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Sturm and Sylvester algorithms revisited via tridiagonal determinantal representations

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ABSTRACT

First, we show that Sturm algorithm and Sylvester algorithm, which compute the number of real roots of a given univariate polynomial, lead to two dual tridiagonal determinantal representations of the polynomial. Next, we show that the number of real roots of a polynomial given by a tridiagonal determinantal representation is greater than the signature of this representation.

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1. Introduction

1.1. Motivations and contributions

There are several methods to count the number of real roots of an univariate polynomial $p(x) \in \mathbb{R}[x]$. Without any doubt, the most famous ones are the Sturm and Sylvester methods. They have been intensively studied and developed.

We show how Sturm and Sylvester methods can be both “coded" by two canonical tridiagonal matrices which can be viewed as dual (Theorem 7).

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Together with Theorem 10, it gives a possible alternate presentation for real roots counting (see [2] for the other existing techniques).

Our work can also be related to a question from Numerical Analysis. Given a monic polynomial $p(x)$ whose roots are all real, how can we construct (with a “reasonable algorithm") a symmetric matrix whose characteristic polynomial is $p(x)$?

An affirmative answer has been given by Fiedler [4] using arrow matrices. Another solution has been proposed by Schmeisser [11] using tridiagonal matrices. It can also be viewed as the so-called Routh-Lanczos algorithm, closely related to the Sturm method.

We generalize this construction to the case of a polynomial $p(x)$ with real or complex roots, and show how to construct a symmetric matrix A such that $p(x)$ is proportional to $\det(xJ - A)$ where J is a signature matrix (i.e. diagonal with ± 1 on the diagonal).

We may also view our work as a contribution to the question of determinantal representation of polynomials. The problem is to write a given polynomial $p(x)$ (with d variables) as

$$p(x) = \lambda \det \left(J - \sum_{i=1}^{d} x_i A_i \right),$$

where $\lambda \in \mathbb{R}$, A_i is a symmetric matrix and J is a signature matrix. It has a lot of applications such as Operator Theory, Control Theory and Linear Matrix Inequalities. Of particular interest for applications is the case of unitary determinantal representation (when J is the identity). See [8,7] for more background and results.

For univariate polynomials ($d = 1$), the question is trivial if we are allowed to use the roots of the polynomial $p(x)$. By considering the reciprocal polynomial of $p(x)$ our method gives an algorithm (not using the roots) for finding a determinantal representation of $p(x)$ via tridiagonal matrices. Although there is a gap between dimension one and higher dimensions, maybe the explicit construction we present, together with the link with the number of real roots, could give some ideas to obtain determinantal representations for some particular cases in higher dimension.

In Theorem 8, we obtain a determinantal expression of $p(x)$ of the form

$$\det(D)p(x) = \det(xD - \mathrm{Td}),$$

where D is (only) diagonal and Td is tridiagonal and symmetric, but all the involved entries belong to the field generated by the coefficients of $p(x)$. For instance, no square root is needed as in the formulas given by [4,11].

As an application, let us mention the following fact: it is well know that a given polynomial with rational coefficients cannot necessarily be written as the characteristic polynomial of a symmetric matrix with rational entries. In this case, the previous determinantal formula could be used as a substitute.

In summary, our results concern several topics: the duality between Sturm and Sylvester algorithms, the generalization of Fielder’s result to any univariate polynomial, the effective construction of a tridiagonal determinantal representation for any univariate polynomial (the entries lying in the field generated by the coefficients of the given polynomial). It enlights some natural connections between questions from different areas: numerical analysis, real roots counting, determinantal representations. To show how naturally arise these connections, let us say some words about the results contained in [5].

This last paper appeared at the same period than [4] (which is one of the main motivations for our work). The techniques involved in [4,5] are very similar. In [4] Fiedler answered a question from numerical analysis, and in [5] he gave (although it is not formulated in the terminology of determinantal representations) the construction of a definite determinantal representation for a *rational* algebraic curve satisfying the so-called *Real Zero condition*. Moreover, he announced that the result is false for a general algebraic curve satisfying the Real Zero condition (hypothesis (b) in [5]), and gave a counterexample.

Unfortunately, it follows from a deep result of Helton and Vinnikov [8] on determinantal representations, that any real algebraic curve satisfying the Real Zero condition does admit a definite determinantal representation. Alternatively, one can check elementary that the announced counterexample in [5] does not satisfy the Real Zero condition, and also that there is a mistake in the computation of the number of real roots in [5, Eq. (5)].

We think that this particular example illustrates how fruitful could be the connections between the different topics involved in our article.

1.2. Organization of the paper

In Section 2, we introduce signed remainder sequences of two given monic polynomials $p(x)$ and $q(x)$ of respective degrees n and $n-1$. We give a presentation of this sequence through a tridiagonal matrix $\mathrm{Td}(p,q)$. Next, we give a decomposition of this tridiagonal matrix as $\mathrm{Td}(p,q) = LC_p^TL^{-1}$ where L is lower triangular and C_p^T is the transpose of the companion matrix associated to $p(x)$.

In Section 3, we introduce the duality between the Sturm and Sylvester algorithm, first when the polynomial $p(x)$ has only simple and real roots, and then in Theorem 7 we generalize it to the generic case.

More precisely, on the one hand we have

$$\begin{cases} p(x) = \det(x\mathbf{Id}_n - \mathrm{Td}(p,q)), \\ q(x) = \det(x\mathbf{Id}_{n-1} - \mathrm{Td}(p,q)_{n-1}) \end{cases}$$

with the conventions that $\mathbf{Id}_n$ (or $\mathbf{Id}$ in short) denotes the identity matrix of $\mathbb{R}^{n\times n}$ and $A_k \in \mathbb{R}^{k\times k}$ (respectively $\overline{A}_k \in \mathbb{R}^{k\times k}$) denotes the *kth principal submatrix* (respectively the *kth antiprincipal submatrix*) of A which corresponds to extracting the first k (respectively the last k) rows and columns in the matrix $A \in \mathbb{R}^{n\times n}$.

On the other hand, we consider a natural Hankel (hence symmetric) matrix $H(q/p) \in \mathbb{R}^{n\times n}$ associated to $p(x)$ and $q(x)$. Generically it admits an LU decomposition of the form $H(q/p) = KJK^T$ where J is a *signature matrix* (a diagonal matrix with coefficients ± 1 on the diagonal) and K is lower triangular. Then, we introduce the tridiagonal matrix $\overline{\mathrm{Td}} = K^{-1}C_p^TK$, which is such that $p(x) = \det(x\mathbf{Id}_n - \overline{\mathrm{Td}})$.

If we consider that the matrices $\mathrm{Td}(p,q)$ and $\overline{\mathrm{Td}}$ represent linear mappings in some basis, then the duality Theorem 7 means that one matrix can be deduced from the other simply by reversing the ordering of the basis.

We shall mention that, in the case when all the roots of $p(x)$ are real, the existence of a tridiagonal and symmetric matrix Td given by the signed remainders sequence of $p(x)$ and $q(x)$ together with the identity $p(x) = \det(x\mathbf{Id}_n - \mathrm{Td})$ corresponds to the Routh-Lanczos algorithm which answers a structured Jacobi inverse problem. Namely, the question to find a real symmetric tridiagonal matrix A with a given characteristic polynomial $p(x)$ such that the characteristic polynomial of its principal minor A_{n-1}, of size $n-1$, is proportional to $p'(x)$. We refer to [3] for a survey on the subject. One aim of Section 4 is to generalize the Routh-Lanczos algorithm to a polynomial all of whose roots are not necessarily real. It provides another answer to a question of Fiedler [4] which proposes a solution using symmetric *arrow* matrices instead of tridiagonal and symmetric ones.

