c \to_{\mathrm{r}} a \in S_\alpha\},$$

$$S_\lambda = \textstyle\bigcup_{\alpha < \lambda} S_\alpha, \text{ where } \lambda \text{ is a limit ordinal.}$$

This process stabilizes at some ordinal β and the required completion is $S = \bigcup_{\alpha \leqslant \beta} S_\alpha$.

We use the process above to prove a well-known result, whose proof though does not appear explicitly in [7] or in the references given there.

Theorem 1.1. *The set of Gabriel topologies, top(Q) on a* (pre) *coherent, right sided quantale Q, ordered by inclusion is a frame.*

Proof. As mentioned above, the intersection of any family of Gabriel topologies is a Gabriel topology so top(Q) is a complete lattice. The supremum of a family of topologies is not their set-theoretic union since it fails to satisfy (T2). It is, however, a pretopology, and the topology it generates is obviously the supremum of that family.

In order to show the infinite distributive law, consider a topology T and a set $\{S_i \mid i \in I\}$ of topologies. We need to show that $(\bigvee S_i) \cap T \subseteq \bigvee (S_i \cap T)$, the other inequality holding trivially. If the process of forming $\bigvee S_i$, by completing the union of the S_i's under (T2), stabilizes at some ordinal β, it suffices to show that, for all ordinals $\alpha \leqslant \beta$, $(\bigcup S_i)_\alpha \cap T \subseteq (\bigcup (S_i \cap T))_\alpha$.

The claim is obviously true at the initial step and follows immediately at a limit ordinal step. So assuming that the claim holds for α, let us show it for $\alpha + 1$. Let $b \in (\bigcup S_i)_{\alpha+1} \cap T$. That means that $b \in T$ and there is an $a \in (\bigcup S_i)_\alpha$ such that, for all $c \leqslant a$, c finite, $c \to_r b \in (\bigcup S_i)_\alpha$. Then $b \vee a \in (\bigcup S_i)_\alpha \cap T \subseteq (\bigcup (S_i \cap T))_\alpha$. Suppose that $f \leqslant b \vee a$ is finite. By the algebraicity of Q, a and b can be written as suprema of finite elements below them, $a = \bigvee \{h_j \mid j \in J\}$, $b = \bigvee \{g_k \mid k \in K\}$, so $f \leqslant g \vee h$ for some finite $g = g_k \leqslant b$ and $h = h_j \leqslant a$. From this we get

$$f \to_r b \geqslant (g \vee h) \to_r b = (g \to_r b) \wedge (h \to_r b) = h \to_r b.$$

The latter equality follows by the right-sidedness of Q, since in this case the top element $\top$ of the quantale equals the multiplicative unit 1, hence $g = 1 \& g \leqslant b$ implies $1 = \top \leqslant g \to_r b$. But $h \to_r b \in (\bigcup S_i)_\alpha$ by the choice of h and $h \to_r b \in T$, since T is a topology. It follows that $f \to_r b \in (\bigcup S_i)_\alpha \cap T$, hence $b \in (\bigcup (S_i \cap T))_{\alpha+1}$. □

Remark. (a) We do not use the fact that the top element of the quantale is finite in the above proof. Hence the word coherent appears prefixed in the statement of the theorem. It appears also as if we have not used the fact that the finite elements of the quantale are closed under &, thus having arrived at a more general result than that in [7]. However, this is not the case, as precoherence is used in showing that the transfinite construction above yields indeed a Gabriel topology.

(b) The proof above can be modified appropriately so that it gives the well-known result that the set of Gabriel topologies on a ring is a frame. It cannot, though, be applied directly as the quantale of right ideals may not be coherent in the non-commutative case (as the product of two finitely generated ideals need not be finitely generated). The result could be obtained using the formalism of quantales but a more general notion of Gabriel topology, introduced in [8]. We do not wish to pursue Gabriel topologies in that generality here.

Definition 1.2. A topology S on an algebraic, right-sided quantale Q is called compact if, whenever $\{a_i \mid i \in I\}$ is a directed family of elements of Q with $\bigvee \{a_i \mid i \in I\} \in S$, then there is $a_i \in S$ for some $i \in I$.

The following is a fundamental technical result concerning the notion of compact topologies and is going to be used extensively in the sequel.

Lemma 1.2. *Let P be a pretopology on a precoherent, right-sided quantale Q satisfying the compactness property, i.e. that whenever $\{a_i \mid i \in I\}$ is a directed family of elements of Q with $\bigvee\{a_i \mid i \in I\} \in P$, then there is $a_i \in P$, for some $i \in I$. Then the topology S generated by P is compact.*

Proof. By the transfinite construction of S it suffices to show that for all ordinals α, all directed sets $\{a_i \mid i \in I\}$, if $\bigvee\{a_i \mid i \in I\} \in S_\alpha$, then there is $a_i \in S_\alpha$, for some $i \in I$. Inspecting the construction and arguing inductively we are only presented with a problem at the successor ordinal step.

