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American lookback option with fixed strike price—2-D parabolic variational inequality [☆]

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ABSTRACT

In this paper we study a 2-dimensional parabolic variational inequality with financial background. We define a suitable weak formula and obtain existence and uniqueness of the problem. Moreover we analyze the behaviors of the free boundary surface.

1. Introduction

In this paper we study a 2-dimensional parabolic variational inequality

$$\begin{cases} \min\{\mathcal{L}u, u-(K-y)^+\}=0, & 0<y<s<+\infty,\ 0<t\leqslant T, \\ u(s,y,0)=(K-y)^+, & 0\leqslant y\leqslant s<+\infty, \\ \partial_y u(y,y,t)=0, & y>0,\ 0<t\leqslant T, \end{cases} \tag{1.1}$$

where

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$$\mathcal{L}u = \partial_t u - \frac{1}{2}\sigma^2 s^2 \partial_{ss} u - (r-q)s\partial_s u + ru, \tag{1.2}$$

and $u(s, y, t)$ is the fair price of the American lookback put option with fixed strike price at time $T-t$, σ, r, q are positive constants, $s = S_\tau$ is the price of stock at time τ, $y = Y_\tau = \min_{0\leqslant\lambda\leqslant\tau} S_\lambda$. In Appendix A we present the formulation and the financial background of the problem (1.1). In [6] the authors examined the monotonicity properties of the option values and stopping regions with respect to the interest rate, dividend yield and time. In [13,17] the efficient methods were presented for computing the value and early exercise boundary of American lookback option with fixed strike price. The asymptotic behaviors at times close to expiration and at infinite time to expiration were examined in [5].

In (1.1) if $(K-y)^+$ and $0 < y < s < +\infty$ are replaced by $(\tilde{y} - K)^+$ and $0 < s < \tilde{y} < +\infty$, respectively, where $\tilde{y} = \tilde{Y}_\tau = \max_{0\leqslant\lambda\leqslant\tau} S_\lambda$, it is called American lookback call option with fixed strike price. Denote the call option price by $u_c(s, \tilde{y}, t)$, let

$$\bar{s} = s^{-1}, \qquad \bar{y} = \tilde{y}^{-1}, \qquad \bar{u}(\bar{s}, \bar{y}, t) = \tilde{y}^{-1} u_c(s, \tilde{y}, t);$$

then we can obtain a similar problem as (1.1) for $\bar{u}(\bar{s}, \bar{y}, t)$.

In (1.1) if $(K-y)^+$ is replaced by $(s-y)^+$ (or $(\tilde{y}-s)^+$), it is called American lookback call (or put) option with floating strike price. In this case it can be reduced to a 1-dimensional problem by a special transformation in [11]. But this method is not useful for fixed strike lookbacks. The solution for American floating strike lookbacks can be found by solving a 1-dimensional variational inequality which governs the options [17], and this was shown to be similar to the valuation of a plain vanilla American option [2,3], with the exception that American floating strikes have a Neumann-type boundary condition and can be solved via simulation techniques [1,4].

Up to now people have not found a special transformation which can reduce the dimension of problem (1.1). Let us analyze the difficulty of dealing with the problem. Note that there are no derivatives with respect to y in the operator (1.2), so we can see y as a parameter, in this sense (1.2) is a 1-dimensional parabolic operator. But we cannot solve the problem for each fixed $y > 0$, since the boundary condition in (1.1) is with respect to $\partial_y u$, so in fact we can only take (1.2) as a 2-dimensional degenerate parabolic (or ultraparabolic) operator [16]. At this point it brings difficulty for proving the existence of solution and analyzing the behaviors of the free boundary surface.

In the next section, we present the definition of weak solution of problem (1.1), and prove the existence of weak solution of problem (1.1). The proof of uniqueness will put in Section 3. In Section 4 we prove $\partial_y u(s, y, t)$ is continuous on $\{s = y,\ t > 0\}$. In Section 5, we analyze the properties of the free boundary $y = h(s, t)$. It is continuous with $h(s, 0) = \min\{s, K\}$, and it is strictly monotonic with respect to s and t. Moreover the inverse function $s = g(y, t)$ is continuous with respect to y and infinitely differentiable with respect to t. Appendix A is the formulation of the model. Appendix B shows that the unique solution to the problem (1.1) coincides with the expected value of the American lookback put option with fixed strike price.

2. The existence of weak solution

First we describe how to absorb the boundary condition $\partial_y u = 0$ into the weak formulation of solution. Denote $\Omega = \{(s, y) \mid s > 0,\ 0 < y < s\}$, suppose u is a classical solution of problem (1.1), then for any $\psi(s, y) \in C_0^1(\overline{\Omega})$, $0 < t \leqslant T$,

$$\begin{aligned}\int_\Omega s^2 \partial_{ss} u \psi \, ds\, dy &= \int_0^{+\infty} dy \int_y^{+\infty} s^2 \partial_{ss} u \psi \, ds = -\int_\Omega \partial_s u \partial_s \big(s^2 \psi\big) \, ds\, dy + \int_0^{+\infty} \partial_s u s^2 \psi \big|_{s=y}^{s=+\infty} dy \\ &= -\int_\Omega \partial_s u \partial_s \big(s^2\psi\big)\, ds\, dy - \int_0^{+\infty} \partial_s u(y, y, t) y^2 \psi(y, y)\, dy.\end{aligned} \tag{2.1}$$

Applying the boundary condition in (1.1), we have

$$\frac{d}{dy}u(y,y,t)=\frac{\partial}{\partial s}u(y,y,t)+\frac{\partial}{\partial y}u(y,y,t)=\frac{\partial}{\partial s}u(y,y,t), \tag{2.2}$$

so the last term in (2.1)

$$-\int_0^{+\infty}\partial_s u(y,y,t)y^2\psi(y,y)\,dy=\int_0^{+\infty}u(y,y,t)\frac{d}{dy}\big[y^2\psi(y,y)\big]\,dy, \tag{2.3}$$

substituting (2.3) into (2.1) we obtain

$$\int_\Omega s^2\partial_{ss}u\psi\,ds\,dy=-\int_\Omega \partial_s u\partial_s\big(s^2\psi\big)\,ds\,dy+\int_0^{+\infty}u(y,y,t)\frac{d}{dy}\big[y^2\psi(y,y)\big]\,dy. \tag{2.4}$$

It can be seen that the equality (2.4) absorbed the boundary condition $\partial_y u=0$ in (1.1).

In spirit of [8] and [14], we introduce the following maximal monotone graph

$$G(\lambda)=\begin{cases}0, & \lambda>0,\\ [0,+\infty), & \lambda=0.\end{cases}$$

Denote $Q_T=\{(s,y,t)\mid 0<y<s,\ 0<t\leqslant T\}$. Define function class for the solution,

$$B=\big\{u\in C(\overline{Q}_T)\ \big|\ u,\partial_s u,\partial_y u\in L^\infty(Q_T),\ \partial_t u\in L^\infty\big(0,T;H^{-1}_{loc}(\Omega)\big)\big\}.$$

Now we can define weak solution of problem (1.1).

Definition of weak solution of problem (1.1). A pair $(u,\xi)\in B\times L^\infty(Q_T)$ is called a weak solution of problem (1.1), if

(1) $u(s,y,t)\geqslant(K-y)^+$,
(2) $u(s,y,0)=(K-y)^+$,
(3) $\xi\in G(u-(K-y)^+)$,
(4) *For any* $\psi(s,y)\in C^1_0(\overline{\Omega})$, *a.e.* $0<t\leqslant T$,

$$\int_\Omega\Big[\partial_t u\psi+\frac{1}{2}\sigma^2(\partial_s u)\partial_s\big(s^2\psi\big)-(r-q)s\partial_s u\psi+ru\psi\Big]\,ds\,dy$$
$$-\frac{1}{2}\sigma^2\int_0^{+\infty}u(y,y,t)\frac{d}{dy}\big[y^2\psi(y,y)\big]\,dy=\int_\Omega\xi\psi\,ds\,dy. \tag{2.5}$$

Note that for $0<y<K$, the boundary condition $\partial_y u=0$ and initial condition $u=(K-y)^+$ are not consistent on $s=y$. In order to have existence of weak solution we construct domain

$$\Omega_\varepsilon=\big\{(s,y)\ \big|\ s>\varepsilon,\ 0<y<f_\varepsilon(s)\big\},$$

which approximates domain Ω as ε goes to zero (see Fig. 1), where

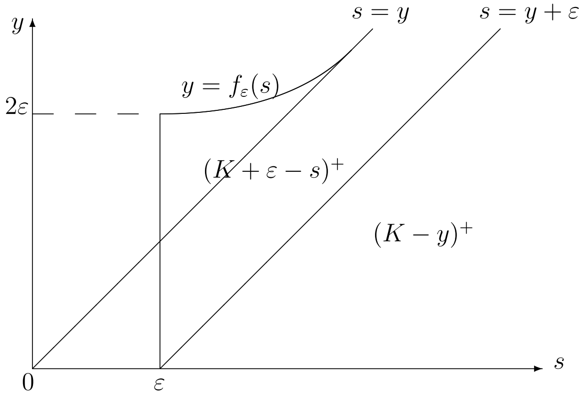

Fig. 1. Ω_ε and $\varphi(s,y)$.

$$f_\varepsilon(s)=\begin{cases}2\varepsilon+\frac{1}{4\varepsilon}(s-\varepsilon)^2, & \varepsilon\leqslant s\leqslant 3\varepsilon,\\ s, & s>3\varepsilon.\end{cases}$$

It is clear that $\lim_{\varepsilon\to 0} f_\varepsilon(s)=s$ for $s>0$ and

$$f_\varepsilon'(\varepsilon)=0,\quad 0\leqslant f_\varepsilon'(s)\leqslant 1. \tag{2.6}$$

Moreover we need following $\varphi(s,y)$ (see Fig. 1) to approximate initial value $(K-y)^+$, that is

$$\varphi(s,y)=\max\bigl\{(K+\varepsilon-s)^+,(K-y)^+\bigr\}=\begin{cases}(K+\varepsilon-s)^+, & \varepsilon\leqslant s\leqslant y+\varepsilon,\ y<f_\varepsilon(s),\\ (K-y)^+, & s>y+\varepsilon.\end{cases}$$

It can be seen that

$$\varphi(s,0)=K,\quad s\geqslant\varepsilon, \tag{2.7}$$

$$\partial_s\varphi(\varepsilon,y)=-1,\quad 0\leqslant y\leqslant 2\varepsilon, \tag{2.8}$$

$$\partial_y\varphi\bigl(s,f_\varepsilon(s)\bigr)=0,\quad s\geqslant\varepsilon, \tag{2.9}$$

and

$$\partial_s\varphi(s,y)=\begin{cases}-H(K+\varepsilon-s), & \varepsilon\leqslant s\leqslant y+\varepsilon,\ y<f_\varepsilon(s),\\ 0, & s>y+\varepsilon,\end{cases}$$

$$\partial_y\varphi(s,y)=\begin{cases}0, & \varepsilon\leqslant s\leqslant y+\varepsilon,\ y<f_\varepsilon(s),\\ -H(K-y), & s>y+\varepsilon,\end{cases}$$

where H is the Heaviside function. It follows that

$$(K-y)^+\leqslant\varphi(s,y)\leqslant K, \tag{2.10}$$

$$-1\leqslant\partial_s\varphi(s,y)\leqslant 0,\qquad -1\leqslant\partial_y\varphi(s,y)\leqslant 0, \tag{2.11}$$

$$-1\leqslant\partial_s\varphi(s,y)+\partial_y\varphi(s,y)\leqslant 0, \tag{2.12}$$

$$\partial_{ss}\varphi(s,y)\geqslant 0,\qquad \partial_{sy}\varphi(s,y)\leqslant 0,\qquad \partial_{yy}\varphi(s,y)\geqslant 0. \tag{2.13}$$

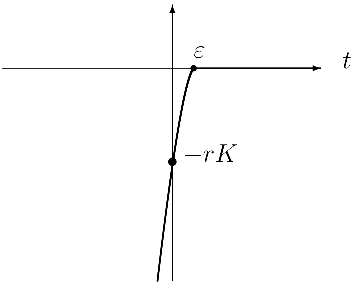

Fig. 2. $\beta_\varepsilon(t)$.