In Section 4, we focus on the question of counting real roots and the question of determinantal representation. We say that $p(x) = \det(J - xA)$ is a determinantal representation of the polynomial $p(x)$ if $J \in \mathbb{R}^{n\times n}$ is a signature matrix and $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix.

Remark that we may transform the identity $p(x) = \det(J - xA)$ into $p^*(x) = \det(xJ - A)$ where $p^*(x)$ is the reciprocal polynomial of $p(x)$. If we write

$$p^*(x) = \det(J) \times \det(x\mathbf{Id} - AJ),$$

then this shows a connection with the results of Section 3 when the matrix AJ is tridiagonal. More precisely, we establish that such a determinantal representation is always possible and we may even find a family of representations for a given polynomial $p(x)$. We also show that given such a determinantal representation for a polynomial $p(x)$, its number of real roots is at least equal to the signature of the signature matrix J.

Finally, in Section 5 we conclude with some worked examples.

2. Tridiagonal representation of signed remainders sequences

2.1. Definitions

Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_{n-1})$ and $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$ be three sequences of real numbers. We set the tridiagonal matrix $\mathrm{Td}(\alpha, \beta, \gamma)$ to be:

$$\mathrm{Td}(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha_n & \gamma_{n-1} & 0 & \ldots & 0 \\ \beta_{n-1} & \alpha_{n-1} & \gamma_{n-2} & \ddots & \vdots \\ 0 & \beta_{n-2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_1 \\ 0 & \ldots & 0 & \beta_1 & \alpha_1 \end{pmatrix}.$$

Let $p(x)$ and $q(x)$ be two monic polynomials of respective degrees n and $n-1$. We set $\mathrm{SRemS}(p, q) = (p_k(x))_k$ to be the *signed remainders sequence* of $p(x)$ and $q(x)$ defined in the following way:

$$\begin{cases} p_0(x) = p(x), \\ p_1(x) = q(x), \\ p_k(x) = q_{k+1}(x)p_{k+1}(x) - \epsilon_{k+1}\beta_{k+1}^2 p_{k+2}(x), \end{cases} \tag{1}$$

where

$$\begin{cases} p_k(x), q_{k+1}(x) \in \mathbb{R}[x], \\ \epsilon_{k+1} \in \{-1, +1\}, \\ \beta_{k+1} \text{ is a positive real number}, \\ p_{k+2}(x) \text{ is monic and } \deg p_{k+2} < \deg p_{k+1}. \end{cases} \tag{2}$$

This is a finite sequence which stops at the step just before we reach the zero polynomial as remainder.

Let us assume that there is no degree breakdown in $\mathrm{SRemS}(p, q)$. Namely:

$$(\forall k \in \{0, \ldots, n\}) \ (\deg p_k = n - k). \tag{3}$$

Then, $q_{k+1}(x)$ is a degree one polynomial which we write $q_{k+1}(x) = (x - \alpha_{k+1})$ with $\alpha_{k+1} \in \mathbb{R}$. Another consequence is that $\gcd(p, q) = 1$.

Let $\gamma_{k+1} = \epsilon_{k+1}\beta_{k+1}$ and consider the following tridiagonal matrix:

$$\mathrm{Td}(p, q) = \mathrm{Td}(\alpha, \beta, \gamma).$$

We may read on this matrix all the informations about the signed remainders sequence $\mathrm{SRemS}(p, q)$.

For a given tridiagonal matrix $\mathrm{Td} = \mathrm{Td}(\alpha, \beta, \gamma) \in \mathbb{R}^{n\times n}$, we define the *first principal lower diagonal* (respectively the *first principal upper diagonal*) of Td to be the sequence $\beta = (\beta_1, \ldots, \beta_{n-1})$ (respectively $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$). We will say that these first principal diagonals are *non-singular* if all the coefficients β_i (respectively γ_i) are non-zero.

Note that the no degree breakdown assumption (3) implies that the principal diagonals of $\mathrm{Td}(p, q)$ are non-singular.

Proposition 1

(*i*) *To any tridiagonal matrix* $\mathrm{Td} = \mathrm{Td}(\alpha, \beta, \gamma)$ *with non-singular principal diagonals, we may canonically associate a (unique) couple of monic polynomials $p(x)$ and $q(x)$ of respective degrees n and $n-1$ such that the sequence* $\mathrm{SRemS}(p, q)$ *has no degree breakdown and such that for all k the characteristic polynomial of* Td_k *is equal to $p_{n-k}(x)$:*

$$\det(x\mathbf{Id}_k - \mathrm{Td}_k) = p_{n-k}(x).$$

(*ii*) *To any couple of monic polynomials $p(x)$ and $q(x)$ of respective degrees n and $n-1$ such that* $\mathrm{SRemS}(p, q)$ *has no degree breakdown, we may associate a unique tridiagonal matrix with non-singular principal diagonals* $\mathrm{Td}(p, q) = \mathrm{Td}(\alpha, \beta, \gamma)$ *satisfying for all k, $\beta_k > 0$ and $\gamma_k = \epsilon_k\beta_k$ where $\epsilon_k = \pm 1$.*

(*iii*) *When we have* (*i*) *and* (*ii*), *the matrix* $\mathrm{Td}(p,q)) \times P$ *is tridiagonal and symmetric, with*

$$P = \begin{pmatrix} \epsilon_{n-1} \times \cdots \times \epsilon_1 & & & & \\ & \ddots & & & \\ & & \epsilon_2 \times \epsilon_1 & & \\ & & & \epsilon_1 & \\ & & & & 1 \end{pmatrix}.$$

(*iv*) *When we have* (*i*) *and* (*ii*), *the sequence of signs in the leading coefficients of the signed remainders sequence* $\mathrm{SRemS}(p,q)$ *is*:

$$(1, 1, \epsilon_1, \epsilon_2, \epsilon_1 \times \epsilon_3, \epsilon_2 \times \epsilon_4, \epsilon_1 \times \epsilon_3 \times \epsilon_5, \ldots, \epsilon_{n-1 \bmod 2} \times \cdots \times \epsilon_{n-3} \times \epsilon_{n-1}).$$

Proof. Concerning (*i*), the polynomials $p(x)$ and $q(x)$ are taken to be $p(x) = \det(x\mathbf{Id}_n - \mathrm{Td})$ and $q(x) = \det(x\mathbf{Id}_{n-1} - \mathrm{Td}_{n-1})$. Then, we set for all k,

$$\delta_{n-k}(x) = \det(x\mathbf{Id}_k - \mathrm{Td}_k)$$

(where Td_k is the kth principal submatrix of Td) and we develop the determinant

$$\delta_0(x) = \det(x\mathbf{Id}_n - \mathrm{Td})$$

with respect to the last row. We get

$$\delta_0(x) = (x - \alpha_1)\delta_1(x) - (\beta_1\gamma_1)\delta_2(x).$$

Repeating this process, we obtain the same recurrence relation as the one defining the sequence $(p_k(x))_k$ in (1). Since $\delta_0(x) = p_0(x)$ and $\delta_1(x) = p_1(x)$, we get the desired identity.