Suppose that the claim holds for α and let $x = \bigvee\{a_i \mid i \in I\} \in S_{\alpha+1}$, where $\{a_i \mid i \in I\}$, is directed. There is a $b \in S_\alpha$ so that for all finite $c \leqslant b$, $c \to_{\mathrm{r}} x \in S_\alpha$. The quantale is algebraic so we can write $b = \bigvee\{c_k \mid k \in K\}$, with the c_k's finite. From the inductive hypothesis, there is a $k \in K$ with $c_k \in S_\alpha$. For that specific c_k now

$$c_k \to_{\mathrm{r}} \bigvee\{a_i \mid i \in I\} = \bigvee\{c_k \to_{\mathrm{r}} a_i \mid i \in I\} \in S_\alpha.$$

(The equality holds in a coherent quantale, for all finite elements c and directed families $\{a_i \mid i \in I\}$: If f is a finite element of Q such that $f \leqslant c \to_{\mathrm{r}} \bigvee\{a_i \mid i \in I\}$, equivalently, $f \& c \leqslant \bigvee\{a_i \mid i \in I\}$, as the finite elements were assumed to be closed under &, there is $j \in I$ such that $f \& c \leqslant a_j$, equivalently, $f \leqslant c \to_{\mathrm{r}} a_j$.) Using once more now the inductive hypothesis, there is $i \in I$ with $c_k \to_{\mathrm{r}} a_i \in S_\alpha$. Then, for all finite $e \leqslant c_k$, $e \to_{\mathrm{r}} a_i \in S_\alpha$, since $e \to_{\mathrm{r}} a_i \geqslant c_k \to_{\mathrm{r}} a_i$. We conclude that $a_i \in S_\alpha$, for that specific $i \in I$. □

The lemma has as a consequence the following result, which provides the focal point of this work.

Theorem 1.3. *The set* $\mathrm{top}_\omega(Q)$ *of compact topologies on a coherent, right-sided quantale Q is a frame (a subframe of that of all Gabriel topologies).*

Proof. First we show that compact topologies are closed in the set of all topologies under the formation of arbitrary suprema. Let $\{S_i \mid i \in I\}$ be a set of compact topologies. Its supremum in $\mathrm{top}(Q)$ is the completion S of the pretopology $\bigcup\{S_i \mid i \in I\}$, which obviously satisfies the compactness property. So by the preceeding lemma S is compact. If I is empty then its supremum, the smallest topology $\{\top\}$, consisting alone of the top element of the quantale, is compact since we assumed that the quantale is coherent.

Next, we show that compact topologies are closed in the set of all topologies under the formation of finite infima. The empty infimum being the largest topology, that is all of Q, is obviously compact. Binary infima are given by intersections, so let $\bigvee\{a_i \mid i \in I\} \in S \cap T$, where $\{a_i \mid i \in I\}$ is directed and S, T are compact. There are $i \in I$

and $j \in I$ with $a_i \in S$ and $a_j \in T$. We can find $k \in I$ with $a_i \leqslant a_k$ and $a_j \leqslant a_k$. Then $a_k \in S \cap T$, so $S \cap T$ is compact. □

2. The coherence of the frame of compact topologies

We investigate here the compactness, local compactness and coherence of the frame of compact Gabriel topologies. In order to show the two latter properties we impose further conditions on the quantale than those discussed in the first section. Compactness though holds more generally by virtue of the next lemma.

Lemma 2.1. *The union of a directed set of compact topologies is a compact topology. In other words, the supremum of a directed family of compact topologies is given by its set-theoretic union.*

Proof. Let $\{S_i \subseteq Q \mid i \in I\}$ be a directed set of compact topologies. We have to argue that $\bigcup S_i$ satisfies (T2) of Definition 1.1. Suppose that $b \in Q$ is such that there is an $a \in \bigcup S_i$ such that, for all finite $c \leqslant a$ we have that $c \rightarrow_{\mathrm{r}} b \in \bigcup S_i$. There is an $i \in I$ such that $a \in S_i$ and for every finite $c \leqslant a$ there is j depending on c, such that $c \rightarrow_{\mathrm{r}} b \in S_j$. Writing a as the directed supremum of elements smaller than it, $a = \bigvee\{c_k \mid k \in K\}$, we find from compactness a $k \in K$ so that $c_k \in S_i$. Let then $j = j(c_k)$ be the index for which $c_k \rightarrow_{\mathrm{r}} b \in S_j$. Let l be such that $S_i \subseteq S_l$ and $S_j \subseteq S_l$. Then $c_k \in S_l$ and, for all finite $e \leqslant c_k$, $e \rightarrow_{\mathrm{r}} b \in S_l$. So $b \in S_l \subseteq \bigcup S_i$, showing that $\bigcup S_i$ satisfies (T2). □

Proposition 2.2. *The frame of compact topologies on a coherent, right-sided quantale is compact.*

Proof. Let the directed set of compact topologies $\{S_i \subseteq Q \mid i \in I\}$ have as a supremum the largest topology, i.e. the quantale Q itself. From the preceeding lemma that means that $\bigcup S_i = Q$. So there is an $i \in I$ with the bottom element of Q belonging to S_i, in which case we have that $S_i = Q$. □

The local compactness of the frame of compact topologies amounts to the existence of sufficiently many finite elements of the frame. We try to produce them as the smallest topologies containing a given finite element of the quantale. In order to ensure that what we get is a compact topology we will have to restrict the class of quantales that we are considering. The well behaving quantales are the ones satisfying either of the following two properties:

(C1) Commutativity, i.e. for all a, b in Q, $a \& b = b \& a$.