In order to prove the existence of solution of (1.1), as in [7] and [9], we construct a penalty function $\beta_\varepsilon(t)$ satisfying (see Fig. 2)

$$\beta_\varepsilon(t) \in C^2(-\infty, +\infty), \qquad \beta_\varepsilon(t) \leqslant 0,$$
$$\beta_\varepsilon(0) = -rK,$$
$$\beta'_\varepsilon(t) \geqslant 0, \qquad \beta''_\varepsilon(t) \leqslant 0,$$

and

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(t) = \begin{cases} 0, & t > 0, \\ -\infty, & t < 0. \end{cases}$$

Let $\varphi_\varepsilon(s, y)$ be the mollification of $\varphi(s, y)$ which still satisfies (2.7)–(2.13). Consider an approximation problem of (1.1)

$$\mathcal{L}_\varepsilon u_\varepsilon + \beta_\varepsilon\big(u_\varepsilon - (K - y)\big) = 0, \quad s > \varepsilon,\ 0 < y < f_\varepsilon(s),\ 0 < t \leqslant T, \tag{2.14}$$

$$u_\varepsilon(s, y, 0) = \varphi_\varepsilon(s, y), \quad s > \varepsilon,\ 0 < y < f_\varepsilon(s), \tag{2.15}$$

$$u_\varepsilon(s, 0, t) = K, \quad s > \varepsilon,\ 0 < t \leqslant T, \tag{2.16}$$

$$\partial_y u_\varepsilon\big(s, f_\varepsilon(s), t\big) = 0, \quad s > \varepsilon,\ 0 < t \leqslant T, \tag{2.17}$$

$$\partial_s u_\varepsilon(\varepsilon, y, t) = 0, \quad 0 < y < 2\varepsilon,\ 0 < t \leqslant T, \tag{2.18}$$

where

$$\mathcal{L}_\varepsilon u_\varepsilon = \mathcal{L} u_\varepsilon - \varepsilon\partial_{yy} u_\varepsilon = \partial_t u_\varepsilon - \frac{1}{2}\sigma^2 s^2 \partial_{ss} u_\varepsilon - \varepsilon\partial_{yy} u_\varepsilon - (r - q)s\partial_s u_\varepsilon + r u_\varepsilon.$$

Note that initial condition (2.15) and boundary condition (2.17) are consistent on $y = f_\varepsilon(s)$ by (2.9), but (2.15) and (2.18) are not consistent on $s = \varepsilon$ by (2.8).

Remark on boundary condition (2.16). Letting $y = 0$ in (1.1), it is deduced that

$$\begin{cases} \min\big\{\mathcal{L}u(s, 0, t), u(s, 0, t) - K\big\} = 0, & 0 < s < +\infty,\ 0 < t \leqslant T, \\ u(s, 0, 0) = K. \end{cases} \tag{2.19}$$

It is clear that $u(s, 0, t) = K$ is the solution of problem (2.19).

Denote $Q_T^\varepsilon = \Omega_\varepsilon \times (0,T)$.

Lemma 2.1. *For fixed $\varepsilon > 0$, problem (2.14)–(2.18) has a solution $u_\varepsilon \in C^{2,1}(Q_T^\varepsilon) \cap C(\overline{Q_T^\varepsilon}) \cap L^\infty(Q_T^\varepsilon)$ with $\partial_y u_\varepsilon \in C(\overline{Q_T^\varepsilon}) \cap L^\infty(Q_T^\varepsilon)$ and $\partial_s u_\varepsilon, \partial_{sy} u_\varepsilon, \partial_{yy} u_\varepsilon \in L^\infty(Q_T^\varepsilon)$, moreover*

$$(K-y)^+ \leqslant u_\varepsilon \leqslant K, \tag{2.20}$$

$$-rK \leqslant \beta_\varepsilon\big(u_\varepsilon - (K-y)\big) \leqslant 0. \tag{2.21}$$

Proof. For fixed $\varepsilon > 0$, it is not hard to show by fixed point theorem that problem (2.14)–(2.18) has a solution $u_\varepsilon \in C^{2,1}(Q_T^\varepsilon) \cap C(\overline{Q_T^\varepsilon}) \cap L^\infty(Q_T^\varepsilon)$.

Applying (2.6) and making even extension of u_ε to $s < \varepsilon$, we see that $\partial_y u_\varepsilon \in C(\overline{Q_T^\varepsilon}) \cap L^\infty(Q_T^\varepsilon)$ and $\partial_s u_\varepsilon, \partial_{sy} u_\varepsilon, \partial_{yy} u_\varepsilon \in L^\infty(Q_T^\varepsilon)$.

Now we prove (2.20). Since

$$\mathcal{L}_\varepsilon 0 + \beta_\varepsilon\big(0-(K-y)\big) = \beta_\varepsilon\big(0-(K-y)\big) \leqslant 0,$$

$$\mathcal{L}_\varepsilon (K-y) + \beta_\varepsilon(0) = r(K-y) - rK \leqslant 0,$$

$$\mathcal{L}_\varepsilon K + \beta_\varepsilon\big(K-(K-y)\big) = rK + \beta_\varepsilon(y) \geqslant rK + \beta_\varepsilon(0) = 0,$$

recalling (2.10), combining initial and boundary conditions we know that 0, $K-y$ are subsolutions, K is a supersolution, so (2.20) holds. (2.21) follows by (2.20) and the definition of β_ε. □

Lemma 2.2. *The following estimates hold*:

$$-1 \leqslant \partial_y u_\varepsilon \leqslant 0, \tag{2.22}$$

$$\partial_s u_\varepsilon \leqslant 0. \tag{2.23}$$

Proof. We first prove inequality (2.22), since $u_\varepsilon(s,0,t) = K$, so

$$\begin{aligned} \partial_{yy} u_\varepsilon(s,0,t) &= \frac{1}{\varepsilon}\left[\partial_t u_\varepsilon - \frac{1}{2}\sigma^2 s^2 \partial_{ss} u_\varepsilon - (r-q)s\partial_s u_\varepsilon + r u_\varepsilon + \beta_\varepsilon\big(u_\varepsilon - (K-y)\big)\right]\bigg|_{y=0} \\ &= \frac{1}{\varepsilon}\big[rK + \beta_\varepsilon(0)\big] = 0. \end{aligned} \tag{2.24}$$

Differentiating Eq. (2.14) with respect to y, denoting $v_1 = \partial_y u_\varepsilon$, then

$$\begin{cases} \partial_t v_1 - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v_1 - \varepsilon \partial_{yy} v_1 - (r-q)s\partial_s v_1 + r v_1 + \beta'_\varepsilon(\cdot)(v_1+1) = 0, \\ v_1(s,y,0) = \partial_y \varphi_\varepsilon(s,y) \in [-1,0] \quad \big(\text{by } (2.11)\big), \\ \partial_y v_1(s,0,t) = 0 \quad \big(\text{by } (2.24)\big), \\ v_1\big(s, f_\varepsilon(s), t\big) = 0, \\ \partial_s v_1(\varepsilon, y, t) = 0 \quad \big(\text{by } (2.18)\big). \end{cases} \tag{2.25}$$

It is clear that 0 is a supersolution and -1 is a subsolution, hence $-1 \leqslant v_1 = \partial_y u_\varepsilon \leqslant 0$.

Next we will prove inequality (2.23), since $\partial_y u_\varepsilon(s, f_\varepsilon(s), t) = 0$, thus

$$\partial_{sy} u_\varepsilon\big(s, f_\varepsilon(s), t\big) + \partial_{yy} u_\varepsilon\big(s, f_\varepsilon(s), t\big) f'_\varepsilon(s) = 0; \tag{2.26}$$

from (2.22) we know $\partial_{yy}u_\varepsilon(s, f_\varepsilon(s), t) \geqslant 0$, hence

$$\partial_y(\partial_s u_\varepsilon)\big(s, f_\varepsilon(s), t\big) = \partial_{sy}u_\varepsilon\big(s, f_\varepsilon(s), t\big) = -\partial_{yy}u_\varepsilon\big(s, f_\varepsilon(s), t\big) f'_\varepsilon(s) \leqslant 0. \tag{2.27}$$

Differentiating Eq. (2.14) with respect to s, denoting $v_2 = \partial_s u_\varepsilon$, then

$$\begin{cases} \partial_t v_2 - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v_2 - \varepsilon \partial_{yy} v_2 - \big(r - q + \sigma^2\big) s \partial_s v_2 + q v_2 + \beta'_\varepsilon(\cdot) v_2 = 0, \\ v_2(s, y, 0) = \partial_s \varphi_\varepsilon(s, y) \leqslant 0 \quad \big(\text{by (2.11)}\big), \\ v_2(s, 0, t) = 0, \\ \partial_y v_2\big(s, f_\varepsilon(s), t\big) \leqslant 0 \quad \big(\text{by (2.27)}\big), \\ v_2(\varepsilon, y, t) = 0, \end{cases} \tag{2.28}$$

and applying maximum principle we have $v_2 = \partial_s u_\varepsilon \leqslant 0$. □

Lemma 2.3. *The solution u_ε of problem* (2.14)–(2.18) *satisfies*

$$\partial_{sy} u_\varepsilon \leqslant 0, \tag{2.29}$$

$$\partial_s u_\varepsilon \geqslant -1. \tag{2.30}$$

Proof. Differentiating equation in (2.25) with respect to s, denoting $w = \partial_{sy} u_\varepsilon$, then

$$\begin{cases} \partial_t w - \frac{1}{2}\sigma^2 s^2 \partial_{ss} w - \varepsilon \partial_{yy} w - \big(r - q + \sigma^2\big) s \partial_s w + q w + \beta'_\varepsilon(\cdot) w = -\beta''_\varepsilon(\cdot)(\partial_y u_\varepsilon + 1)\partial_s u_\varepsilon, \\ w(s, y, 0) = \partial_{sy} \varphi_\varepsilon(s, y) \leqslant 0 \quad \big(\text{by (2.13)}\big), \\ w(s, 0, t) \leqslant 0 \quad \big(\text{by (2.23)}\big), \\ w\big(s, f_\varepsilon(s), t\big) \leqslant 0 \quad \big(\text{by (2.27)}\big), \\ w(\varepsilon, y, t) = 0 \quad \big(\text{by (2.18)}\big). \end{cases}$$

Since $-\beta''_\varepsilon(\cdot)(\partial_y u_\varepsilon + 1)\partial_s u_\varepsilon \leqslant 0$, so 0 is a supersolution, therefore $\partial_{sy} u_\varepsilon \leqslant 0$.