Point (*ii*) follows straightforward from the beginning of the section, whereas (*iii*) and (*iv*) follow from elementary computation. □

To the tridiagonal matrix $\mathrm{Td}(p,q)$, we may associate also another natural polynomial remainder sequence: $\overline{\mathrm{SRemS}(p,q)} = \mathrm{SRemS}(p,\bar{q})$ where

$$\begin{cases} p(x) = \det(x\mathbf{Id}_n - \mathrm{Td}), \\ \bar{q}(x) = \det(x\mathbf{Id}_{n-1} - \overline{\mathrm{Td}}_{n-1}), \end{cases}$$

with the convention that $\overline{\mathrm{Td}}_k$ is the kth antiprincipal submatrix of Td.

The signed remainders sequence $\overline{\mathrm{SRemS}(p,q)}$ will be considered as the *dual* signed remainders sequence of $\mathrm{SRemS}(p,q)$. This only means that we may read on a tridiagonal matrix from the top left rather than from the bottom right !

For cosmetic reasons we will write $\overline{\mathrm{Td}(p,q)}$ in place of $\mathrm{Td}(p,\bar{q})$. We obviously have:

$$\overline{\mathrm{Td}(p,q)} = \mathbf{Ad} \times \mathrm{Td}(p,q) \times \mathbf{Ad}, \tag{4}$$

where $\mathbf{Ad}_n \in \mathbb{R}^{n\times n}$ ($\mathbf{Ad}$ in short) stand for the anti-identity matrix of size n:

$$\mathbf{Ad}_n = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}.$$

2.2. Companion matrix

We denote by A^T the transpose of the matrix $A \in \mathbb{R}^{n\times n}$ and we define the *companion matrix* of the polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ to be

$$C_p = \begin{pmatrix} 0 & \dots & \dots & 0 & -a_0 \\ 1 & \ddots & & \vdots & -a_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -a_{n-2} \\ 0 & \dots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

We recall a well-known identity (see for instance [3]):

Proposition 2. *Let $p(x)$ and $q(x)$ be two monic polynomials of respective degrees n and $n-1$ such that* $\mathrm{SRemS}(p,q)$ *has no degree breakdown.*

Then there is a lower triangular matrix L such that

$$\mathrm{Td}(p,q) = LC_p^T L^{-1}. \tag{5}$$

To be self-contained, we also give a brief proof:

Proof. With the notation of Section 2.1, let $\mathcal{P}(x) = (\gamma_1 \dots \gamma_{n-1} p_n(x), \dots, \gamma_1 p_2(x), p_1(x))$. A direct computation gives

$$\mathcal{P}(x)\,(\mathrm{Td}(p,q))^T = x\mathcal{P}(x) + (0, \dots, 0, -p(x)).$$

Let U be the upper triangular matrix whose columns are the coefficients of the polynomials of $\mathcal{P}(x)$ in the canonical basis $\mathcal{C}(x) = (1, x, \dots, x^{n-1})$. In other words:

$$\mathcal{C}(x)U = \mathcal{P}(x).$$

Besides, we have

$$\mathcal{C}(x)C_p = x\mathcal{C}(x) + (0, \dots, 0, -p(x)).$$

Thus

$$\begin{aligned} \mathcal{C}(x)C_pU &= x\mathcal{C}(x)U + (0, \dots, 0, -p(x)) \text{ since } p_1(x) \text{ is monic} \\ &= \mathcal{P}(x)\,(\mathrm{Td}(p,q))^T \\ &= \mathcal{C}(x)U\,(\mathrm{Td}(p,q))^T. \end{aligned}$$

We deduce the identity

$$V(x_1, \dots, x_n)C_pU = V(x_1, \dots, x_n)U\,(\mathrm{Td}(p,q))^T$$

for any Vandermonde matrix $V(x_1, \dots, x_n)$ whose lines are $(1, x_i, \dots, x_i^{n-1})$ for $i = 1 \dots n$. If we choose the n reals $x_1, \dots, x_n$ to be distinct, then $V(x_1, \dots, x_n)$ becomes invertible and we get:

$$\mathrm{Td}(p,q) = LC_p^T L^{-1},$$

where L is the lower triangular matrix defined by $L = U^T$. $\square$

The following result says that the decomposition generically exists for any tridiagonal matrix, and is also unique:

Proposition 3. *Any tridiagonal matrix* Td *with non-singular principal diagonals can be written* $\mathrm{Td} = LC_p^T L^{-1}$ *where* $p(x) = \det(x\mathbf{Id} - \mathrm{Td})$ *and L is a lower triangular matrix. Moreover the matrix L is unique up to multiplication by a real number.*

Proof. The existence is given by Propositions 1 and 2.

We come now to the unicity. Assume that $L_1 C_p^T L_1^{-1} = L_2 C_p^T L_2^{-1}$ where L_1 and L_2 are lower triangular. Then, $L = L_2^{-1} L_1$ is a lower triangular matrix which commute with C_p^T.

If $L = (t_{i,j})_{1 \leqslant i,j \leqslant n}$, then

$$LC_p^T = \begin{pmatrix} 0 & t_{1,1} & 0 & \dots & 0 \\ \vdots & t_{2,1} & t_{2,2} & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & t_{n-1,1} & \dots & \dots & t_{n-1,n-1} \\ ? & \dots & \dots & \dots & ? \end{pmatrix}$$

and

$$C_p^T L = \begin{pmatrix} t_{2,1} & t_{2,2} & 0 & \dots & 0 \\ t_{3,1} & t_{3,2} & t_{3,3} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ t_{n,1} & \dots & \dots & t_{n,n-1} & t_{n,n} \\ ? & \dots & \dots & \dots & ? \end{pmatrix}.$$

Thus $t_{1,1} = t_{2,2} = \cdots = t_{n,n}$ and $t_{2,1} = t_{3,2} = \cdots = t_{n,n-1} = 0$ and $t_{3,1} = t_{4,2} = \cdots = t_{n,n-2} = 0$, and so on until $t_{n,1} = 0$. We deduce that $L = \lambda\mathbf{Id}$ and we are done. $\square$

2.3. Sturm algorithm

As a particularly important case of signed remainders sequences, we shall mention the Sturm sequence which is $\mathrm{SRemS}(p, q)$ where q is taken to be the derivative of the polynomial $p(x)$ up to normalization, i.e. $q = p'/\deg(p)$.

For a given finite sequence $\nu = (\nu_1, \dots, \nu_k)$ of elements in $\{-1, +1\}$, we recall the *Permanence minus Variations* number:

$$\mathrm{PmV}(\nu_1, \dots, \nu_k) = \sum_{i=1}^{k-1} \nu_i \nu_{i+1}.$$

Here the sequence ν will stand for the sequence of signs of leading coefficients in $\mathrm{SRemS}(p, q)$. Then the Sturm Theorem [2, Theorem 2.50] says that the number $\mathrm{PmV}(\nu)$ is exactly the number of real roots of $p(x)$.

If we assume that the polynomial $p(x)$ has n distinct real roots, then the Sturm sequence has no degree breakdown and for all k we have $\nu_k = 1$. Hence we get a *symmetric* tridiagonal matrix $\mathrm{Td}(p, q)$ which has the decomposition $\mathrm{Td}(p, q) = LC_p^T L^{-1}$ where L is the lower triangular matrix defined as in Section 2.2. In particular, the last row of L gives the list of coefficients of the polynomial $q(x)$ in the canonical basis.