(C2) For all finite elements c, d of Q, $c \rightarrow_{\mathrm{r}} d$ is again finite.

The examples of quantales that we intend to capture with the first condition are those of ideals of a commutative ring or quotients of such by finitary quantic nuclei, as well as that of coherent frames. Also the results apply to the quantale of subalgebras of the free algebra on one generator $F(1)$ for an algebraic theory the unary operations of which are central. In this case the notion of Gabriel topology discussed here coincides with the generalized notion of topology on a category of algebras as presented in [1], a situation studied in [5]. The notion of product of two subalgebras, which makes the complete lattice of subalgebras into a quantale, is the following: Using the correspondance between elements of $F(1)$ and unary operations of the theory, let $s(x).t(y)$, where $x \in S$, $y \in T$, and S, T are subalgebras, denote the element of $F(1)$ corresponding to the composite of the operations $s(x)$ and $t(y)$, where now s, t are endomorphisms of $F(1)$ corresponding to unary operations. The product $S \& T$ is then the subalgebra spanned by all such elements.

The condition (C2) is inspired by the quantale of right ideals of a ring with the property that for every finitely generated right ideal F, every element a of the ring, $F : a$ is again finitely generated. Such rings are called coherent [10, I.10]. If the ring is non-commutative then the quantale of right ideals will usually fail to be coherent. The arguments that follow though apply *mutatis mutandis* to the case of (right-) coherent rings, showing that the frame of compact Gabriel topologies on such a ring is locally compact. The result in this form appears also in [5].

Lemma 2.3. *Let Q be a coherent, right sided quantale satisfying either one of the conditions* (C1), (C2) *above. Then the smallest topology containing a finite element $f \in Q$, denoted by S_f, is compact. Moreover, it is a finite element of the frame of compact topologies.*

Proof. If Q satisfies (C1) then the principal filter $\uparrow(f)$ is a pretopology because, if $c \in Q$ is finite then $f \leqslant c \to_{\mathrm{r}} f$, since $f \& c = c \& f \leqslant f$. Then, in view of the Lemma 1.2, S_f is compact.

If Q satisfies (C2) then $\uparrow(f)$ need not be a pretopology but there is a smallest such that contains it given by

$$P_f = \{b \in Q \mid \forall c \in Q,\ c \text{ finite},\ c \to_{\mathrm{r}} f \leqslant b\}$$

(the reader could consult [5] for the details of the anyway easy argument). Then (C2) ensures that P_f has the compactness property so that, by Lemma 1.2 again, S_f is compact.

In either case let $S_f \subseteq \bigcup T_i$, where $\{T_i \mid i \in I\}$ is a directed set of compact topologies (so, by Lemma 2.1, $\bigcup T_i$ is its supremum). There is $i \in I$ with $f \in T_i$ so, since S_f is the smallest topology containing f, $S_f \subseteq T_i$, proving the second claim. □

Proposition 2.4. *If Q is a coherent, right sid ed quantale satisfying either of* (C1) *or* (C2), *then* $\mathrm{top}_\omega(Q)$ *is a locally compact frame.*

Proof. We want to show that every compact topology T is the supremum of the topologies of the form S_f which it contains. In other words, if R is another topology and for all S_f, $S_f \subseteq T$ implies $S_f \subseteq R$, then $\mathrm{T} \subseteq R$. Let $a \in T$. Writing a as the supremum of the finite elements f which it contains, we find $f \leqslant a$ with $f \in T$, or equivalently $S_f \subseteq T$. That implies $S_f \subseteq R$, or, $f \in R$, from which we get that $a \in R$. □

From this point on we turn our attention exclusively to commutative coherent quantales. We analyze the topologies S_f in this case and arrive at the proof of coherence for $\mathrm{top}_\omega(Q)$.

Lemma 2.5. *If f is a finite element of the commutative, coherent, right sided quantale Q and b any element of Q then, with the above notations, $b \in S_f$ iff there is an $n \in \mathbb{N}_0$ with $f^n \leqslant b$, where $f^n = f \& \cdots \& f$* (n times).

Proof. Suppose $b \in S_f = D$. If $b \in \uparrow(f)$ then $f \leqslant b$. If $b \in \uparrow(f)_\lambda = \bigcup_{\alpha<\lambda} \uparrow(f)_\alpha$ then, there is an $\alpha < \lambda$, such that $T \in \uparrow(f)_\alpha$ and the conclusion is true of all $\alpha < \lambda$.