Now we apply (2.29) for proving (2.30), denote $v_3 = \partial_s u_\varepsilon + \partial_y u_\varepsilon$. Applying the equations in (2.25) and (2.28) we have

$$\begin{aligned} &\partial_t v_3 - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v_3 - \varepsilon \partial_{yy} v_3 - (r - q) s \partial_s v_3 + r v_3 + \beta'_\varepsilon(\cdot) v_3 \\ &\quad = -\beta'_\varepsilon(\cdot) + \sigma^2 s \partial_{ss} u_\varepsilon + (r - q)\partial_s u_\varepsilon. \end{aligned} \tag{2.31}$$

Notice that

$$\partial_{ss} u_\varepsilon = \partial_s(\partial_s u_\varepsilon + \partial_y u_\varepsilon) - \partial_{sy} u_\varepsilon = \partial_s v_3 - \partial_{sy} u_\varepsilon, \tag{2.32}$$

$$\partial_s u_\varepsilon = v_3 - \partial_y u_\varepsilon. \tag{2.33}$$

Substituting (2.32) and (2.33) into (2.31),

$$\begin{aligned} &\partial_t v_3 - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v_3 - \varepsilon \partial_{yy} v_3 - (r - q) s \partial_s v_3 + r v_3 + \beta'_\varepsilon(\cdot) v_3 \\ &\quad = -\beta'_\varepsilon(\cdot) + \sigma^2 s(\partial_s v_3 - \partial_{sy} u_\varepsilon) + (r - q)(v_3 - \partial_y u_\varepsilon), \end{aligned}$$

i.e.,

$$\begin{aligned}&\partial_t v_3 - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v_3 - \varepsilon \partial_{yy} v_3 - \big(r - q + \sigma^2\big) s \partial_s v_3 + q v_3 + \beta_\varepsilon'(\cdot) v_3\\ &\quad = -\beta_\varepsilon'(\cdot) - \sigma^2 s \partial_{sy} u_\varepsilon + (q - r)\partial_y u_\varepsilon\\ &\quad \geqslant -\beta_\varepsilon'(\cdot) - q \quad \big(\text{by (2.29) and (2.22)}\big).\end{aligned}$$

So v_3 is a supersolution of the equation

$$\begin{aligned}&\partial_t v_3 - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v_3 - \varepsilon \partial_{yy} v_3 - \big(r - q + \sigma^2\big) s \partial_s v_3 + q v_3 + \beta_\varepsilon'(\cdot) v_3\\ &\quad = -\beta_\varepsilon'(\cdot) - q. \end{aligned} \tag{2.34}$$

It is clear that -1 is a solution of (2.34). On the other hand the initial and boundary conditions for v_3 are

$$\begin{cases} v_3(s, y, 0) = \partial_s \varphi_\varepsilon(s, y) + \partial_y \varphi_\varepsilon(s, y) \geqslant -1 \quad \big(\text{by (2.12)}\big),\\ v_3(s, 0, t) = \partial_y u_\varepsilon(s, 0, t) \geqslant -1 \quad \big(\text{by (2.22)}\big),\\ \partial_y v_3\big(s, f_\varepsilon(s), t\big) \geqslant 0 \quad \big(\text{see (2.35) below}\big),\\ v_3(\varepsilon, y, t) = \partial_y u_\varepsilon(\varepsilon, y, t) \geqslant -1 \quad \big(\text{by (2.22)}\big),\end{cases}$$

where

$$\begin{aligned}\partial_y v_3\big(s, f_\varepsilon(s), t\big) &= \partial_{sy} u_\varepsilon\big(s, f_\varepsilon(s), t\big) + \partial_{yy} u_\varepsilon\big(s, f_\varepsilon(s), t\big)\\ &= \partial_{sy} u_\varepsilon\big(s, f_\varepsilon(s), t\big) + \partial_{yy} u_\varepsilon\big(s, f_\varepsilon(s), t\big) f_\varepsilon'(s)\\ &\quad - \partial_{yy} u_\varepsilon\big(s, f_\varepsilon(s), t\big) f_\varepsilon'(s) + \partial_{yy} u_\varepsilon\big(s, f_\varepsilon(s), t\big)\\ &= \partial_{yy} u_\varepsilon\big(s, f_\varepsilon(s), t\big)\big[1 - f_\varepsilon'(s)\big] \geqslant 0 \quad \big(\text{by (2.26) and (2.6)}\big).\end{aligned} \tag{2.35}$$

Hence $v_3 = \partial_s u_\varepsilon + \partial_y u_\varepsilon \geqslant -1$, therefore

$$\partial_s u_\varepsilon \geqslant -1 - \partial_y u_\varepsilon \geqslant -1. \qquad \square$$

For parabolic equation, the boundedness of derivatives of solution with respect to spacial variables deduces 1/2 Hölder continuity of solution with respect to t. Now we prove this result.

Lemma 2.4. *The solution u_ε of problem* (2.14)–(2.18) *satisfies*

$$\big|u_\varepsilon(s, y, t_1) - u_\varepsilon(s, y, t_2)\big| \leqslant C|t_1 - t_2|^{1/2}, \tag{2.36}$$

where C is independent of ε and depends on the upper bound of s.

Proof. Suppose $t_1 < t_2$, define

$$I(s, y, t_1, t_2) = \int_s^{s+\sqrt{t_2 - t_1}} d\xi \int_y^{y+\sqrt{t_2 - t_1}} \big[u_\varepsilon(\xi, \eta, t_2) - u_\varepsilon(\xi, \eta, t_1)\big]\, d\eta.$$

In one hand, applying the mean value theorem,

$$I(s,y,t_1,t_2)=\big[u_\varepsilon\big(s^*,y^*,t_2\big)-u_\varepsilon\big(s^*,y^*,t_1\big)\big](t_2-t_1), \tag{2.37}$$

where $s<s^*<s+\sqrt{t_2-t_1}$, $y<y^*<y+\sqrt{t_2-t_1}$. On the other hand, using Eq. (2.14),

$$\begin{aligned}
I(s,y,t_1,t_2)={}&\int_s^{s+\sqrt{t_2-t_1}}d\xi\int_y^{y+\sqrt{t_2-t_1}}d\eta\int_{t_1}^{t_2}\partial_t u_\varepsilon(\xi,\eta,\tau)\,d\tau\\
={}&\int_s^{s+\sqrt{t_2-t_1}}d\xi\int_y^{y+\sqrt{t_2-t_1}}d\eta\\
&\times\int_{t_1}^{t_2}\left[\frac{1}{2}\sigma^2\xi^2\partial_{ss}u_\varepsilon+\varepsilon\partial_{yy}u_\varepsilon+(r-q)\xi\partial_s u_\varepsilon-ru_\varepsilon-\beta_\varepsilon(\cdot)\right]d\tau\\
={}&I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

Since

$$\begin{aligned}
I_1={}&\frac{1}{2}\sigma^2\int_y^{y+\sqrt{t_2-t_1}}d\eta\int_{t_1}^{t_2}d\tau\int_s^{s+\sqrt{t_2-t_1}}\xi^2\partial_{ss}u_\varepsilon(\xi,\eta,\tau)\,d\xi\\
={}&\frac{1}{2}\sigma^2\int_y^{y+\sqrt{t_2-t_1}}d\eta\int_{t_1}^{t_2}\xi^2\partial_s u_\varepsilon(\xi,\eta,\tau)\Big|_{\xi=s}^{\xi=s+\sqrt{t_2-t_1}}\,d\tau\\
&-\frac{1}{2}\sigma^2\int_y^{y+\sqrt{t_2-t_1}}d\eta\int_{t_1}^{t_2}d\tau\int_s^{s+\sqrt{t_2-t_1}}2\xi\partial_s u_\varepsilon(\xi,\eta,\tau)\,d\xi,
\end{aligned}$$

applying the boundedness of $\partial_s u_\varepsilon$, we obtain

$$|I_1|\leqslant C(t_2-t_1)^{3/2},$$

where C depends on the upper bound of s. In a similar way, we have

$$|I_i|\leqslant C(t_2-t_1)^{3/2},\quad i=2,3,4,5,$$

therefore

$$|I|\leqslant C(t_2-t_1)^{3/2}. \tag{2.38}$$

Applying (2.37) and (2.38) we obtain

$$\big|u_\varepsilon\big(s^*,y^*,t_2\big)-u_\varepsilon\big(s^*,y^*,t_1\big)\big|\leqslant C(t_2-t_1)^{1/2}.$$

Thus

$$\begin{aligned}\big|u_\varepsilon(s,y,t_2)-u_\varepsilon(s,y,t_1)\big| &\leqslant \big|u_\varepsilon(s,y,t_2)-u_\varepsilon\big(s^*,y^*,t_2\big)\big|+\big|u_\varepsilon\big(s^*,y^*,t_2\big)-u_\varepsilon\big(s^*,y^*,t_1\big)\big|\\ &\quad+\big|u_\varepsilon\big(s^*,y^*,t_1\big)-u_\varepsilon(s,y,t_1)\big|\\ &\leqslant 2\big(\big|s-s^*\big|+\big|y-y^*\big|\big)+C(t_2-t_1)^{1/2}\leqslant C(t_2-t_1)^{1/2}. \qquad \square\end{aligned}$$

Theorem 2.5. *The problem* (1.1) *has a weak solution* $(u,\xi)\in B\times L^\infty(Q_T)$, *and*

$$(K-y)^+\leqslant u\leqslant K, \tag{2.39}$$

$$-1\leqslant \partial_s u\leqslant 0, \tag{2.40}$$

$$-1\leqslant \partial_y u\leqslant 0, \tag{2.41}$$

$$\big|u(s,y,t_2)-u(s,y,t_2)\big|\leqslant C(t_2-t_1)^{1/2}, \tag{2.42}$$

where C *depends on the upper bound of* s.

Proof. Since $\{u_\varepsilon\}$ satisfy (2.20)–(2.23), (2.30) and (2.36), applying Arzelà–Ascoli theorem we know there exists a subsequence of $\{u_\varepsilon\}$ (still denoted by $\{u_\varepsilon\}$), $u\in C(\overline{Q}_T)\cap L^\infty(Q_T)$ and $\xi\in L^\infty(Q_T)$, such that

$$\begin{aligned} u_\varepsilon\to u &\quad \text{in } C_{loc}(\overline{Q}_T),\\ \partial_s u_\varepsilon \rightharpoonup \partial_s u &\quad \text{in } L^\infty_{loc}(Q_T)\text{ weakly}^*,\\ \partial_y u_\varepsilon \rightharpoonup \partial_y u &\quad \text{in } L^\infty_{loc}(Q_T)\text{ weakly}^*,\\ -\beta_\varepsilon\big(u_\varepsilon-(K-y)\big)\rightharpoonup \xi &\quad \text{in } L^\infty_{loc}(Q_T)\text{ weakly}^*.\end{aligned}$$

Moreover applying Eq. (2.14), we see that

$$\partial_t u_\varepsilon \rightharpoonup \partial_t u \quad \text{in } L^\infty\big(0,T;H^{-1}_{loc}(\Omega)\big)\text{ weakly}^*,$$

thus $u\in B$, $0\leqslant\xi\leqslant rK$.