3. Duality between Sturm and Sylvester algorithms

3.1. Sylvester algorithm

Let us introduce the symmetric matrix $\mathrm{Newt}_p(n) = (n_{i,j})_{0 \leqslant i,j \leqslant n-1}$ defined as

$$n_{i,j} = \mathrm{Trace}\ (C_p^{i+j}) = N_{i+j}$$

which is nothing but the $(i + j)$th Newton sum of the polynomial $p(x)$. To be more explicit, if $\alpha_1, \dots, \alpha_n$ denote all the complex roots of the polynomial $p(x)$, then the kth Newton sum is the real number $N_k = \alpha_1^k + \cdots + \alpha_n^k$.

Recall that the *signature* $\mathrm{sign}(A)$ of a real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is defined to be the number $p - q$, where p is the number of positive eigenvalues of A (counted with multiplicity) and q the number of negative eigenvalues of A (counted with multiplicity). The Sylvester Theorem (which has been generalized later by Hermite: [2, Theorem 4.57]) says that the matrix $\mathrm{Newt}_p(n)$ is invertible if and only if $p(x)$ has only simple roots, and also that $\operatorname{sgn}(\mathrm{Newt}_p(n))$ is exactly the number of distinct real roots of $p(x)$.

In particular, if the polynomial $p(x)$ has n distinct real roots, then the matrix $\mathrm{Newt}_p(n)$ is positive definite. Thus, by the Choleski decomposition algorithm, we can find a lower triangular matrix K such that $\mathrm{Newt}_p(n) = KK^T$. Let us show how to exploit this decomposition.

First, we write

$$p(x) = \det(x\mathbf{Id} - C_p^T).$$

Then, we introduce a useful identity (which will be discussed in more details in the forthcoming section):

$$\mathrm{Newt}_p(n)C_p = C_p^T\mathrm{Newt}_p(n),$$

So, we get:

$$p(x) = \det(x\mathbf{Id} - K^{-1}C_p^T K).$$

Note that the matrix $K^{-1}C_p^T K$ is tridiagonal. Our purpose in the following is to establish a connection with the identity

$$p(x) = \det(x\mathbf{Id} - LC_p^T L^{-1})$$

obtained in Proposition 3.

More generally, we will point out a connection between tridiagonal representations associated to signed remainders sequences on one hand, and tridiagonal representations derived from decompositions of some Hankel matrices on the other hand.

3.2. Hankel matrices and intertwining relation

Roughly speaking, the idea of the previous section is to start with the canonical companion identity

$$p(x) = \det(x\mathbf{Id} - C_p^T)$$

and then to use a symmetric invertible matrix H satisfying the so-called *intertwining relation*

$$HC_p = C_p^T H. \tag{6}$$

Since H is supposed to be symmetric invertible, Eq. (6) only says that the matrix HC_p is symmetric. It is a classical and elementary result that a matrix H satisfying Eq. (6) is necessarily an *Hankel matrix*.

Definition 1. We say that the matrix $H = (h_{i,j})_{0\leqslant i,j\leqslant n-1} \in \mathbb{R}^{n\times n}$ is an Hankel matrix if $h_{i,j} = h_{i',j'}$ whenever $i + j = i' + j'$. Then, it makes sense to introduce the real numbers $a_{i+j} = h_{i,j}$ which allow to write in short $H = (a_{i+j})_{0\leqslant i,j\leqslant n-1}$.

Let $s = (s_k)$ be a sequence of real numbers. We denote by $H_n(s)$ or by $H_n(s_0, \ldots, s_{2n-2})$ the following Hankel matrix of $\mathbb{R}^{n\times n}$:

$$H_n(s) = (s_{i+j})_{0\leqslant i,j\leqslant n-1} = \begin{pmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & & \iddots & s_n \\ \vdots & \iddots & \iddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{pmatrix}.$$

We get from [2, Theorem 9.17]:

Proposition 4. *Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $s = (s_k)$ be a sequence of real numbers. The following assertions are equivalent*

(*i*) $(\forall k \geqslant n)\ (s_k = -a_{n-1}s_{k-1} - \ldots - a_0 s_{k-n})$.
(*ii*) *There is a polynomial $q(x)$ of degree* $\deg q < \deg p$ *such that*

$$\frac{q(x)}{p(x)} = \sum_{j=0}^{\infty} \frac{s_j}{x^{j+1}}.$$

(*iii*) *There is an integer* $r \leqslant n$ *such that* $\det(H_r(s)) \neq 0$, *and for all* $k > r$, $\det(H_k(s)) = 0$.

Whenever these conditions are fulfilled, we denote by $H_n(q/p)$ *the Hankel matrix* $H_n(s)$.

Back to the intertwining relation (6): it is immediate that an Hankel matrix H is a solution if and only if the (finite) sequence $(s_0, \dots, s_{2n-2})$ satisfies the linear recurrence relation of Proposition 4 (*i*), for $k = n, \dots, 2n-2$.

For further details and developments about the intertwining relation, we refer to [8].

The vector subspace of Hankel matrices in $\mathbb{R}^{n\times n}$ satisfying relation (6) has dimension n and contains a remarkable element, that is the Hankel matrix $\mathrm{Newt}_p(n)$ that was considered in Section 3.1 about Sylvester algorithm. Indeed, it is a well-known and elementary fact that the N_k's are real numbers which verify the Newton identities:

$$(\forall k \geqslant n) \ \ (N_k + a_{n-1}N_{k-1} + \cdots + a_0 N_{k-n} = 0)\,.$$

3.3. Barnett formula

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $q(x)$ a (non-necessarily monic) polynomial in $\mathbb{R}[x]$ whose degree is $\leqslant n-1$.

Among all the bases of $\mathbb{R}[z]/p(z)$ that will be interesting for the following, let us mention the canonical basis $\mathcal{C} = (1, x, \dots, x^{n-1})$ and also the (degree decreasing) Horner basis $\mathcal{H}(x) = (h_0, \dots, h_{n-1})$ associated to the polynomial $p(x)$ which is defined by:

$$\begin{cases} h_0(x) = x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1, \\ \vdots \\ h_i(x) = x^{n-1-i} + a_{n-1}x^{n-2-i} + \cdots + a_{i+1} = xh_{i+1}(x) + a_{i+1}, \\ \vdots \\ h_{n-2}(x) = x + a_{n-1}, \\ h_{n-1}(x) = 1. \end{cases}$$

We come to a central proposition which is a consequence of the Barnett formula. It has been established in [1] using direct matrix computations. For the convenience of the reader, we will give here another proof which has the particularity of not using the notion of Bezoutian, as it is classical to proceed.

Proposition 5. *Let* $p(x)$ *and* $q(x)$ *be two polynomials such that* $\deg q < \deg p = n$ *and let* P_{CH} *be the change of basis matrix from the canonical basis* $\mathcal{C}$ *to the Horner basis* $\mathcal{H}$. *We have*

$$q(C_p^T) = H_n(q/p)P_{CH}.$$

Proof. We first note that it is enough to check the formula for $q(x) = 1$, since the formula is linear in the polynomial $q(x)$ and also stable by multiplication by x since we have

Lemma 6. *If* $\deg q < (\deg p) - 1$, *then* $H_n(xq/p) = C_p^T H_n(q/p)$.

Proof. It is a direct application of Proposition 4(i). □

Now we check the formula when $q(x) = 1$.