If $b \in \uparrow(f)_{\alpha+1}$, then there is an $a \in \uparrow(f)_\alpha$ such that, for all finite $c \leqslant a$, $c \to b \in \uparrow(f)_\alpha$. There is a $k \in \mathbb{N}_0$ such that $f^k \leqslant a$ and $f^m \leqslant c \to b$. In particular, for the finite $c = f^k \leqslant a$ we get that $f^m \& f^k = f^{m+k} \leqslant b$.

Conversely, suppose that there is a natural number n such that $f^n \leqslant b$. We claim that $b \in \uparrow(f)_n$. This is so because for all finite $c_1 \leqslant f$, we have that $c_1 \to b \in \uparrow(f)_{n-1}$. We verify this by fixing c_1 and examining whether $c_2 \to (c_1 \to b) \in \uparrow(f)_{n-2}$, for all finite $c_2 \leqslant f$. After $n-1$ steps the problem reduces to whether, for all finite $c_{n-1} \leqslant f$, it is the case that $c_{n-1} \to (\ldots(c_2 \to (c_1 \to b)\ldots) \in \uparrow(f)$. This is indeed the case because $c_{n-1} \& \cdots \& c_1 \& f \leqslant f^n \leqslant b$. □

Proposition 2.6. *The topologies of the form S_f are closed under intersection. More precisely, for two finite elements f, g of a coherent commutative, right sided Q, $S_f \cap S_g = S_{f \vee g}$.*

Proof. It is immediate that $S_{f \vee g} \subseteq S_f \cap S_g$. For the other inequality, suppose that $b \in S_f \cap S_g$. From the lemma above there are $m, n \in \mathbb{N}_0$ with $f^n \leqslant b$ and $g^m \leqslant b$. So $f^n \vee g^m \leqslant b$ and we can use this as to show, exploiting the distributive law of the quantale, that $(f \vee g)^{\max(m,n)} \leqslant b$. Once more the previous lemma gives that $b \in S_{f \vee g}$. □

Summarizing the Propositions 2.2, 2.4 and 2.6 we have

Theorem 2.7. *The frame of compact topologies on a commutative, coherent, right sided quantale is coherent.*

We close this section giving a more precise description of the lattice of finite elements of the frame of compact topologies.

Proposition 2.8. *If f and g are finite elements of a coherent, commutative, right sided quantale Q then $S_f \vee S_g = S_{f \& g}$.*

Proof. Since $f \& g \leqslant f$ (commutativity plus right-sidedness) follows that $S_f \subseteq S_{f\&g}$ and similarly $S_g \subseteq S_{f\&g}$, so $S_f \vee S_g \subseteq S_{f\&g}$. For the other inequality it suffices to show that $f\&g \in S_f \vee S_g$. The latter is the completion of the pretopology $S_f \cup S_g$ (set-theoretic union). But for all finite $c \leqslant f$, $g \leqslant c \to (f \& g)$. So $f \& g \in (S_f \cup S_g)_1$. □

Corollary 2.9. *The frame of compact topologies on a commutative, coherent, right sided quantale Q is the ideal completion of the lattice $L = \{S_f \mid f \in Q,\ f$ finite$\}$ (with the operations of meet and join as in Propositions* 2.6 *and* 2.8).

The frame of compact topologies on a coherent, commutative, right sided quantale is coherent, so [4, Theorem II.3.4] it has enough points. We turn to characterize the points of the frame of compact topologies as those filters that are inaccessible by all joins.

Proposition 2.10. *A topology P corresponds to a point of the frame of compact topologies* $\mathrm{top}_\omega(Q)$, *where Q is a coherent, commutative, right sided quantale, iff, whenever the join $a \vee b$ of two elements is in P, then either a or b is in P.*

Proof. We use the identification of points of a frame with its meet irreducible elements [4, p. 41].

Let P be a meet-irreducible element of $\mathrm{top}_\omega(Q)$, and suppose that $a \vee b$ is in P. Assume first that a and b are finitely generated. Then $S_a \cap S_b = S_{a \vee b} \subseteq P$, where, as above, S_a is the smallest topology containing a. So, either $S_a \subseteq p$, or $S_b \subseteq p$. Equivalently, either a, or b are in P. For general a and b then, write them as directed suprema of finitely generated elements, $a = \bigvee f_i$, $b = \bigvee g_j$. Then $a \vee b = \bigvee f_i \vee \bigvee g_j = \bigvee (f_i \vee g_j)$, where the supremum is directed, so some $f_i \vee g_j$ is in P, so one of the f_i's or the g_j's is in P, and so is the corresponding a or b.

Conversely, suppose that P is a join-inaccessible set, and $S \cap T \subseteq P$. If S is not contained in P, then there is an element a of Q, which is in S but not in P. Then, for any b in T, $a \vee b$ is both in S (being larger than a) and in T (being larger than b). So, it is in P. But a is not in P and, P being join-inaccessible, b has to be in P. So, $T \subseteq P$. □

3. Applications to ring theory and logic

3.1. Universal property of the frame of compact Gabriel topologies and constructions of spectra

We give here a number of examples where the construction of the frame of compact Gabriel topologies is available and we identify it with well-known constructions of spectra in ring theory. We start by indicating a universal property that the frame of compact Gabriel topologies possesses, on which we will rely for exhibiting the connections with the various ring spectra.