Now we prove that (u,ξ) satisfies (1)–(4) in the definition of weak solution. Due to $u_\varepsilon(s,y,t)\geqslant (K-y)^+$, $u_\varepsilon(s,y,0)=\varphi_\varepsilon(s,y)$, letting $\varepsilon\to 0$, we have

$$u(s,y,t)\geqslant (K-y)^+,\qquad u(s,y,0)=(K-y)^+.$$

Next we want to prove $\xi\in G(u-(K-y)^+)$. According to the definition, we only need to prove that if $u(s_0,y_0,t_0)>(K-y_0)^+$, then $\xi=0$. In fact, if $u(s_0,y_0,t_0)>(K-y_0)^+$, then there exist $\lambda>0$ and a δ neighborhood $B_\delta(s_0,y_0,t_0)$ of (s_0,y_0,t_0), if ε is small enough, we have

$$u_\varepsilon(s,y,t)>(K-y)^++\lambda,\quad (s,y,t)\in B_\delta(s_0,y_0,t_0),$$

thus, if ε is small enough,

$$\beta_\varepsilon\big(u_\varepsilon(s,y,t)-(K-y)\big)\geqslant\beta_\varepsilon(\lambda)=0,\quad (s,y,t)\in B_\delta(s_0,y_0,t_0).$$

It shows that

$$\xi(s,y,t)=0,\quad (s,y,t)\in B_\delta(s_0,y_0,t_0),$$

hence

$$\xi\in G\big(u-(K-y)^+\big).$$

Eventually we prove condition (2.5). For any $\psi(s,y)\in C_0^1(\overline{\Omega})$, *a.e.* $0<t\leqslant T$, multiplying Eq. (2.14) by ψ, integrating over Ω, then

$$\begin{aligned}
0&=\int\limits_\Omega \mathcal{L}_\varepsilon u_\varepsilon\psi\,ds\,dy\\
&=\int\limits_\Omega\bigg[\partial_t u_\varepsilon\psi+\frac{1}{2}\sigma^2(\partial_s u_\varepsilon)\partial_s\big(s^2\psi\big)-\varepsilon\partial_{yy}u_\varepsilon\psi-(r-q)s\partial_s u_\varepsilon\psi+ru_\varepsilon\psi\\
&\quad+\beta_\varepsilon\big(u_\varepsilon-(K-y)\big)\psi\bigg]ds\,dy-\frac{1}{2}\sigma^2\int\limits_0^{+\infty}u_\varepsilon(y,y,t)\frac{d}{dy}\big[y^2\psi(y,y)\big]dy,
\end{aligned}$$

thus

$$\begin{aligned}
&\int\limits_\Omega\bigg[\partial_t u_\varepsilon\psi+\frac{1}{2}\sigma^2(\partial_s u_\varepsilon)\partial_s\big(s^2\psi\big)-(r-q)s\partial_s u_\varepsilon\psi+ru_\varepsilon\psi\bigg]ds\,dy\\
&\qquad-\frac{1}{2}\sigma^2\int\limits_0^{+\infty}u_\varepsilon(y,y,t)\frac{d}{dy}\big[y^2\psi(y,y)\big]dy\\
&\quad=\int\limits_\Omega\big[\varepsilon\partial_{yy}u_\varepsilon-\beta_\varepsilon\big(u_\varepsilon-(K-y)\big)\big]\psi\,ds\,dy\\
&\quad=\int\limits_0^{+\infty}\varepsilon\partial_y u_\varepsilon\psi\big|_{y=0}^{y=s}ds-\int\limits_\Omega\big[\varepsilon\partial_y u_\varepsilon\partial_y\psi+\beta_\varepsilon\big(u_\varepsilon-(K-y)\big)\psi\big]ds\,dy.
\end{aligned}$$

Since $-1\leqslant\partial_y u_\varepsilon\leqslant 0$, letting $\varepsilon\to 0$, we obtain

$$\begin{aligned}
&\int\limits_\Omega\bigg[\partial_t u\psi+\frac{1}{2}\sigma^2(\partial_s u)\partial_s\big(s^2\psi\big)-(r-q)s\partial_s u\psi+ru\psi\bigg]ds\,dy\\
&\qquad-\frac{1}{2}\sigma^2\int\limits_0^{+\infty}u(y,y,t)\frac{d}{dy}\big[y^2\psi(y,y)\big]dy=\int\limits_\Omega\xi\psi\,ds\,dy.
\end{aligned}$$

Moreover, (2.39)–(2.42) are deduced from (2.20), (2.22), (2.23), (2.30) and (2.36). □

3. Uniqueness of weak solution

Denote $\Omega_n = \{(s, y) \mid n^{-1} < s < n,\ 0 < y < s\}$. Note that (2.5) in the definition of weak solution is equivalent to

(4)′ *For any* $\psi(s, y) \in C^1_{loc}(\overline{\Omega})$, *a.e.* $0 < t \leqslant T$, *a.e.* $n > 1$,

$$\int_{\Omega_n} \partial_t u\psi\,ds\,dy + \int_{\Omega_n}\left[\frac{1}{2}\sigma^2(\partial_s u)\partial_s\big(s^2\psi\big) - (r-q)s\partial_s u\psi + ru\psi\right]ds\,dy$$
$$+ \frac{1}{2}\sigma^2 n^{-2}\int_0^{n^{-1}} \partial_s u\big(n^{-1}, y, t\big)\psi\big(n^{-1}, y\big)\,dy - \frac{1}{2}\sigma^2 n^2 \int_0^n \partial_s u(n, y, t)\psi(n, y)\,dy$$
$$+ \frac{1}{2}\sigma^2 n^2 u(n, n, t)\psi(n, n) - \frac{1}{2}\sigma^2 n^{-2} u\big(n^{-1}, n^{-1}, t\big)\psi\big(n^{-1}, n^{-1}\big)$$
$$- \frac{1}{2}\sigma^2 \int_{n^{-1}}^n u(y, y, t)\frac{d}{dy}\big[y^2\psi(y, y)\big]\,dy = \int_{\Omega_n} \xi\psi\,ds\,dy. \tag{3.1}$$

The difference of (2.5) and (3.1) is that in (3.1) there are four more terms on the boundaries $s = n^{-1}$ and $s = n$ of Ω_n. We will employ (3.1) to prove the uniqueness of weak solution.

Theorem 3.1. *The weak solution of* (1.1) *is unique.*

Proof. Suppose both (u_1, ξ_1) and (u_2, ξ_2) are two weak solutions in $B \times L^\infty(Q_T)$. Denote $u = u_1 - u_2$, applying (3.1) for (u_1, ξ_1) and (u_2, ξ_2), subtracting the resulting equalities each other, we have for any $\psi(s, y) \in C^1_{loc}(\overline{\Omega})$, *a.e.* $0 < t \leqslant T$, *a.e.* $n > 1$,

$$\int_{\Omega_n} \partial_t u\psi\,ds\,dy + \int_{\Omega_n}\left[\frac{1}{2}\sigma^2(\partial_s u)\partial_s\big(s^2\psi\big) - (r-q)s\partial_s u\psi + ru\psi\right]ds\,dy$$
$$+ \frac{1}{2}\sigma^2 n^{-2}\int_0^{n^{-1}} \partial_s u\big(n^{-1}, y, t\big)\psi\big(n^{-1}, y\big)\,dy - \frac{1}{2}\sigma^2 n^2 \int_0^n \partial_s u(n, y, t)\psi(n, y)\,dy$$
$$+ \frac{1}{2}\sigma^2 n^2 u(n, n, t)\psi(n, n) - \frac{1}{2}\sigma^2 n^{-2} u\big(n^{-1}, n^{-1}, t\big)\psi\big(n^{-1}, n^{-1}\big)$$
$$- \frac{1}{2}\sigma^2 \int_{n^{-1}}^n u(y, y, t)\frac{d}{dy}\big[y^2\psi(y, y)\big]\,dy = \int_{\Omega_n} (\xi_1 - \xi_2)\psi\,ds\,dy. \tag{3.2}$$

Let $u_\delta \in C^1(\overline{Q}_T)$ and when $\delta \to 0^+$,

$$u_\delta \to u \quad \text{in } C(\overline{Q}_T),$$
$$|\partial_s u_\delta| + |\partial_y u_\delta| \leqslant 4,$$
$$u_\delta \to u \quad \text{in } H^1_{loc}(\Omega) \text{ weakly}.$$

Taking $\psi(s,y)=u_\delta(s,y,t)\frac{1}{s^2[(s^2-1)^++1]}$ in (3.2) yields

$$\begin{aligned}
&\int_{\Omega_n}\partial_t u u_\delta\frac{1}{s^2[(s^2-1)^++1]}\,ds\,dy+\frac{1}{2}\sigma^2\int_{\Omega_n}(\partial_s u)\partial_s\left(u_\delta\frac{1}{(s^2-1)^++1}\right)ds\,dy\\
&\quad-\int_{\Omega_n}\left[(r-q)\partial_s u u_\delta\frac{1}{s[(s^2-1)^++1]}-ruu_\delta\frac{1}{s^2[(s^2-1)^++1]}\right]ds\,dy\\
&\quad+\frac{1}{2}\sigma^2\int_0^{n^{-1}}\partial_s u\big(n^{-1},y,t\big)u_\delta\big(n^{-1},y,t\big)\,dy-\frac{1}{2}\sigma^2n^{-2}\int_0^n\partial_s u(n,y,t)u_\delta(n,y,t)\,dy\\
&\quad+\frac{1}{2}\sigma^2n^{-2}u(n,n,t)u_\delta(n,n,t)-\frac{1}{2}\sigma^2u\big(n^{-1},n^{-1},t\big)u_\delta\big(n^{-1},n^{-1},t\big)\\
&\quad-\frac{1}{2}\sigma^2\int_{n^{-1}}^n u(s,s,t)\frac{d}{ds}\left[\frac{u_\delta(s,s,t)}{(s^2-1)^++1}\right]ds\\
&=\int_{\Omega_n}u_\delta\frac{1}{s^2[(s^2-1)^++1]}(\xi_1-\xi_2)\,ds\,dy. \qquad (3.3)
\end{aligned}$$

Notice that the last term in the left-hand side of equality (3.3)

$$\begin{aligned}
&-\frac{1}{2}\sigma^2\int_{n^{-1}}^n u(s,s,t)\frac{d}{ds}\left[\frac{u_\delta(s,s,t)}{(s^2-1)^++1}\right]ds\\
&\quad=-\frac{1}{2}\sigma^2\int_{n^{-1}}^n u_\delta(s,s,t)\frac{d}{ds}\left[\frac{u_\delta(s,s,t)}{(s^2-1)^++1}\right]ds\\
&\qquad+\frac{1}{2}\sigma^2\int_{n^{-1}}^n (u_\delta-u)(s,s,t)\frac{d}{ds}\left[\frac{u_\delta(s,s,t)}{(s^2-1)^++1}\right]ds.
\end{aligned}$$

For fixed n, the last term goes to zero if $\delta\to0^+$. Thus

$$\begin{aligned}
&-\frac{1}{2}\sigma^2\int_{n^{-1}}^n u(s,s,t)\frac{d}{ds}\left[\frac{u_\delta(s,s,t)}{(s^2-1)^++1}\right]ds\\
&=-\frac{1}{2}\sigma^2\int_{n^{-1}}^n u_\delta(s,s,t)\frac{d}{ds}\left[\frac{u_\delta(s,s,t)}{(s^2-1)^++1}\right]ds+\varepsilon(\delta)\\
&=-\frac{1}{2}\sigma^2\frac{(u_\delta(s,s,t))^2}{(s^2-1)^++1}\bigg|_{n^{-1}}^n+\frac{1}{2}\sigma^2\int_{n^{-1}}^n\frac{u_\delta(s,s,t)}{(s^2-1)^++1}\frac{d}{ds}u_\delta(s,s,t)\,ds+\varepsilon(\delta)
\end{aligned}$$

$$= -\frac{1}{2}\sigma^2 \frac{(u_\delta(s,s,t))^2}{(s^2-1)^+ + 1}\bigg|_{n^{-1}}^{n} + \frac{1}{4}\sigma^2 \int_{n^{-1}}^{n} \frac{1}{(s^2-1)^+ + 1} \frac{d}{ds}\big(u_\delta(s,s,t)\big)^2\, ds + \varepsilon(\delta)$$

$$= -\frac{1}{4}\sigma^2 \frac{(u_\delta(s,s,t))^2}{(s^2-1)^+ + 1}\bigg|_{n^{-1}}^{n} + \frac{1}{2}\sigma^2 \int_{1}^{n} \big(u_\delta(s,s,t)\big)^2 \frac{1}{s^3}\, ds + \varepsilon(\delta). \tag{3.4}$$