The change of basis matrix P_{CH} is in fact the following Hankel matrix

$$P_{CH} = H_n(a_1, \dots, a_{n-1}, 1, 0, \dots, 0) \in \mathbb{R}^{n\times n}$$

with the usual notation $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

We remark also that $H_n(1/p) = H_n(0, \ldots, 0, 1, s_n, \ldots, s_{2n-2})$ for some real numbers $s_n, \ldots, s_{2n-2}$ which satisfy the following relations given by Proposition 4(i):

$$\begin{cases} s_n = -a_{n-1}, \\ s_{n+1} = -a_{n-1}s_n - a_{n-2}, \\ \vdots \\ s_{2n-2} = -a_{n-1}s_{2n-3} - \ldots - a_1. \end{cases}$$

It enables us to check that

$$\mathbf{Id}_n = H_n(0, \ldots, 0, 1, c_1, \ldots, c_{n_1}) \times H_n(a_1, \ldots, a_{n-1}, 1, 0, \ldots, 0) = H_n(1/p) \times P_{CH}$$

and get the desired Formula. $\square$

To end the section, we show how Sturm and Sylvester algorithms can be considered as dual, in the case where all the roots of $p(x)$ are real and simple, say $x_1 < \cdots < x_n$. Then, $q(x) = p'(x)/n$ has also $n-1$ simple real roots $y_1 < \cdots < y_{n-1}$ which are interlacing those of $p(x)$. Namely

$$x_1 < y_1 < x_2 < y_2 < \cdots < y_{n-1} < x_n.$$

We may repeat the argument to see that this interlacing property of real roots remains for any two consecutive polynomials $p_k(x)$ and $p_{k+1}(x)$ of the sequence $\mathrm{SRemS}(p, q)$. In particular, $\mathrm{SRemS}(p, q)$ does not have any degree breakdown, all the ϵ_k are equal to $+1$, and $H(q/p)$ is positive definite.

We have, by Proposition 5

$$q(C_p^T) = H_n(q/p)P_{CH}.$$

Since $H_n(q/p)$ is positive definite, the Cholesky algorithm gives a decomposition

$$H_n(q/p) = KK^T,$$

where $K \in \mathbb{R}^{n\times n}$ is lower triangular. So we can write

$$p(x) = \det(x\mathbf{Id} - K^{-1}C_p^T K).$$

We shall remark at this point that the matrix $K^{-1}C_p^T K$ is tridiagonal and symmetric.

We get $q(C_p^T) = K\mathbf{Ad}L$ where $L = \mathbf{Ad}K^T P_{CH}$. Then, we observe that L is a lower triangular matrix (since $P_{CH}\mathbf{Ad}$ is upper triangular) and $K\mathbf{Ad}L$ commute with C_p^T. Thus, we have the identity:

$$LC_p^T L^{-1} = \mathbf{Ad}(K^{-1}C_p^T K)\mathbf{Ad}.$$

We denote by Td this tridiagonal matrix. Let $(p_k(x))$ be the signed remainders sequence associated to Td as given in Proposition 1(i). The first row of $K\mathbf{Ad}L$ is proportional to the last row of the matrix L which is proportional to $p_1(x)$. It remains to observe that the first row of $K\mathbf{Ad}L = q(C_p^T)$ gives exactly the coefficients of the polynomial $q(x)$ in the canonical basis. Then, $p_1(x) = q(x)$.

We have shown that, if $p(x)$ has n simple real roots and $q(x) = p'(x)/n$, then $H_n(q/p)$ is positive definite with Cholesky decomposition $H_n(q/p) = KK^T$, and if we denote by $\tilde{q}(x)$ the monic polynomial whose coefficients are proportional to the last row of K^{-1}, then $\mathrm{Td}(p, \tilde{q}) = \overline{\mathrm{Td}(p, q)}$. This establishes the announced duality.

3.4. Generic case

We turn now to the generic situation. Let $p(x)$ and $q(x)$ be monic polynomials of respective degrees n and $n-1$ such that $\mathrm{SRemS}(p, q)$ does not have any degree breakdown. We will see that this condition is equivalent to saying that all the principal minors of the Hankel matrix $H_n(q/p)$ do not vanish. We then say that we are in the *non-defective* situation.

At this point, we shall remark also that the non-vanishing of all the principal minors of the Hankel matrix $H_n(q/p)$ is also equivalent to saying that the matrix $H_n(q/p)$ admits an invertible LU decomposition. Namely, there exists a lower triangular matrix L with entries 1 on the diagonal, and an upper invertible triangular matrix U such that $H_n(q/p) = LU$. Moreover this decomposition is unique and since $H_n(q/p)$ is symmetric we may write it as

$$H_n(q/p) = LDL^T, \tag{7}$$

where D is diagonal and L is a lower triangular matrix with 1 on the diagonal.

In fact, for our purpose, we will sometimes prefer the unique decomposition

$$H_n(q/p) = KJK^T, \tag{8}$$

where J is a signature matrix and K is lower triangular.

Generalizing the previous section, we get:

Theorem 7. *Let $p(x)$ and $q(x)$ be two monic polynomials of respective degrees n and $n-1$ such that all the principal minors of the matrix $H_n(q/p)$ are invertible. Let us denote $H_n(q/p) = KJK^T$ its symmetric LU-decomposition, where J is a signature matrix and K a lower triangular matrix, and denote by $\tilde{q}(x)$ the monic polynomial whose coefficients in the canonical basis are proportional to the last row of K^{-1}. Then, the sequence* $\mathrm{SRemS}(p, q)$ *does not have any degree breakdown and*

$$\mathrm{Td}(p, \tilde{q}) = \overline{\mathrm{Td}(p, q)}.$$

Proof. We start with the companion identity:

$$p(x) = \det(x\mathbf{Id} - C_p^T).$$

Because of Proposition 4(i), we notice that the matrix $H_n(q/p)$ verifies the intertwining relation

$$H_n(q/p)C_p = C_p^T H_n(q/p).$$

Then, we write the symmetric LU-decomposition of $H_n(q/p)$:

$$H_n(q/p) = KJK^T$$

which gives the identity

$$p(x) = \det(x\mathbf{Id} - K^{-1}C_p^T K).$$

We have, by Proposition 5

$$q(C_p^T) = H_n(q/p)P_{CH} = K\mathbf{Ad}L$$

where

$$L = \mathbf{Ad}JK^T P_{CH}.$$

We observe first that L is an invertible lower triangular matrix (since $P_{CH}\mathbf{Ad}$ is upper triangular), and second that $K\mathbf{Ad}L$ commute with C_p^T. Thus, we have the identity:

$$LC_p^T L^{-1} = \mathbf{Ad}(K^{-1}C_p^T K)\mathbf{Ad}.$$

Then we can deduce that $\mathrm{SRemS}(p, \tilde{q})$ has no degree breakdown and by Proposition 3 we have

$$\overline{\mathrm{Td}(p, \tilde{q})} = \mathbf{Ad}(K^{-1}C_p^T K)\mathbf{Ad}.$$

Moreover, the first row of $K\mathbf{Ad}L$ is proportional to the last row of the matrix L. It remains to observe that the first row of $K\mathbf{Ad}L = q(C_p^T)$ gives exactly the coefficients of the polynomial $q(x)$ in the canonical basis. Thus, by Proposition 3 we get

$$LC_p^T L^{-1} = \mathrm{Td}(p, q)$$

Which completes the proof. $\square$

Remark 2. Note that $K^{-1}C_p^T KJ$, and hence $LC_p^T L^{-1}\bar{J}$, is symmetric, where $\bar{J} = \mathbf{Ad}J\mathbf{Ad}$.

4. Tridiagonal determinantal representations

4.1. Notation

We say that an univariate polynomial $p(x) \in \mathbb{R}[x]$ of degree n such that $p(0) \neq 0$ has a *determinantal representation* if

$$\text{(DR)} \quad p(x) = \alpha \det(J - Ax),$$

where $\alpha \in \mathbb{R}^*$, J is a signature matrix in $\mathbb{R}^{n\times n}$, and A is a symmetric matrix in $\mathbb{R}^{n\times n}$ (we obviously have $\alpha = \det(J)p(0)$).