Recall from [7, Proposition 4.1.4] that a coherent quantale has the form Idl(M), where M is a join semilattice with a top element and a binary associative operation, denoted abusively by &, such that $a\& -$ and $-\& a$ preserve finite suprema and $a\& T = a$, for all $a \in M$.

Proposition 3.1. *Let M be a multiplicative join semilattice as above, and L the distributive lattice of finite elements of the frame of compact topologies on* Idl(M). *Then the (order reversing) map $\theta : M \to L$, given by $\theta(a) = S_a$, is universal among maps from M to distributive lattice satisfying*

$$\text{(i)} \quad \theta(1) = \bot_L, \qquad \theta(a \vee b) = \theta(a) \wedge \theta(b),$$

$$\text{(ii)} \quad \theta(0) = \top_L, \qquad \theta(a \& b) = \theta(a) \vee \theta(b).$$

In other words, the (order preserving) map $\theta : M \to L^{\mathrm{op}}$ is the unit of the (forgetful-free) adjunction between multiplicative join semilattices (and maps preserving & and $\vee$) and distributive lattices.

Proof. The fact that θ defined above satisfies (i) and (ii) is the content of Propositions 2.6 and 2.8.

Now let $\varphi : M \to D$ be any other such map into a distributive lattice D. We extend φ along θ by $\varphi^{\#} : L \to D$, defined on S_a by $\varphi^{\#}(S_a) = \varphi(a)$. We show that $\varphi^{\#}$ is well defined, i.e. when $S_a = S_b$ then $\varphi(a) = \varphi(b)$. But $S_a = S_b$ means that b is in S_a so, by Lemma 2.4 that there is $k \in \mathbb{N}_0$ such that $b \geqslant a^k$. Applying then φ we get $\varphi(b) \geqslant \varphi(a^k) = \varphi(a)$. By symmetry, the other inequality holds as well, proving that $\varphi^{\#}$ is well defined. That $\varphi^{\#}$ is a lattice homomorphism follows immediately from its definition and the fact that φ satisfies (i) and (ii). □

Remark. Relatively recently Sun presented in [11] a construction of the left adjoint to the forgetful functor from distributive lattices to multiplicative semilattices (called there monoidal lattices) in terms of multiplicatively prime ideals on monoidal lattices.

Recall [4, V.3] that in order to give a constructive description of the Zariski spectrum (not relying on the existence of prime ideals), Joyal defined *a notion of zeros* for a commutative ring R to be a distributive lattice L and a map $\zeta : R \to L$ satisfying

$$\text{(Z)} \quad \text{(i)} \quad \zeta(1) = \bot_L, \qquad \zeta(a + b) \geqslant \zeta(a) \wedge \zeta(b),$$

$$\text{(ii)} \quad \zeta(0) = \top_L, \qquad \zeta(a . b) = \zeta(a) \vee \zeta(b).$$

Then the frame of opens for the Zariski spectrum is Idl(L^{op}), where L is the *universal notion of zeros* for R, i.e. the universal solution to the problem of finding such a map ζ. Taking the ideal completion of the lattice L itself leads to what is called the domain spectrum or the coZariski spectrum (in [4]) of the ring. As a space, it has the same points as the Zariski spectrum and as subbasic opens the complements of the basic opens for the Zariski topology. It is used for representing the ring by a sheaf of integral domains, while the Zariski spectrum is used for representing the ring by a sheaf of local rings.

With Proposition 3.1 at hand it is easy to deduce the following:

Theorem 3.2. *When Q is the quantale of ideals of a commutative ring R then the lattice L of finite elements of the frame* $\mathrm{top}_\omega(Q)$ *is the universal notion of zeros for the ring R.*

Proof. Let M be the multiplicative join semilattice of finitely generated ideals of R. The mapping $\eta : R \to M$, given by $\eta(a) = \langle a \rangle$, where $\langle a \rangle$ is the principal ideal generated by a, is universal among mappings from R into multiplicative join semilattices, satisfying the following properties:

(i) $\eta(1) = R, \quad \eta(a + b) \geqslant \eta(a) \vee \eta(b),$

(ii) $\eta(0) = \langle 0 \rangle, \quad \eta(a.b) = \eta(a) \& \eta(b).$

The proof of that fact is a straightforward verification that we will not give here. Then the result follows immediately from Proposition 3.1 by defining the universal map as the composite $\zeta = \theta \circ \eta$ and noticing that a distributive lattice is just a multiplicative join semilattice with idempotent multiplication. □

Corollary 3.3. *When Q is the quantale of ideals of a commutative ring, the frame* $\mathrm{top}_\omega(Q)$ *is isomorphic to that of opens for the coZariski spectrum of the ring.*

Following the analysis of [4], we come to consider the construction of the Brumfiel spectrum of a commutative ordered ring, i.e. a commutative ring equipped with a partial order compatible with the ring structure: $a \leqslant b$ implies $a + c \leqslant b + c$, $a \leqslant b$ and $c \geqslant 0$ implies $a.c \leqslant b.c$ and $a^2 \geqslant 0$. The Brumfiel spectrum is defined as (the ideal completion of) the (dual of the) distributive lattice L universal among the ones admitting a mapping from R so that the conditions (Z), above, are satisfied as well as the following:

(iii) $0 \leqslant a \leqslant b$ implies $\zeta(b) \leqslant \zeta(a)$.