Substituting (3.4) into (3.3) and letting δ go to zero, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_n} u^2 \frac{1}{s^2[(s^2-1)^+ + 1]}\, ds\, dy + \frac{1}{2}\sigma^2 \int_{\Omega_n} (\partial_s u)\partial_s\left(u \frac{1}{(s^2-1)^+ + 1}\right) ds\, dy \\
&\quad - \int_{\Omega_n} \left[(r-q)(\partial_s u) u \frac{1}{s[(s^2-1)^+ + 1]} - ru^2 \frac{1}{s^2[(s^2-1)^+ + 1]}\right] ds\, dy \\
&\quad + \frac{1}{2}\sigma^2 \int_0^{n^{-1}} \partial_s u\big(n^{-1}, y, t\big) u\big(n^{-1}, y, t\big)\, dy - \frac{1}{2}\sigma^2 n^{-2} \int_0^{n} \partial_s u(n, y, t) u(n, y, t)\, dy \\
&\quad + \frac{1}{2}\sigma^2 n^{-2}\big(u(n,n,t)\big)^2 - \frac{1}{2}\sigma^2\big(u\big(n^{-1}, n^{-1}, t\big)\big)^2 \\
&\quad - \frac{1}{4}\sigma^2 \frac{(u(s,s,t))^2}{(s^2-1)^+ + 1}\bigg|_{n^{-1}}^{n} + \frac{1}{2}\sigma^2 \int_1^n \big(u(s,s,t)\big)^2 \frac{1}{s^3}\, ds \\
&= \int_{\Omega_n} u \frac{1}{s^2[(s^2-1)^+ + 1]} (\xi_1 - \xi_2)\, ds\, dy.
\end{aligned} \tag{3.5}$$

Notice that in the last term,

$$u(\xi_1 - \xi_2) = (u_1 - u_2)(\xi_1 - \xi_2),$$

and if $u_1 > u_2$, then $u_1 > (K-y)^+$; in this case $\xi_1 = 0 \leqslant \xi_2$, in the same reason if $u_1 < u_2$, then $\xi_1 \geqslant \xi_2$. So in any case $u(\xi_1 - \xi_2) \leqslant 0$, hence the right-hand side of (3.5) is not positive.

Moreover, the second term in the left-hand side of equality (3.5)

$$\begin{aligned}
&\frac{1}{2}\sigma^2 \int_{\Omega_n} (\partial_s u)\partial_s\left(u \frac{1}{(s^2-1)^+ + 1}\right) ds\, dy \\
&\quad \geqslant \frac{1}{4}\sigma^2 \int_{\Omega_n} (\partial_s u)^2 \frac{1}{(s^2-1)^+ + 1}\, ds\, dy \\
&\qquad - C \int_{\Omega_n} u^2 \frac{1}{s^2[(s^2-1)^+ + 1]}\, ds\, dy,
\end{aligned} \tag{3.6}$$

where C is independent of n.

Applying $|\partial_s u| \leqslant 2$, $|u| \leqslant 2K$,

$$\left| \int_0^{n^{-1}} \partial_s u\left(n^{-1}, y, t\right) u\left(n^{-1}, y, t\right) dy \right| \leqslant 4Kn^{-1}, \tag{3.7}$$

$$\left| n^{-2} \int_0^{n} \partial_s u(n, y, t) u(n, y, t)\, dy \right| \leqslant 4Kn^{-1}, \tag{3.8}$$

$$-\left(u\left(n^{-1}, n^{-1}, t\right)\right)^2 \geqslant -\varepsilon(n), \tag{3.9}$$

where $\lim_{n\to+\infty} \varepsilon(n) = 0$ by $u_1(s,0,t) = u_2(s,0,t) = K$ and $u(0,0,t) = 0$. Substituting (3.6)–(3.9) into (3.5), we know that

$$\frac{d}{dt} \int_{\Omega_n} u^2 \frac{1}{s^2[(s^2-1)^+ + 1]}\, ds\, dy \leqslant C \int_{\Omega_n} u^2 \frac{1}{s^2[(s^2-1)^+ + 1]}\, ds\, dy + \varepsilon(n).$$

It follows that

$$\int_{\Omega_n} u^2 \frac{1}{s^2[(s^2-1)^+ + 1]}\, ds\, dy \leqslant \frac{\varepsilon(n)}{C}\left(e^{Ct} - 1\right).$$

Let $n \to +\infty$,

$$\int_{\Omega} u^2 \frac{1}{s^2[(s^2-1)^+ + 1]}\, ds\, dy = 0, \quad 0 \leqslant t \leqslant T.$$

Hence the uniqueness is proved. □

4. Continuity of $\partial_y u$ on $s = y$, $t > 0$

Lemma 4.1. *If $t > 0$, u is always positive, i.e.*

$$u(s, y, t) > 0, \quad t > 0,\ s \geqslant y. \tag{4.1}$$

Proof. For any $y > 0$,

$$\mathcal{L}u \geqslant 0, \quad (s,t) \in (y, \infty) \times (0, T],$$
$$u(y, y, t) \geqslant (K - y)^+ \geqslant 0,$$
$$u(s, y, 0) = (K - y)^+ \geqslant 0,$$

therefore $u(s, y, t) > 0$, $t \in (0, T]$, $s \in (y, \infty)$ by the strong maximum principle, moreover since $\partial_s u(s, y, t) \leqslant 0$, we have

$$u(y, y, t) \geqslant u(s, y, t) > 0, \quad t > 0. \qquad \square \tag{4.2}$$

Lemma 4.2. *$u(s, s, t) - (K - s)$ is monotonic increasing with respect to s.*

Proof. Notice that

$$\frac{d}{ds}\big[u_\varepsilon\big(s, f_\varepsilon(s), t\big) - (K-s)\big] = \partial_s u_\varepsilon\big(s, f_\varepsilon(s), t\big) + \partial_y u_\varepsilon\big(s, f_\varepsilon(s), t\big) f'_\varepsilon(s) + 1$$
$$= \partial_s u_\varepsilon\big(s, f_\varepsilon(s), t\big) + 1 \geqslant 0,$$

thus $u_\varepsilon(s, f_\varepsilon(s), t) - (K-s)$ is monotonic increasing with respect to s. Letting $\varepsilon \to 0$, we have $u(s,s,t) - (K-s)$ is monotonic increasing with respect to s. □

Lemma 4.3. *$u(s, y, t)$ is monotonic increasing with respect to t, i.e.,*

$$\partial_t u(s, y, t) \geqslant 0, \quad \textit{a.e. in } Q_T. \tag{4.3}$$

Proof. For any $\delta > 0$, set $v(s, y, t) = u(s, y, t+\delta)$, then $v(s, y, t)$ satisfies

$$\begin{cases} \min\big\{\mathcal{L}v, v - (K-y)^+\big\} = 0, & 0 < y < s < +\infty,\ 0 < t \leqslant T-\delta, \\ v(s, y, 0) = u(s, y, \delta), & 0 \leqslant y \leqslant s < +\infty, \\ \partial_y v(y, y, t) = \partial_y u(y, y, t+\delta) = 0, & y > 0,\ 0 < t \leqslant T-\delta. \end{cases}$$

In view of

$$v(s, y, 0) = u(s, y, \delta) \geqslant (K-y)^+ = u(s, y, 0),$$

applying the monotonicity of solution of variational inequality with respect to initial value, we have

$$v(s, y, t) \geqslant u(s, y, t), \quad t > 0,$$

which implies $u(s, y, t)$ is monotonic increasing with respect to t. □

Lemma 4.4.

$$u(y, y, t) > (K-y)^+, \quad t > 0,\ y > 0. \tag{4.4}$$

Proof. If there the conclusion is false, there exist $t_0 > 0,\ y_0 > 0$, such that

$$u(y_0, y_0, t_0) = (K-y_0)^+.$$

From (4.1) we see that $y_0 < K$. In view of Lemma 4.2, for any $y < y_0$, we have

$$u(y, y, t_0) - (K-y) = 0.$$

Applying the right-hand side of (2.40) and (4.3) deduces that

$$u(s, y, t) - (K-y) = 0, \quad s > y,\ y < y_0,\ 0 \leqslant t \leqslant t_0. \tag{4.5}$$

Thus

$$\partial_y u(y, y, t) = -1, \quad 0 \leqslant t \leqslant t_0,\ y < y_0, \tag{4.6}$$

which contradicts the definition of weak solution (2.5). In fact, from (4.5),

$$\partial_t u = \partial_s u = 0, \quad s > y,\ y < y_0,\ 0 \leqslant t \leqslant t_0.$$

Let $\psi(s, y) \in C_0^1(\Omega)$ and $\psi \equiv 0$ for $y \geqslant y_0$ in (2.5), then (2.5) becomes, for $0 < t \leqslant t_0$,

$$\int_\Omega r(K-y)\psi\, ds\, dy = \int_\Omega \xi\psi\, ds\, dy.$$

It means

$$\xi = r(K-y), \quad s > y,\ y < y_0,\ 0 \leqslant t \leqslant t_0. \tag{4.7}$$

Once more, let $\psi(s, y) \in C_0^1(\overline{\Omega})$ and $\psi \equiv 0$ for $y \geqslant y_0$ in (2.5), then, for $0 < t \leqslant t_0$,

$$\int_\Omega r(K-y)\psi\, ds\, dy - \frac{1}{2}\sigma^2 \int_0^{+\infty} (K-y)\frac{d}{dy}\big[y^2\psi(y,y)\big]\, dy = \int_\Omega \xi\psi\, ds\, dy.$$

It follows that, by (4.7),

$$\int_0^{+\infty} (K-y)\frac{d}{dy}\big[y^2\psi(y,y)\big]\, dy = 0.$$

But

$$\int_0^{+\infty} (K-y)\frac{d}{dy}\big[y^2\psi(y,y)\big]\, dy = \int_0^{+\infty} y^2\psi(y,y)\, dy,$$

which cannot always be zero for any $\psi \in C_0^1(\overline{\Omega})$ and $\psi \equiv 0$. □

Theorem 4.5. *$\partial_y u(s, y, t)$ is continuous on the surface $\{y = s,\ 0 < t \leqslant T\}$.*

Proof. Since $u(y, y, t) > (K-y)^+$ for any $y, t > 0$, we have $u_\varepsilon(y, y, t) > (K-y)^+$ if ε is small enough. For any y_0, $t_0 > 0$, letting $\delta_0 > 0$ be small enough such that $t_0 - \delta_0 > 0$, denote

$$D(y_0, t_0) = \left\{(s, y, t) \in Q_T\colon\ |y - y_0| < \frac{y_0}{2},\ 0 < s - y < \delta,\ 0 < t_0 - t < \delta_0\right\},$$

where δ is small enough, such that

$$u_\varepsilon(s, y, t) > (K-y)^+, \quad (s, y, t) \in D(y_0, t_0).$$

Denoting $v_1 = \partial_y u_\varepsilon$, then from (2.25), if ε is small enough,

$$\begin{cases} \partial_t v_1 - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v_1 - \varepsilon \partial_{yy} v_1 - (r-q)s\partial_s v_1 + r v_1 = 0, & (s,y,t) \in D(y_0,t_0), \\ v_1(s,s,t) = 0, & s \in \left(\frac{1}{2}y_0, \frac{3}{2}y_0\right), \\ v_1(s,y,t) \geqslant -1, & \text{on } \partial D(y_0,t_0). \end{cases} \tag{4.8}$$