Likewise, we say that $p(x)$ has a *weak determinantal representation* if

$$\text{(WDR)} \quad p(x) = \alpha \det(S - Ax),$$

where $\alpha \in \mathbb{R}^*$, S is symmetric invertible and A is symmetric.

Of course the existence of (DR) is obvious for univariate polynomials, but we will focus on the problem of *effectivity*. Namely, we want an algorithm (say of polynomial complexity with respect to the coefficients and the degree of $p(x)$) which produces the representation. Typically, we do want to avoid the use of the roots of $p(x)$.

One result in that direction can be found in [10] (which is inspired from [4]). It uses arrow matrices as a “model", whereas in the present article we make use of tridiagonal matrices.

When all the roots of $p(x)$ are real, the effective construction of determinantal representation for univariate real polynomials exists even if we add the condition that $J = \mathbf{Id}$. It has been discussed in several places, although not exactly with the determinantal representation formulation. Indeed, in place of looking for DR we may consider the equivalent problem of determining a symmetric matrix whose characteristic polynomial is given. Indeed, if the size of the matrix A is fixed to be the degree n of the polynomial, the condition

$$p(x) = \det(\mathbf{Id} - xA)$$

is equivalent to

$$p^*(x) = \det(x\mathbf{Id} - A),$$

where $p^*(x)$ is the reciprocal polynomial of $p(x)$. In [4], arrow matrices are used to answer this last problem. On the other hand, the Routh-Lanczos algorithm (which can be viewed as Proposition 1) gives also an answer, using a tridiagonal model. Note that the problem may also be reformulated as a structured Jacobi inverse problem (confer [3] for a survey).

In the following, we generalize the tridiagonal model to any polynomial $p(x)$, possibly having complex roots. Doing that, general signature matrices J appear, whose entries depend on the number of real roots of $p(x)$.

4.2. Over a general field

A lot of identities in Section 3 are still valid over a general field k. For instance, if $p(x)$ and $q(x)$ are monic polynomials of respective degrees n and $n-1$, we may still associate the Hankel matrix $H(q/p) = (s_{i+j})_{0\leqslant i,j\leqslant n-1} \in k^{n\times n}$ defined by the identity

$$\frac{q(x)}{p(x)} = \sum_{j=0}^{\infty} \frac{s_j}{x^{j+1}}.$$

Then, we have the following:

Theorem 8. *Let $p(x) \in k[x]$, $q(x) \in k[x]$ be two monic polynomials of respective degrees n and $n-1$, and set $H = H_n(q/p)$. Then, the matrix $C_p^T H$ is symmetric and we have the WDR:*

$$\det(H) \times p(x) = \det(xH - C_p^T H).$$

Moreover, if we consider the LU-decomposition of type (7): $H = LDL^T$ where $L \in k^{n\times n}$ is lower triangular with entries 1 *on the diagonal and $D \in k^{n\times n}$ a diagonal matrix, then we have*

$$\det(D) \times p(x) = \det(xD - \mathrm{Td}), \tag{9}$$

where $\mathrm{Td} = L^{-1}C_p^T LD$ *is a tridiagonal and symmetric matrix.*

Proof. We exactly follow the proof of Theorem 7. □

Note that the condition for H to be invertible is equivalent to the fact that the polynomials $p(x)$ and $q(x)$ are coprime, since we have

$$\mathrm{rk}(H(q/p)) = \deg(p) - \deg(\gcd(p, q)).$$

To see this, we may refer to the first assertion of [2, Theorem 9.4] whose proof is valid over any field.

The WDR of Theorem 8 has the advantage that the considered matrices have entries in the ring generated by the coefficients of the polynomial $p(x)$. This point is not satisfied in the methods proposed in [10] or in the Routh-Lanczos algorithm.

In fact, the use of Hankel matrices satisfying the intertwining relation seems to be more convenient since we are able to “stop the algorithm at an earlier stage" than the Routh-Lanczos algorithm, namely before having to compute squares roots.

Of course, at the time we want to derive a DR, then we have to add some conditions on the field k, for instance we shall work over an ordered field where square roots of positive elements exist. And hence, in this case, we may use LU-decomposition of type (8) by taking a square root of the matrix D.

To end the section, we may summarize that, for a given polynomial $p(x)$, we have an obvious but non-effective (i.e. using factorization) DR with entries in the splitting field of $p(x)$ over k, to compare with an effective WDR given by Theorem 8 where entries are in the field generated by the coefficients of $p(x)$.

4.3. Symmetric tridiagonal representation and real roots counting

If $p(x)$ and $r(x)$ are two real polynomials, we recall the number known as the Tarski Query:

$$\mathrm{TaQ}(r, p) = \#\{x \in \mathbb{R} | p(x) = 0 \wedge r(x) > 0\} - \#\{x \in \mathbb{R} | p(x) = 0 \wedge r(x) < 0\}.$$

We also recall the definition of the Permanences minus variations number of a given sequence of signs $\nu = (\nu_1, \ldots, \nu_k)$:

$$\mathrm{PmV}(\nu) = \sum_{i=1}^{k-1} \nu_i \nu_{i+1}.$$

We summarize, from [2, Theorem 4.32, Proposition 9.25, Corollary 9.8] some useful properties of these numbers,

Proposition 9. *Let $p(x)$ and $q(x)$ be two monic polynomials of respective degrees n and $n - 1$, such that the sequence* $\mathrm{SRemS}(p, q)$ *has no degree breakdown. Let $r(x)$ be another polynomial such that $q(x)$ is the remainder of $p'(x)r(x)$ modulo $p(x)$. Then,*

$$\mathrm{PmV}(\nu) = \mathrm{sgn}(H_n(q/p)) = \mathrm{TaQ}(r, p),$$

where ν is the sequence of signs of the leading coefficients in the signed remainders sequence $\mathrm{SRemS}(p, q)$.

We come now to our main result about real roots counting:

Theorem 10. *Let* $\mathrm{Td} \in \mathbb{R}^{n\times n}$ *be a tridiagonal and symmetric matrix with non-singular first principal diagonals. Let also $p(x) \in \mathbb{R}[x]$ be a real polynomial with no multiple root such that*

$$p(x) = \det(J)\det(xJ - \mathrm{Td}),$$

where J is a signature matrix whose last entry on the diagonal is $+1$.

Then, the number of real roots of $p(x)$ is greater than $\mathrm{sgn}(J)$.

Proof. We have

$$p(x) = \det(x\mathbf{Id}_n - \mathrm{Td} \times J)$$

and we set

$$q(x) = \det\left(x\mathbf{Id}_{n-1} - (\mathrm{Td} \times J)_{n-1}\right).$$

The matrix $\mathrm{Td} \times J$ is still tridiagonal with non-singular first principal diagonals. We then consider the sequence $\mathrm{SRemS}(p, q)$ and denote by ν the associated sequence of signs of leading coefficients.