We are going to obtain the Brumfiel spectrum by applying our construction to the quantale of convex ideals of the ring. Recall that an ideal I is called convex if whenever $a \leqslant b \leqslant c$ and $a \in I, c \in I$ then $b \in I$. For every ideal I there is a smallest convex ideal $j(I)$ containing I, which is given by means of the following inductive process:

$$j'(I) = \{b \in A \mid \exists a, c \in I \text{ so that } a \leqslant b \leqslant c\},$$

$$j^{(n+1)}(I) = j^{(n)\prime}(I),$$

$$j(I) = \textstyle\bigcup_{n<\omega} j^{(n)}(I).$$

Again it is a straightforward verification that j is a quantic nucleus in the sense of [7] and that it is finitary, i.e. it commutes with directed unions of ideals. Actually it does so at every step n of the inductive construction. It follows from [7, Proposition 4.1.3], that the set of convex ideals is a coherent (commutative) quantale. Then we obtain:

Proposition 3.4. *Let Q be the set of convex ideals of a commutative ordered ring. Then the lattice of finite elements of the frame* $\mathrm{top}_\omega(Q)$ *is isomorphic to the (dual of) the Brumfiel spectrum of the ring.*

Proof. In view of Proposition 3.1 and the proof of Theorem 3.2, it suffices to prove that the mapping $\eta : R \to M$ to the set of finite elements of the quantale Q, given by $\eta(a) = j(\langle a \rangle)$, where $j(\langle a \rangle)$ is the convex hull of the principal ideal generated by a, satisfies condition (iii) above. But if $0 \leqslant a \leqslant b$ then $a \in j(\langle b \rangle)$, so $j(\langle a \rangle) \subseteq j(\langle b \rangle)$. □

Further, we come to the description of the real Zariski spectrum of a commutative ring [2]. Here we are not aware of any defining universal property of the spectrum, which is usually described by restricting the Zariski topology to the set of real prime ideals. Recall that an ideal I is real closed when $x_1^2 + \cdots + x_n^2 \in I$ implies that, for all $i = 1, \ldots, n$, $x_i \in I$. Again there is a smallest real ideal containing any given ideal I, which is

$$r(I) = \{a \in R \mid \exists n \in \mathbb{N}_0, \exists x_1, \ldots, x_n \in R \text{ so that } a^2 + x_1^2 + \cdots + x_n^2 \in I\}.$$

(cf. [2, Proposition I.3]). Once more r is a finitary quantic nucleus on the quantale Q of ideals of R, so it gives rise to a coherent commutative quantale, to which our construction can be applied. The identification of $\mathrm{top}_\omega(Q)$ with (the dual of) the real Zariski spectrum is obtained then via our description of the points of $\mathrm{top}_\omega(Q)$ in Proposition 2.10 and by comparing the Zariski topology with the one on the spatial reflection of the frame (cf. [4, p. 41]).

3.2. A logical interpretation of compact Gabriel topologies

We indicate here a connection between compact Gabriel topologies on a coherent quantale on the one hand and deductively closed sets of sentences for a certain system of logic on the other. The logical system in mind is one that when presented in terms of Gentzen sequents it lacks the contraction rule and possibly the exchange rule, i.e. when α, β, γ are formulae and Γ, Δ are finite sequences of formulae the following two deductions are not in general valid:

$$\frac{\Gamma, \alpha, \alpha, \Delta \vdash \gamma}{\Gamma, \alpha, \Delta \vdash \gamma} \qquad \frac{\Gamma, \alpha, \beta, \Delta \vdash \gamma}{\Gamma, \beta, \alpha, \Delta \vdash \gamma}$$

The two rules above amount to the identities $a \& a = a$ and $a \& b = b \& a$ in the algebra of propositions. The lack of these identities brings us to the realm of quantales. Indeed in all the present work we have been dealing with structures failing in general to satisfy the former identity (non idempotent quantales), while we have tried to give results holding in the generality of noncommutative quantales (primarily Propositions 2.2 and 2.4), thus avoiding to adopt the latter identity. Notice that having dropped as well the following rule (called the weakening rule):

$$\frac{\Gamma, \alpha, \Delta \vdash \gamma}{\Gamma, \alpha, \beta, \Delta \vdash \gamma}$$

we would have fallen into the realm of linear logic. That would mean though that our algebra of propositions would fail to satisfy the inequality $a \& b \leqslant a$, equivalently $a \leqslant b \to_{\mathrm{r}} a$, which we have used repeatedly so far under the name of right-sidedness.