Define

$$w(s,y,t) = \frac{2}{\delta_0}(t-t_0) + B(s-y)^2 - C(y-y_0)^2 - D(s-y),$$

where $B, C, D > 0$ are to be determined. Thus, if we take $B\delta^2 = 1$, $C = \frac{8}{y_0^2}$, $D = \frac{2}{\delta}$,

$$\begin{cases} w(s,y,t_0-\delta_0) = -2 + B(s-y)^2 - C(y-y_0)^2 - D(s-y) \leqslant -2 + B\delta^2 = -1, \\ w(y,y,t) = \frac{2}{\delta_0}(t-t_0) - C(y-y_0)^2 \leqslant -C(y-y_0)^2 \leqslant 0, \\ w(y+\delta,y,t) = \frac{2}{\delta_0}(t-t_0) + B\delta^2 - C(y-y_0)^2 - D\delta \leqslant B\delta^2 - D\delta = -1, \\ w\left(s,\frac{1}{2}y_0,t\right) = \frac{2}{\delta_0}(t-t_0) + B\left(s-\frac{1}{2}y_0\right)^2 - \frac{1}{4}Cy_0^2 - D\left(s-\frac{1}{2}y_0\right) \leqslant B\delta^2 - \frac{1}{4}Cy_0^2 = -1, \\ w\left(s,\frac{3}{2}y_0,t\right) = \frac{2}{\delta_0}(t-t_0) + B\left(s-\frac{3}{2}y_0\right)^2 - \frac{1}{4}Cy_0^2 - D\left(s-\frac{3}{2}y_0\right) \leqslant B\delta^2 - \frac{1}{4}Cy_0^2 = -1. \end{cases}$$

Moreover

$$\begin{aligned} &\partial_t w - \frac{1}{2}\sigma^2 s^2 \partial_{ss} w - \varepsilon \partial_{yy} w - (r-q)s\partial_s w + rw \\ &\leqslant \frac{2}{\delta_0} - \sigma^2 s^2 B + 2\varepsilon C - (r-q)s\big[2B(s-y) - D\big] + rB(s-y)^2 \\ &\leqslant \frac{2}{\delta_0} + \left[-\frac{1}{4}\sigma^2 y_0^2 + 2|r-q|\left(\frac{3}{2}y_0+\delta\right)\delta + r\delta^2\right]B + 2\varepsilon C + |r-q|D\left(\frac{3}{2}y_0+\delta\right) \\ &= \frac{2}{\delta_0} + \left[-\frac{1}{4}\sigma^2 y_0^2 + 2|r-q|\left(\frac{3}{2}y_0+\delta\right)\delta + r\delta^2\right]\frac{1}{\delta^2} + 16\varepsilon\frac{1}{y_0^2} + \frac{2}{\delta}|r-q|\left(\frac{3}{2}y_0+\delta\right) \\ &= -\frac{1}{4\delta^2}\sigma^2 y_0^2 + \frac{6}{\delta}|r-q|y_0 + 4|r-q| + \frac{16\varepsilon}{y_0^2} + \frac{2}{\delta_0} + r \leqslant 0, \end{aligned}$$

if δ is small enough. Hence w is a subsolution of (4.8), i.e.,

$$\partial_y u_\varepsilon \geqslant w \quad \text{in } D(y_0,t_0),$$

and letting $\varepsilon \to 0$, we obtain

$$\partial_y u \geqslant w \quad \text{in } D(y_0,t_0).$$

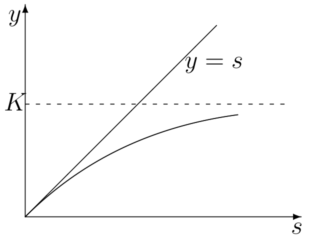

Fig. 3. $h(s, t_0)$.

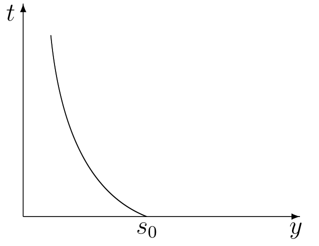

Fig. 4. $h(s_0, t)$.

Note that $w(y_0, y_0, t_0) = 0$, so $\overline{\lim}_{(s,y,t)\to(y_0,y_0,t_0)} \partial_y u(s, y, t) \geqslant 0$. Recalling $\partial_y u \leqslant 0$, we conclude

$$\lim_{(s,y,t)\to(y_0,y_0,t_0)} \partial_y u(s, y, t) = 0. \qquad \square$$

5. Properties of free boundary

From (2.40), (2.41) and (4.3) we have

$$\partial_s\big(u - (K - y)\big) \leqslant 0, \tag{5.1}$$

$$\partial_y\big(u - (K - y)\big) \geqslant 0, \tag{5.2}$$

$$\partial_t\big(u - (K - y)\big) \geqslant 0. \tag{5.3}$$

We notice that, by (4.1), $u > (K - y)^+$ is equivalent to $u > K - y$, so we define coincidence set

$$\mathcal{C} = \big\{(s, y, t) \bigm| u(s, y, t) = K - y\big\},$$

and noncoincidence set

$$\mathcal{N} = \big\{(s, y, t) \bigm| u(s, y, t) > K - y\big\}.$$

From (5.2) we can define free boundary

$$h(s, t) = \max\big\{y \bigm| u(s, y, t) = K - y\big\}, \quad t \in (0, T],\ s \in [0, +\infty).$$

Since $(K - y)^+ = 0$ if $y \geqslant K$, by (4.1) we know that

$$\{y < K\} \supset \mathcal{C}, \qquad \{y \geqslant K\} \subset \mathcal{N},$$

thus

$$h(s, t) < K, \quad t > 0. \tag{5.4}$$

Theorem 5.1. *For any fixed $t \in (0, T]$, $h(s, t)$ is monotonic increasing with respect to $s \in [0, +\infty)$; for any fixed $s \in [0, +\infty)$, $h(s, t)$ is monotonic decreasing with respect to $t \in (0, T]$.*

Proof. Applying (5.1) and (5.2) we see that $h(s, t)$ is monotonic increasing with respect to s (see Fig. 3), (5.2) and (5.3) show that $h(s, t)$ is monotonic decreasing with respect to t (Fig. 4). $\square$

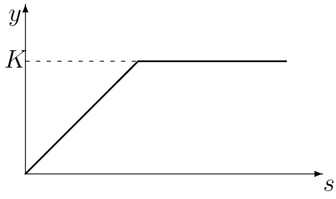

Fig. 5. $h(s, 0)$.

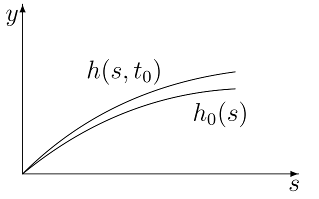

Fig. 6. $h(s, t_0)$ and $h_0(s)$.

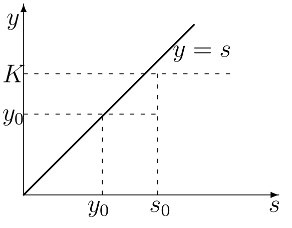

Fig. 7. U_0.

Remark 1. Since $h(s,t)$ is monotonic with respect to $t \in (0, T]$, we can define $h(s,0) = \lim_{t\to 0^+} h(s,t)$.

Theorem 5.2.

$$h(s,0) = \min\{s, K\}, \quad s \in [0, +\infty), \tag{5.5}$$

$$h_0(s) \leqslant h(s,t) < \min\{s, K\}, \quad s \in [0, +\infty),\ t \in (0, T], \tag{5.6}$$

where $y = h_0(s)$ is the free boundary of variational inequality

$$\begin{cases} \mathcal{L}_1 u_\infty \geqslant 0, \quad u_\infty - (K-y)^+ \geqslant 0, & s \in \mathbb{R}^+,\ y \in (0,s), \\ [\mathcal{L}_1 u_\infty]\big[u_\infty - (K-y)^+\big] = 0, & s \in \mathbb{R}^+,\ y \in (0,s), \\ \partial_y u_\infty(s,s) = 0, & s \in \mathbb{R}^+, \end{cases} \tag{5.7}$$

where

$$\mathcal{L}_1 u_\infty = -\frac{1}{2}\sigma^2 s^2 \partial_{ss} u_\infty - (r-q) s \partial_s u_\infty + r u_\infty.$$

Proof. We first prove (5.5). From inequalities (4.4), (5.4) and Theorem 5.1 we know

$$h(s,t) < \min\{s, K\}, \quad t > 0, \tag{5.8}$$

so $h(s,0) \leqslant \min\{s,K\}$. If (5.5) is false, then there exists s_0, such that $h(s_0,0) < \min\{s_0, K\}$, denoting $y_0 = h(s_0, 0)$, then for any $s \in (y_0, s_0)$, we have $y = h(s,0) \leqslant h(s_0,0) = y_0$. (See Figs. 5 and 6.)

Denote $U \triangleq (y_0, s_0) \times (y_0, \min(s,K)) \times (0,T)$, $U_0 \triangleq (y_0, s_0) \times (y_0, \min(s,K)) \times \{0\}$ (see Fig. 7).

By the definition of $h(s,0)$ and $h(s,t)$ being monotonic decreasing with respect to t, we know that

$$\mathcal{L} u = 0 \quad \text{in } U,$$

and $u(s,y,0) = K - y$ in U_0, thus

$$\partial_s u = \partial_{ss} u = 0 \quad \text{in } U_0,$$

hence

$$\partial_t u(s, y, 0) = -ru(s, y, 0) = -r(K - y) < 0 \quad \text{in } U_0,$$

which is a contradiction with $\partial_t u \geqslant 0$.

Next we aim to prove (5.6). From (5.8) we only need to prove the left part of (5.6). Let $s = g_0(y)$ be the free boundary of the problem (5.7). The problem (5.7) can be solved as follows: find $u_\infty(s, y)$, $g_0(y)$ satisfying

$$\begin{cases} \frac{1}{2}\sigma^2 s^2 \partial_{ss} u_\infty + (r - q)s\partial_s u_\infty - ru_\infty = 0, & y < s < g_0(y),\ y \in (0, +\infty), \\ u_\infty\big(g_0(y), y\big) = (K - y)^+, & y \in (0, +\infty), \\ \partial_s u_\infty\big(g_0(y), y\big) = 0, & y \in (0, +\infty), \\ \partial_y u_\infty(s, s) = 0, & s \in \mathbb{R}^+, \end{cases}$$

then $g_0(y)$ is the unique solution to the differential equation (which had been proved in [10])

$$\gamma_1\left(\frac{y}{g_0(y)}\right)^{\gamma_0} - \gamma_0\left(\frac{y}{g_0(y)}\right)^{\gamma_1} = \gamma_0\gamma_1(K - y)\frac{g_0'(y)}{g_0(y)}\left[\left(\frac{y}{g_0(y)}\right)^{\gamma_1} - \left(\frac{y}{g_0(y)}\right)^{\gamma_0}\right],$$

where

$$\gamma_0 = \left(\frac{1}{2} - \frac{r - q}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{r - q}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0,$$

$$\gamma_1 = \left(\frac{1}{2} - \frac{r - q}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{r - q}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0.$$

Since

$$g_0'(y) = \frac{\gamma_1(\frac{y}{g_0(y)})^{\gamma_0} - \gamma_0(\frac{y}{g_0(y)})^{\gamma_1}}{\gamma_0\gamma_1(K - y)[(\frac{y}{g_0(y)})^{\gamma_1} - (\frac{y}{g_0(y)})^{\gamma_0}]} g_0(y) > 0,$$

thus $g_0(y)$ is strictly increasing with respect to y, then we can define $y = g_0^{-1}(s) \triangleq h_0(s)$. In view of $\partial_t u_\infty(s, y) = 0$, then $u_\infty(s, y)$ satisfies evolutionary system

$$\begin{cases} \mathcal{L}u_\infty \geqslant 0, \quad u_\infty - (K - y)^+ \geqslant 0, & s \in \mathbb{R}^+,\ y \in (0, s),\ t > 0, \\ [\mathcal{L}u_\infty]\big[u_\infty - (K - y)^+\big] = 0, & s \in \mathbb{R}^+,\ y \in (0, s),\ t > 0, \\ u_\infty(s, y)|_{t=0} = u_\infty(s, y), & s \in \mathbb{R}^+,\ y \in (0, s), \\ \partial_y u_\infty(s, s) = 0, & s \in \mathbb{R}^+,\ t > 0. \end{cases}$$

Since $u_\infty(s, y)|_{t=0} = u_\infty(s, y) \geqslant (K - y)^+ = u(s, y, 0)$, applying the monotonicity of solution of variational inequality with respect to initial value, we have

$$u_\infty(s, y) \geqslant u(s, y, t),$$

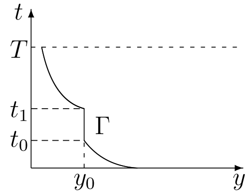

Fig. 8. $h(s_0, t)$.