Since $\gcd(p, p') = 1$, we set $r(x)$ to be the unique polynomial of degree $< n$ such that

$$r \equiv \frac{q}{p'} \bmod p.$$

Then,

$$p'r \equiv q \bmod p$$

and from Proposition 9, we get:

$$\mathrm{PmV}(\nu) = \mathrm{TaQ}(r, p) \leqslant \#\{x \in \mathbb{R} | p(x) = 0\}.$$

Let us introduce some notation at this step. Let $\mathrm{Td} = \mathrm{Td}(\alpha, \beta, \gamma)$, $\epsilon(a)$ be the sign in $\{-1, +1\}$ of the non-zero real number a, and finally let

$$J = \begin{pmatrix} \theta_{n-1} & & & \\ & \ddots & & \\ & & \theta_1 & \\ & & & 1 \end{pmatrix}.$$

Then, we can write

$$p(x) = \det\left(x\mathbf{Id}_n - P(\mathrm{Td} \times J)P^{-1}\right),$$

where

$$P = \begin{pmatrix} (\theta_{n-1} \dots \theta_1) \times (\epsilon(\gamma_{n-1}) \dots \epsilon(\gamma_1)) & & & \\ & \ddots & & \\ & & \theta_1 \times \epsilon(\gamma_1) & \\ & & & 1 \end{pmatrix}.$$

We note that in fact $P(\mathrm{Td} \times J)P^{-1} = \mathrm{Td}(p, q)$. Indeed, all the coefficients on the first lower principal diagonal are positive. Moreover, all the coefficients on the first upper principal diagonal are given by the sequence

$$(\theta_{n-1} \times \theta_{n-2}, \dots, \theta_2 \times \theta_1, \theta_1).$$

We deduce from Proposition 1(iv) that the sequence of signs of leading coefficients in the signed remainders sequence $\mathrm{SRemS}(p, q)$ is the following:

$$\nu = (\theta_{n-1} \times \dots \times \theta_1, \dots, \theta_2 \times \theta_1, \theta_1, 1, 1).$$

Thus

$$\mathrm{PmV}(\nu) = 1 + \sum_{k=1}^{n-1} \theta_k = \mathrm{sgn}(J)$$

and we are done. □

Remark 3. Another way, maybe less constructive, to prove the result is to use the duality of Theorem 7. Indeed, replacing as in the previous proof the matrix $\mathrm{Td} \times J$ with $P(\mathrm{Td} \times J)P^{-1}$, we write the identity

$$\mathrm{Td} \times J = LC_p^T L^{-1}.$$

Then, by duality, we have

$$LC_p^T L^{-1} = \mathbf{Ad} K^{-1} C_p^T K J' \mathbf{Ad},$$

where we have used the LU-decomposition

$$H_n(q/p) = KJ'K^T.$$

Let us introduce $\overline{J'} = \mathbf{Ad} J' \mathbf{Ad}$; we get:

$$\left(LC_p^T L^{-1} J\right) \times (J\overline{J'}) = \mathbf{Ad} K^{-1} C_p K J' \mathbf{Ad}.$$

We remark that the matrices $LC_p^T L^{-1} J$ and $K^{-1} C_p K J'$ are both tridiagonal and symmetric with non-singular principal diagonals, so we necessarily have

$$J\overline{J'} = \pm \mathbf{Id}.$$

Notice that by assumption the last coefficient of J is $+1$ and that the first coefficient of J' is always $+1$ (since it is the leading coefficient of $\frac{q(x)}{p(x)}$). Thus

$$J\overline{J'} = \mathbf{Id}.$$

By Proposition 9, it completes an another proof for Theorem 10.

An alternative way to make use of the computation of this last remark is to notice that we get another proof of the equality

$$\mathrm{PmV}(\nu) = \mathrm{sgn}(H_n(q/p))$$

which appears in the sequence of identities

$$\mathrm{sgn}(H_n(q/p)) = \mathrm{sgn}(J') = \mathrm{sgn}(J) = \mathrm{PmV}(\nu) = \mathrm{TaQ}(r, p).$$

Remark 4. It is possible to extend Theorem 10 in the case where principal diagonals of $\mathrm{Td} = \mathrm{Td}(\alpha, \beta, \beta)$ are singular. Namely, for all k such that $\beta_k = 0$, we have to assume that the corresponding kth entry on the diagonal of J is equal to $+1$. Then, we get that the number of real roots of $p(x)$, *counted with multiplicity*, is greater than $\mathrm{sgn}(J)$.

To see this, it suffices to note that the polynomial defined by $p(x) = \det(J)\det(xJ - \mathrm{Td})$ factorizes through

$$p(x) = \det(J_1)\det(xJ_1 - \mathrm{Td}_k) \times \det(J_2)\det(xJ_2 - \overline{\mathrm{Td}}_{n-k}).$$

Moreover, the matrices Td_k and $\overline{\mathrm{Td}}_{n-k}$ remain tridiagonal and symmetric and J_1, J_2 remain signature matrices. If we denote by $\bigoplus$ the usual direct sum of matrices, we have $J = J_1 \bigoplus J_2$ and $\mathrm{Td} = \mathrm{Td}_k \bigoplus \overline{\mathrm{Td}}_{n-k}$.

Thus, we may proceed by induction on the degree of $p(x)$.

Before stating the converse property of Theorem 10, we establish a genericity lemma.

Lemma 11. *Let $p(x)$ be a monic polynomial of degree n with only simple roots and $q(x) = x^{n-1} + b_1 x^{n-1} + \cdots + b_{n-1}$. Then, the set of all $(n-1)$-tuples $(b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1}$ such that there is an integer $k \in \{1, \ldots, n\}$ satisfying $\det(H_k(q/p)) = 0$, is a proper subvariety of $\mathbb{R}^{n-1}$.*

Proof. We only have to show that for all k, $\det(H_k(q/p))$, viewed as a polynomial in the variables $b_1, \ldots, b_{n-1}$, is not the zero polynomial.

Let $H_n(q/p) = (s_{i+j})_{0 \leqslant i,j \leqslant n-1}$ where

$$\frac{q(x)}{p(x)} = \sum_{j=0}^{\infty} \frac{s_j}{x^{j+1}}$$

and denote by $\{\alpha_1, \ldots, \alpha_n\}$ the set of all (possibly complex) roots of $p(x)$. Then,

$$s_j = \sum_{i=1}^{n} \frac{q(\alpha_i)}{p'(\alpha_i)} \alpha_i^j.$$

Let us introduce the real numbers defined as

$$u_j = \sum_{i=1}^{n} \frac{\alpha_i^j}{p'(\alpha_i)}.$$

We obviously have $u_j = 0$ whenever $j \leqslant n-2$ and also $u_{n-1} = 1$ (look at $\lim_{x \to +\infty} \frac{x^j q(x)}{p(x)}$). So we deduce:

$$\begin{cases} s_0 = 1, \\ s_1 = b_1 + u_n, \\ \text{and more generally} \\ (\forall j \in \{1, \ldots, 2n-2\})\ (s_j = b_j + b_{j-1}u_n + \cdots + b_1 u_{n+j-2} + u_{n+j-1}). \end{cases}$$

Then, it becomes clear that $H_{k+1}(q/p) \not\equiv 0$ for any k such that $k \leqslant \lfloor \frac{n-1}{2} \rfloor = r$, since $s_{2k} \in \mathbb{R}[b_1, \ldots, b_{2k}]$ has degree 1 in the variable b_{2k}.

Next, for $r < k \leqslant n$, we develop the determinant $H_k(q/p)$ successively according to the first columns, and we remark that its degree in the variable b_{n-1} is equal to $2k-n$ (with leading coefficient equal to -1). This completes the proof. $\square$

The lemma above says that the condition

$$(\forall k \in \{1, \ldots, n\})\ (\det(H_k(q/p)) = 0)$$

is *generic* with respect to the space of coefficients of the polynomial $q(x)$. Because of the relations between coefficients and roots, the condition is also generic with respect to the (possibly complex) roots of the polynomial $q(x)$.