Such logical systems have attracted the attention of various logicians, either out of philosophical considerations related to the concept of implication (relevant logics), or as resource conscious logics in computer science (linear logic), or as logics of inexact concepts (many valued logics). One work in the latter connection is [3], where the reader will find detailed arguments on how such a system provides a solid qualitative ground for dealing with vagueness and for resolving classical paradoxes deriving from the unprecautious use of rules of classical logic for manipulating sentences involving inexact concepts.

To become more precise let us consider the system $\mathbf{L}_{\mathrm{BCK}}$ presented in [6] by Ono and Komori, which captures exactly the type of logic we have in mind, at least in the commutative case. Their noncommutative system $\mathbf{L}_{\mathrm{BCC}}$ does not serve our purposes as it has a single implication operator and extra deductive rule (apart from modus ponens). In the sequel we confine ourselves to the commutative case. Recall also from [12] that $\mathbf{L}_{\mathrm{BCK}}$ is the $\{\&, \vee, \wedge, \to\}$-fragment of (commutative) intuitionistic linear logic. Thus, following the remarks in Chapter 8 of [12], the algebra of provably equivalent sentences of it is an implicational multiplicative lattice, meaning a multiplicative semilattice as explained prior to Proposition 3.1 above, which is further a lattice and is equipped with an implication operation $\to$, so that, for all elements a of it, $a \to -$ is right adjoint to $a \& -$.

By a *theory* for such a system $\mathbf{L}_{\mathrm{BCK}}$ we mean a set of sentences containing T and closed under modus ponens (such a set is called a pretheory in [6], where the term theory is reserved for sets of sentences closed under the extra deductive rule of $\mathbf{L}_{\mathrm{BCC}}$). We obtain then the following interpretation of compact Gabriel topologies:

Proposition 3.5. *Let $\mathscr{L}$ be the ideal completion of the multiplicative lattice L of provably equivalent sentences of the system $\mathbf{L}_{\mathrm{BCK}}$. Then $\mathrm{Th}(\mathbf{L}_{\mathrm{BCK}})$, the set of theories on $\mathbf{L}_{\mathrm{BCK}}$, is order isomorphic to $\mathrm{top}_{\omega}(\mathscr{L})$, the set of compact Gabriel topologies on $\mathscr{L}$.*

Proof. We construct maps $\mu : \mathrm{top}_{\omega}(\mathscr{L}) \to \mathrm{Th}(\mathbf{L}_{\mathrm{BCK}})$ and $\nu : \mathrm{Th}(\mathbf{L}_{\mathrm{BCK}}) \to \mathrm{top}_{\omega}(\mathscr{L})$ assigning to a compact Gabriel topology S the theory $\mu(S) = \{\xi \in \mathbf{L}_{\mathrm{BCK}} \mid \exists \beta \ \xi \in [\![\beta]\!] \in S\}$ and to the theory T the Gabriel topology $\nu(T) = \{x \in \mathscr{L} \mid \xi \in T \text{ and } x \geqslant [\![\xi]\!]\}$, where $[\![\text{-}]\!]$ means the equivalence class of (-). We really have to argue only that the two maps take values in the sets indicated, as the fact that they are inverse to each other is immediate. Notice that we avoid to distinguish between elements of L and finite elements of $\mathscr{L}$.

First if S is a Gabriel topology then $\mu(S)$ is a theory: $\mu(S)$ contains T. It is closed under modus ponens because if $\beta \to \gamma$ is in $\mu(S)$ then, for all $c \in L$ with $c \leqslant [\![\beta]\!]$, since $c \to [\![\gamma]\!] \geqslant [\![\beta]\!] \to [\![\gamma]\!] = [\![\beta \to \gamma]\!]$, $c \to [\![\gamma]\!]$ will be in S, from which follows that $[\![\gamma]\!] \in S$ by axiom (T2) for Gabriel topologies. Hence $\gamma \in \mu(S)$.

Conversely, if T is a theory then $\nu(T)$ is a compact Gabriel topology: It satisfies (T1) since, if $c = [\![\gamma]\!] \in L$ and $x \geqslant [\![\xi]\!] \in \nu(T)$ (so that $\xi \in T$), then $c \to x \geqslant c \to [\![\xi]\!] = [\![\gamma \to \xi]\!]$ and the latter is in $\nu(T)$, as $\xi \to (\gamma \to \xi)$ is a theorem of $\mathbf{L_{BCK}}$ and then applying modus ponens $\gamma \to \xi$ is in T. Further, let $x \geqslant [\![\xi]\!] \in \nu(T)$ and assume that, for all $c \leqslant x$ with $c = [\![\gamma]\!] \in L$, it is the case that $c \to y \in \nu(T)$, so that in particular $b \leqslant [\![\xi]\!] \to y$ for some $b = [\![\beta]\!] \in \nu(T)$ depending on c. Consequently, we have $b\&c = [\![\beta]\!] \& [\![\gamma]\!] = [\![\beta \& \gamma]\!] \leqslant y$. But $b\&c$ is in $\nu(T)$ from which follows that $y \in \nu(T)$, so that (T2) is satisfied. As for compactness, if $\bigvee x_i \in \nu(T)$, where the supremum is directed, by the very definition of $\nu(T)$, there is $c \in L$ so that $c \leqslant \bigvee x_i$. But c is finite, so that, for some i, we have that $c \leqslant x_i$. □