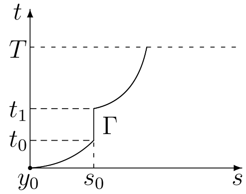

Fig. 9. $y = y_0$.

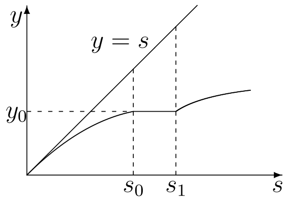

Fig. 10. $t = t_0$.

in particular,

$$u\big(s, h_0(s), t\big) \leqslant u_\infty\big(s, h_0(s)\big) = K - h_0(s),$$

hence we have $h(s,t) \geqslant h_0(s)$ by the definition of $h(s,t)$. □

Theorem 5.3. *For any fixed $s \in (0, +\infty)$, $h(s,t)$ is strictly decreasing with respect to $t \in [0, T]$; for any fixed $t \in (0, T]$, $h(s,t)$ is strictly increasing with respect to $s \in [0, +\infty)$.*

Proof. First we want to prove that $h(s,t)$ is strictly decreasing with respect to t. If the conclusion is false, then there exists s_0, such that $y = h(s_0, t)$ has a vertical part $\Gamma = \{s = s_0,\ y = y_0,\ t_0 < t < t_1\}$ (see Fig. 8), then in the plane $\{y = y_0\}$ (Fig. 9), we have

$$u|_\Gamma = K - y_0, \qquad \partial_s u|_\Gamma = 0,$$

thus

$$\partial_t u|_\Gamma = \partial_{st} u|_\Gamma = 0.$$

Since $\partial_t u \geqslant 0$ and $\mathcal{L}(\partial_t u) = 0$ on $\{y = y_0,\ s < s_0,\ t > t_0\}$, by the strong maximum principle, we have $\partial_{st} u|_\Gamma < 0$, otherwise $\partial_t u \equiv 0$, but both come to contradiction.

Next we aim to prove that $h(s,t)$ is strictly increasing with respect to s. If the conclusion is false, then there exists $t_0 \in (0, T]$, such that $h(s, t_0)$ is not strictly increasing (Fig. 10).

That is, there exists y_0 such that

$$h(s, t_0) = y_0, \quad s \in (s_0, s_1),$$

thus

$$\mathcal{L}u = 0 \quad \text{in } (s_0, s_1) \times (y_0, s) \times (t_0, T),$$

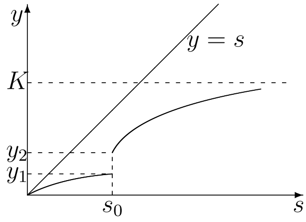

Fig. 11. $t = t_0$.

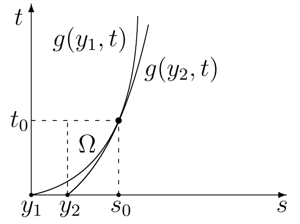

Fig. 12. $g(y_1, t)$ and $g(y_2, t)$.

moreover $u(s, y_0, t_0) = K - y_0$, $s \in (s_0, s_1)$. Thus

$$\partial_t u(s, y_0, t_0) = -ru(s, y_0, t_0) = -r(K - y_0) < 0, \quad s \in (s_0, s_1),$$

which is again in contradiction with $\partial_t u \geqslant 0$. $\square$

Remark 2. Since for $t \in (0, T]$, $y = h(s, t)$ is strictly increasing with respect to $s \in [0, +\infty)$, then we can define $s = g(y, t)$, moreover $g(y, t)$ is strictly increasing with respect to $y \in [0, K)$.

Theorem 5.4. *For any* $y \in [0, K)$, $g(y, t) \in C^\infty(0, T]$ *with* $g(y, 0) = y$.

Proof. For any fixed y, the solution $u(s, y, t)$ of the problem (1.1) satisfies

$$\begin{cases} \min\{\mathcal{L}u(\cdot, y, \cdot), u(\cdot, y, \cdot) - (K - y)^+\} = 0, & \text{in } \Omega_y, \\ u(s, y, 0) = (K - y)^+, & y \leqslant s < +\infty, \\ u(y, y, t) \geqslant 0, & 0 < t \leqslant T, \end{cases}$$

where $\Omega_y = \{(s, t) \mid y < s < +\infty,\ 0 < t \leqslant T\}$. Following the idea of studying a 1-dimensional variational inequality [9], we know $u(\cdot, y, \cdot) \in W^{2,1}_{p,loc}(\Omega_y)$ for any $p > 1$.

In the case of $y = 0$, we have $u \equiv K$ and $g(0, t) \equiv 0$. In the case of $y > 0$, due to $\partial_t u \geqslant 0$ and $(K - y)^+$ is the lower obstacle, then for any fixed $y \in (0, K)$, it can be proved $g(y, \cdot) \in C^{0,1}(0, T]$ by a method developed by Friedman in [9]. Moreover $g(y, \cdot) \in C^\infty(0, T]$ by the bootstrap argument.

Moreover from (5.5), we can obtain

$$g(y, 0) = y, \quad y \in [0, K). \qquad \square$$

Theorem 5.5. $y = h(s, t)$ *is continuous on* $[0, +\infty) \times [0, T]$.

Proof. Now we will prove that, for any $t \in [0, T]$, $h(s, t)$ is continuous with respect to s. If this is not true, there exists $t_0 \in (0, T]$, such that $h(s, t_0)$ is not continuous, i.e. there exists s_0, such that $y_1 \triangleq \lim_{s \to s_0^-} h(s, t_0) < \lim_{s \to s_0^+} h(s, t_0) \triangleq y_2$ (see Fig. 11).

For any $y \in (y_1, y_2)$, $u(s_0, y, t_0) = K - y$, thus $g(y_1, t_0) = g(y_2, t_0) = s_0$. Since $g(y_1, t) \leqslant g(y_2, t)$, we put $s = g(y_1, t)$ and $s = g(y_2, t)$ on the same (s, t) plane (see Fig. 12), set

$$v_1(s, t) = u(s, y_1, t) - (K - y_1), \qquad v_2(s, t) = u(s, y_2, t) - (K - y_2),$$

then

$$\mathcal{L}(v_1 - v_2) = -r(K - y_1) + r(K - y_2) = r(y_1 - y_2) < 0 \quad \text{in } \Omega,$$

where $\Omega = \{(s,t) \mid y_2 < s < g(y_1,t),\ 0 < t \leqslant t_0\}$. Since $u - (K - y)$ is monotonic increasing with respect to y, then $v_1 \leqslant v_2$, thus

$$v_1 < v_2 \quad \text{in } \Omega \quad \text{and} \quad v_1(s_0,t_0) - v_2(s_0,t_0) = 0.$$

We know $g(y_1,\cdot) \in C^\infty(0,T]$, thus we have $\partial_s(v_1 - v_2)|_{(s_0,t_0)} > 0$, but this is impossible, because $\partial_s v_1(s_0,t_0) = \partial_s v_2(s_0,t_0) = 0$, hence $h(s,t)$ is continuous with respect to s.

In a way similar to which when we prove $h(s,0) = \min\{s,K\}$, we can prove for any $s \in [0,+\infty)$, $h(s,t)$ is continuous with respect to t, combining the monotonicity of $h(s,t)$ with respect to t (Theorem 5.1), we know $y = h(s,t)$ is continuous on $[0,+\infty) \times [0,T]$. □

Remark on the continuity of $g(y,t)$ with respect to y. The continuity of $g(y,t)$ with respect to y can be obtained from the continuity of $h(s,t)$ with respect to s.

Appendix A. Formulation of the model

An American lookback option whose underlying asset is the stock which has the price S_τ given by

$$dS_\tau = (r-q)S_\tau\, d\tau + \sigma S_\tau\, dW_\tau, \tag{A.1}$$

where τ is calendar time, r, q, σ are positive constants representing riskless interest rate, dividend rate and volatility, respectively, W_τ is a standard Brownian motion under risk neutral measure $\tilde{P}$.

Let $\mathcal{T}_{\tau,T}$ be the set of all stopping time in $[\tau,T]$, $Y_\tau = \min_{0\leqslant\lambda\leqslant\tau} S_\lambda$ and the value of American lookback put option with fixed strike price at time τ is defined as

$$V(\tau) = \max_{\lambda\in\mathcal{T}_{\tau,T}} \tilde{E}\big[e^{-r(\lambda-\tau)}(K - Y_\lambda)^+ \bigm| \mathcal{F}_\tau\big], \tag{A.2}$$

where K is the strike price. Because that the pair of processes (S_τ, Y_τ) has the Markov property, there exists a function $v(s,y,\tau)$ such that

$$V(\tau) = v(S_\tau, Y_\tau, \tau).$$

It is obvious that

$$v(S_\tau, Y_\tau, \tau) \geqslant (K - Y_\tau)^+. \tag{A.3}$$

By the definition (A.2),

$$\begin{aligned}
e^{-r\tau} v(S_\tau, Y_\tau, \tau) &= \max_{\lambda\in\mathcal{T}_{\tau,T}} \tilde{E}\big[e^{-r\lambda}(K - Y_\lambda)^+ \bigm| \mathcal{F}_\tau\big] \\
&= \max_{\lambda\in\mathcal{T}_{\tau,T}} \tilde{E}\big[\tilde{E}\big[e^{-r\lambda}(K - Y_\lambda)^+ \bigm| \mathcal{F}_{\tau+h}\big] \bigm| \mathcal{F}_\tau\big] \\
&\geqslant \tilde{E}\Big[\max_{\lambda\in\mathcal{T}_{\tau+h,T}} \tilde{E}\big[e^{-r\lambda}(K - Y_\lambda)^+ \bigm| \mathcal{F}_{\tau+h}\big] \Bigm| \mathcal{F}_\tau\Big] \\
&= \tilde{E}\big[e^{-r(\tau+h)} v(S_{\tau+h}, Y_{\tau+h}, \tau+h) \bigm| \mathcal{F}_\tau\big],
\end{aligned}$$

which implies $e^{-r\tau} v(S_\tau, Y_\tau, \tau)$ is a supermartingale under $\tilde{P}$. By the definition of Y_τ, as the analysis in [15] (pp. 309–312) we know $dY_\tau\, dY_\tau = dY_\tau\, dS_\tau = 0$. Moreover, when $S_\tau > Y_\tau$, we have $dY_\tau = 0$, and when $S_\tau = Y_\tau$, we have $\partial_y v(Y_\tau, Y_\tau, \tau) = 0$. It follows that, by the Itô formula,

$$\partial_\tau v(S_\tau, Y_\tau, \tau) + \frac{1}{2}\sigma^2 S_\tau^2 \partial_{ss} v(S_\tau, Y_\tau, \tau) + (r-q) S_\tau \partial_s v(S_\tau, Y_\tau, \tau) - r v(S_\tau, Y_\tau, \tau) \leqslant 0. \quad (A.4)$$