Recall from [6, §3], that a *frame* (generalization of the notion of *Kripke frame*) is a couple $\langle M, K \rangle$, where M is a $\wedge$-semilattice ordered monoid with the monoid operation distributing over $\wedge$ and K a distinguished subset satisfying the following two conditions (which essentially ensure that the poset K is adequate as a carrier of valuations):

(i) if $a \in K$ and $b.d.c \leqslant a$ then there is $d' \in K$ with $d \leqslant d'$and $b.d'.c \leqslant a$ and

(ii) if $a \in K$ and $b \wedge c \leqslant a$, then there is $b' \in K$ with $b' \wedge c \leqslant a$.

Under the above equivalence and using the characterization of our Proposition 2.10 above together with the spatiality of the frame of compact Gabriel topologies (which comes as a consequence of its local compactness, cf. [4, Theorem 4.3]) we obtain the following result that extends Lemma 5.8, Theorem 5.9 and Theorem 5.12 of [6] and resolves any possible conflict of terminology.

Corollary 3.6. *The set* $\mathrm{Th}(\mathbf{L_{BCK}})$ *of theories for* $\mathbf{L_{BCK}}$ *is a frame having enough points. The couple* $\langle \mathrm{Th}\,\mathbf{L_{BCK}}), \mathscr{P} \rangle$, *where* $\mathscr{P}$ *is the set of its points of* $\mathrm{Th}(\mathbf{L_{BCK}})$, *is a frame in the above sense.*

Proof. The monoid operation in this case is simply the join of theories (which, in the commutative case, coincides with the multiplication of theories defined in [6, §5]). As noted in Lemma 5.11(i) of [6], $\mathrm{Th}(\mathbf{L_{BCK}})$ satisfies condition (ii) above simply by being distributive. As for condition (i) we can simply reproduce the argument given in Theorem 5.12 of [6] in the case of $\mathbf{L_{BCK}}$ by simply using our Proposition 2.10. □

Recall also from [6] that, given a frame $\langle M, K \rangle$, a valuation is a relation $\Vdash$ between members of K and propositional variables of $\mathbf{L_{BCK}}$ satisfying that if $a \Vdash p$, $b \Vdash p$ and

$a \wedge b \leqslant c$ then $c \Vdash p$. A valuation can be extended to a relation $a \Vdash \alpha$ between elements of K and sentences of $\mathbf{L}_{\mathbf{BCK}}$ in the way we just turn to indicate below. A sentence α is valid for a valuation if, for all $a \in K$, $a \Vdash \alpha$.

The set of $\mathscr{P}$ prime theories of $\mathbf{L}_{\mathbf{BCK}}$ can be used for defining a valuation for propositional variables and for extending it to all sentences of $\mathbf{L}_{\mathbf{BCK}}$ as follows:
– When p is a propositional variable and $P \in \mathscr{P}$ then $P \Vdash p$ iff $p \in P$;
– $P \Vdash \alpha \wedge \beta$ iff $P \Vdash \alpha$ and $P \Vdash \beta$;
– $P \Vdash \alpha \& \beta$ iff $P \Vdash \alpha$ and $P \Vdash \beta$;
– $P \Vdash \alpha \vee \beta$ iff either $P \Vdash \alpha$ or $P \Vdash \beta$ and
– $P \Vdash \alpha \rightarrow \beta$ iff for all $R \in \mathscr{P}$ $P \Vdash \alpha$ implies $P \vee R \Vdash \beta$.
Then, following once more [6], we conclude that, for all sentence α, $P \Vdash \alpha$ iff $\alpha \in P$.

This allows us to state the final result of this work:

Theorem 3.7. *The logic $\mathbf{L}_{\mathbf{BCK}}$ is complete in the sense that a sentence α is a theorem of $\mathbf{L}_{\mathbf{BCK}}$ iff α is valid for every valuation in every frame $\langle M, K \rangle$ as above.*

Proof. Having kept ourselves away from syntactical details we do not give an argument for the "only if" part. For the converse, suppose that α is not a theorem of $\mathbf{L}_{\mathbf{BCK}}$. Then the smallest Gabriel topology that contains it, $S_{[\![\alpha]\!]}$, is not contained in the smallest topology $\{\top\}$. From the spatiality of $\mathscr{L}$ follows that there exists a prime theory P such that $\{\top\} \subseteq P$ and $S_{[\![\alpha]\!]} \not\subseteq P$. Hence $\alpha \notin P$ and, from the comments above, α is not valid. □

It seems that everything said above could apply to a logical system lacking the exchange rule as well as the contraction rule. By modifying appropriately the notion of Gabriel topology, so as to ensure closure under both implications, the collection of them would still be a locally compact frame and the correspondence between theories and compact right Gabriel topologies would remain valid. The local compactness of the frame of theories would then still lead to a completeness result. We, however, refrain from carrying out here any details justifying this claim.

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