When $v(S_\tau, Y_\tau, \tau) > (K - Y_\tau)^+$ which implies there exists $h > 0$ small enough such that the optimal stopping time $\lambda^* \in \mathcal{T}_{\tau+h,T}$, thus

$$\begin{aligned} e^{-r\tau} v(S_\tau, Y_\tau, \tau) &= \max_{\lambda \in \mathcal{T}_{\tau,T}} \tilde{E}\big[e^{-r\lambda}(K - Y_\lambda)^+ \,\big|\, \mathcal{F}_\tau\big] \\ &= \tilde{E}\big[e^{-r\lambda^*}(K - Y_{\lambda^*})^+ \,\big|\, \mathcal{F}_\tau\big] \\ &= \max_{\lambda \in \mathcal{T}_{\tau+h,T}} \tilde{E}\big[\tilde{E}\big[e^{-r\lambda}(K - Y_\lambda)^+ \,\big|\, \mathcal{F}_{\tau+h}\big] \,\big|\, \mathcal{F}_\tau\big] \\ &= \tilde{E}\big[e^{-r(\tau+h)} v(S_{\tau+h}, Y_{\tau+h}, \tau+h) \,\big|\, \mathcal{F}_\tau\big], \end{aligned}$$

it implies that $e^{-r(\tau\wedge\lambda^*)} v(S_{\tau\wedge\lambda^*}, Y_{\tau\wedge\lambda^*}, \tau\wedge\lambda^*)$ is a martingale under $\tilde{P}$, applying Itô formula to get

$$\partial_\tau v(S_\tau, Y_\tau, \tau) + \frac{1}{2}\sigma^2 S_\tau^2 \partial_{ss} v(S_\tau, Y_\tau, \tau) + (r-q) S_\tau \partial_s v(S_\tau, Y_\tau, \tau) - r v(S_\tau, Y_\tau, \tau) = 0. \quad (A.5)$$

According to the above arguments (A.3)–(A.5), $v(s, y, \tau)$ satisfies

$$\begin{cases} v(s, y, \tau) \geqslant (K - y)^+, \\ \partial_\tau v + \frac{1}{2}\sigma^2 s^2 \partial_{ss} v + (r-q) s \partial_s v - r v \leqslant 0, \\ \left[\partial_\tau v + \frac{1}{2}\sigma^2 s^2 \partial_{ss} v + (r-q) s \partial_s v - r v\right] \cdot \left[v(s, y, \tau) - (K - y)^+\right] = 0. \end{cases} \quad (A.6)$$

Similarly to the analysis of Y_τ in [15], we obtain the boundary condition

$$\partial_y v(y, y, \tau) = 0. \quad (A.7)$$

Its financial meaning is that the option price is not sensitive to the level of the minimum of the asset price (see p. 294 in [11]). By the definition of $v(s, y, \tau)$, we obtain

$$v(s, y, T) = (K - y)^+. \quad (A.8)$$

Hence (A.6)–(A.8) tells us $v(s, y, \tau)$ satisfying

$$\begin{cases} \max\left\{\partial_\tau v + \frac{1}{2}\sigma^2 s^2 \partial_{ss} v + (r-q) s \partial_s v - r v, (K - y)^+ - v\right\} = 0, & \text{in } \tilde{Q}_T, \\ \partial_y v(y, y, \tau) = 0, & y > 0,\ 0 \leqslant \tau < T, \\ v(s, y, T) = (K - y)^+, & 0 < y < s, \end{cases} \quad (A.9)$$

where $\tilde{Q}_T = \{(s, y, \tau) \mid 0 < y < s,\ 0 \leqslant \tau < T\}$. Letting $t = T - \tau$, $u(s, y, t) = v(s, y, \tau)$, then $u(s, y, t)$ satisfies (1.1).

Appendix B. Verification

Letting $\tau = T - t$, denote $v(s, y, \tau) = u(s, y, t)$ and $V(\tau) = v(S_\tau, Y_\tau, \tau)$. In this appendix we prove $V(\tau)$ is the price of American lookback put option at time τ with fixed strike price, i.e., (A.2) holds. Since $u(s, y, t)$ is the solution of (1.1), so $v(s, y, \tau)$ satisfies (A.9).

As in the proof of Theorem 5.4, we know for *a.e.* $y \in \mathbb{R}^+$, $v(\cdot, y, \cdot) \in W^{2,1}_{p,loc}((y, +\infty) \times (0, T))$, so $v(\cdot, y, \cdot)$ and $\partial_s v(\cdot, y, \cdot)$ are continuous in $(y, +\infty) \times (0, T)$ by embedding theorem [12]. Since $\partial_\tau v = -\partial_t u \leqslant 0$ by (4.3) and $v = (K - y)^+$ is the lower obstacle, we can take advantage of the same argument as in [7] to obtain that $\partial_\tau v(\cdot, y, \cdot)$ is continuous across $s = g(y, T - \tau)$, where $g(y, t)$ is the free boundary of problem (1.1). Therefore $\partial_\tau v(\cdot, y, \cdot)$ is continuous in $(y, +\infty) \times (0, T)$. Moreover, $\partial_\tau v(\cdot, y, \cdot)$ and $\partial_{ss} v(\cdot, y, \cdot)$ are locally bounded in $(y, +\infty) \times (0, T)$.

By the definition of $Y_\tau = \min_{0 \leqslant \lambda \leqslant \tau} S_\lambda$, we know $dY_\tau \, dY_\tau = dY_\tau \, dS_\tau = 0$ and $dY_\tau \leqslant 0$. When $dY_\tau < 0$, then $S_\tau = Y_\tau$, and by the boundary condition in (A.9) we have $\partial_y v(Y_\tau, Y_\tau, \tau) = 0$, thus $\partial_y v dY = 0$. For any $\lambda \in \mathcal{T}_{\tau,T}$, applying Itô formula,

$$
\begin{aligned}
e^{-r(\lambda-\tau)} v(S_\lambda, Y_\lambda, \lambda) = v(S_\tau, Y_\tau, \tau) + \int_\tau^\lambda e^{-r(t-\tau)} \Big(& \partial_\tau v(S_t, Y_t, t) \\
& + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss} v(S_t, Y_t, t) + (r - q) S_t \partial_s v(S_t, Y_t, t) \\
& - r v(S_t, Y_t, t) \Big) dt + \int_\tau^\lambda e^{-r(t-\tau)} \sigma S_t \partial_s v(S_t, Y_t, t) \, dW_t.
\end{aligned} \tag{B.1}
$$

Owing to $(A.9)_{(1)}$, we have

$$
\begin{aligned}
v(S_\tau, Y_\tau, \tau) &\geqslant \tilde{E}\big[e^{-r(\lambda-\tau)} v(S_\lambda, Y_\lambda, \lambda) \,\big|\, \mathcal{F}_\tau\big] \\
&\geqslant \tilde{E}\big[e^{-r(\lambda-\tau)} (K - Y_\lambda)^+ \,\big|\, \mathcal{F}_\tau\big].
\end{aligned}
$$

Hence

$$
v(S_\tau, Y_\tau, \tau) \geqslant \max_{\lambda \in \mathcal{T}_{\tau,T}} \tilde{E}\big[e^{-r(\lambda-\tau)} (K - Y_\lambda)^+ \,\big|\, \mathcal{F}_\tau\big]. \tag{B.2}
$$

On the other hand, define

$$
\lambda^* = \begin{cases} \min\{\tau \in [0, T) \mid v(S_\tau, Y_\tau, \tau) = (K - Y_\tau)^+\}, \\ T, \quad \text{if } v(S_\tau, Y_\tau, \tau) > (K - Y_\tau)^+, \ \tau \in [0, T). \end{cases} \tag{B.3}
$$

If $S_\tau \geqslant g(Y_\tau, T - \tau)$, where $g(y, t)$ is defined in Theorem 5.4, then $\lambda^* = \tau$, and

$$
v(S_\tau, Y_\tau, \tau) = (K - Y_\tau)^+ = \tilde{E}\big[e^{-r(\lambda^*-\tau)} (K - Y_{\lambda^*})^+ \,\big|\, \mathcal{F}_\tau\big].
$$

If $Y_\tau \leqslant S_\tau < g(Y_\tau, T - \tau)$, choose λ^* which is given in (B.3), then

$$
\begin{cases} \partial_\tau v(S_t, Y_t, t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss} v(S_t, Y_t, t) + (r - q) S_t \partial_s v(S_t, Y_t, t) - r v(S_t, Y_t, t) = 0, \quad t \in (\tau, \lambda^*), \\ v(S_{\lambda^*}, Y_{\lambda^*}, \lambda^*) = (K - Y_{\lambda^*})^+. \end{cases} \tag{B.4}
$$

Plugging (B.4) back into (B.1), we have

$$\begin{aligned} v(S_\tau, Y_\tau, \tau) &= \tilde{E}\left[e^{-r(\lambda^*-\tau)} v\left(S_{\lambda^*}, Y_{\lambda^*}, \lambda^*\right) \,\middle|\, \mathcal{F}_\tau\right] \\ &= \tilde{E}\left[e^{-r(\lambda^*-\tau)} (K - Y_{\lambda^*})^+ \,\middle|\, \mathcal{F}_\tau\right]. \end{aligned}$$

Combining with (B.2), we have

$$v(S_\tau, Y_\tau, \tau) = \max_{\lambda \in \mathcal{T}_{\tau,T}} \tilde{E}\left[e^{-r(\lambda-\tau)} (K - Y_\lambda)^+ \,\middle|\, \mathcal{F}_\tau\right].$$

References

[1] S. Babbs, Binomial valuation of lookback options, J. Econom. Dynam. Control 24 (2000) 1499–1525.
[2] A. Bensoussan, On the theory of option pricing, Acta Appl. Math. 2 (1984) 139–158.
[3] A. Blanchet, On the regularity of the free boundary in the parabolic obstacle problem. Application to American options, Nonlinear Anal. 65 (2006) 1362–1378.
[4] M. Dai, A modified binomial tree method for currency lookback options, Acta Math. Sinica 16 (2000) 445–454.
[5] M. Dai, Y.K. Kwok, American option with lookback payoff, SIAM J. Appl. Math. 66 (2005) 206–227.
[6] M. Dai, Y.K. Kwok, Characterization of optimal stopping regions of American Asian and lookback options, Math. Finance 16 (2006) 445–454.
[7] A. Friedman, Parabolic variational inequalities in one space dimension and smoothness of the free boundary, J. Funct. Anal. 18 (1975) 151–176.
[8] A. Friedman, The Stefan problem in several space variables, Trans. Amer. Math. Soc. 133 (1968) 51–87.
[9] A. Friedman, Variational Principles and Free Boundary Problems, John Wiley and Sons, New York, 1982.
[10] X. Guo, L. Shepp, Some optimal stopping problems with non-trivial boundaries for pricing exotic options, J. Appl. Probab. 38 (2001) 647–658.
[11] L. Jiang, Mathematical Modeling and Methods of Option Pricing, World Scientific Publishing Co. Pte. Ltd., 2005.
[12] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasi-linear Equations of Parabolic Type, American Mathematical Society, Providence, RI, 1968.
[13] T.L. Lai, T.W. Lim, Exercise regions and efficient valuation of American lookback options, Math. Finance 14 (2004) 249–269.
[14] J.F. Rodrigues, F. Yi, On a two-phase continuous casting Stefan problem with nonlinear flux, European J. Appl. Math. 1 (1990) 259–278.
[15] S.E. Shreve, Stochastic Calculus for Finance II, Continuous-Time Models, Springer-Verlag, Beijing, 2007.
[16] W. Wang, L. Zhang, The C^α regularity of a class of non-homogeneous ultraparabolic equations, Sci. China Ser. A 52 (2009) 1589–1606.
[17] H. Yu, Y.K. Kwok, L. Wu, Early exercise policies of American floating strike and fixed strike lookback options, Nonlinear Anal. 47 (2001) 4591–4602.