Here is our converse statement about real roots counting:

Theorem 12. *Let $p(x)$ be a monic polynomial of degree n which has exactly s real roots counted with multiplicity. We can find effectively a generic family of symmetric tridiagonal matrices* Td *and signature matrices J with* $\operatorname{sgn}(J) = s$, *such that*

$$p(x) = \det(J) \times \det(xJ - \mathrm{Td}).$$

Proof. If $p(x)$ has multiple roots, then we may factorize it by $\gcd(p, p')$ and use the multiplicative property of the determinant to argue by induction on the degree. Now, we assume that $p(x)$ has only simple roots.

We take for $q(x)$ any monic polynomials of degree $n-1$ which has exactly $s-1$ real roots interlacing those of $p(x)$. Namely, if we denote by $x_1 < \cdots < x_s$ all the real roots of $p(x)$ and by $y_1 < \cdots < y_{s-1}$ all the real roots of $q(x)$, we ask that $x_1 < y_1 < x_1 < y_2 < \cdots < y_{s-1} < x_s$.

Let $r(x)$ be the unique polynomial of degree $< n$ such that $r(x) \equiv \frac{q(x)}{p'(x)} \bmod p(x)$ (since $p'(x)$ is invertible modulo $p(x)$).

From $p'r \equiv q \bmod p$ and $p'(x_i) = q(x_i)$ for all real root x_i of $p(x)$, we get

$$\mathrm{TaQ}(r, p) = s = \#\{x \in \mathbb{R} | p(x) = 0\}.$$

We now assume that $q(x)$ satisfies an additional condition; namely, $\mathrm{SRemS}(p, q)$ does not have any degree breakdown, or equivalently that $H(q/p)$ shall admit a LU-decomposition $H_n(q/p) = KJK^T$.

According to Lemma 11, this hypothesis is generically satisfied, although it may not be always satisfied for the natural candidate $q(x) = p'(x)/n$.

Then, we get from Theorem 7

$$p(x) = \det(xJ - K^{-1}C_p^T KJ),$$

where $\mathrm{Td} = K^{-1}C_p^T KJ$ is tridiagonal and symmetric and J is a signature matrix.

By the proof of Proposition 10, we get moreover that

$$\mathrm{sgn}(J) = \mathrm{TaQ}\,(r, p) = \mathrm{sgn}(H_n(q/p)).$$

This completes the proof since $\mathrm{TaQ}\,(r, p) = s$. □

Remark 5

(i) In order to choose such polynomials $q(x)$ with the interlacing roots property, we need to count and localize the real roots of $p(x)$. It can be done via Sturm sequences for instance.

(ii) Although the polynomial $q(x) = p'(x)/n$ has not necessarily the interlacing property in general, it is the case when all the roots of $p(x)$ are real and simple. Moreover, in this case, the interlacing roots condition is equivalent to the no degree breakdown condition. Indeed, $\mathrm{TaQ}\,(p'q \bmod p, p) = n$ if and only if $p'(x)$ and $q(x)$ have the same sign at each root of $p(x)$.

5. Some worked examples

In order to avoid square roots, in our examples we decided to work with weak determinantal representation as in (9). If one wants to deduce determinantal representations with signature matrices, it suffices to normalize.

(1) Let $p(x) = x^3 + sx + t$ with $s \neq 0$, and $q(x) = p'(x) = 3x^2 + s$. Let us introduce the discriminant of $p(x)$ as $\Delta = -4s^3 - 27t^2$. Consider the decomposition of the Hankel matrix

$$H(q/p) = \begin{pmatrix} 3 & 0 & -2s \\ 0 & -2s & -3t \\ -2s & -3t & 2s^2 \end{pmatrix} = LDL^T,$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2s}{3} & \frac{3t}{2s} & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2s & 0 \\ 0 & 0 & \frac{-\Delta}{6s} \end{pmatrix}.$$

We recover the well-known fact that $p(x)$ has three distinct real roots if and only if $s < 0$ and $\Delta > 0$, which obviously reduces to the single condition $\Delta > 0$. Then, we have the determinantal representation

$$\Delta \times p(x) = \det(xD - \mathrm{Td}),$$

where

$$\mathrm{Td} = \begin{pmatrix} 0 & -2s & 0 \\ -2s & -3t & \frac{-\Delta}{6s} \\ 0 & \frac{-\Delta}{6s} & \frac{t\Delta}{4s^2} \end{pmatrix}.$$

(2) Consider the polynomial $p(x) = x^5 - 5x^3 + 4x$. In fact, it factorizes through $p(x) = x(x-1)(x+1)(x-2)(x+2)$ but we obviously make no use of this observation to construct a determinantal

representation ! We only use it to check the consistency of the computation. Let $q(x) = p'(x)/5$. We have

$$H(q/p) = \begin{pmatrix} 5 & 0 & 10 & 0 & 34 \\ 0 & 10 & 0 & 34 & 0 \\ 10 & 0 & 34 & 0 & 130 \\ 0 & 34 & 0 & 130 & 0 \\ 34 & 0 & 130 & 0 & 514 \end{pmatrix},$$

$$\mathrm{Td} = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{\frac{7}{5}} & 0 & 0 \\ 0 & \sqrt{\frac{7}{5}} & 0 & \sqrt{\frac{36}{35}} & \\ 0 & 0 & \sqrt{\frac{36}{35}} & 0 & \sqrt{\frac{4}{7}} \\ 0 & 0 & 0 & \sqrt{\frac{4}{7}} & 0 \end{pmatrix},$$

$$p(x) = \det(x\mathbf{Id} - \mathrm{Td}).$$

In order to get some parametrized identities, let us introduce the following family of polynomials

$$q_a(x) = (x-a)\left(x+\frac{3}{2}\right)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right).$$

We write the LU-decomposition

$$H(q_a/p) = \begin{pmatrix} 1 & \frac{3}{2}-a & \frac{-3a}{2}+\frac{19}{4} & \frac{57}{8}-\frac{19a}{4} & \frac{-57a}{8}+\frac{79}{4} \\ \frac{3}{2}-a & \frac{-3a}{2}+\frac{19}{4} & \frac{57}{8}-\frac{19a}{4} & \frac{-57a}{8}+\frac{79}{4} & \frac{237}{8}-\frac{79a}{4} \\ \frac{-3a}{2}+\frac{19}{4} & \frac{57}{8}-\frac{19a}{4} & \frac{-57a}{8}+\frac{79}{4} & \frac{237}{8}-\frac{79a}{4} & \frac{-237a}{8}+\frac{319}{4} \\ \frac{57}{8}-\frac{19a}{4} & \frac{-57a}{8}+\frac{79}{4} & \frac{237}{8}-\frac{79a}{4} & \frac{-237a}{8}+\frac{319}{4} & \frac{957}{8}-\frac{319a}{4} \\ \frac{-57a}{8}+\frac{79}{4} & \frac{237}{8}-\frac{79a}{4} & \frac{-237a}{8}+\frac{319}{4} & \frac{957}{8}-\frac{319a}{4} & \frac{-957a}{8}+\frac{1279}{4} \end{pmatrix} = L_a D_a L_a^T,$$

where the associated diagonal matrix D_a is equal to

$$\begin{pmatrix} 1 & & & & \\ & -\frac{1}{2}(a+1)(2a-5) & & & \\ & & \left(\frac{15}{16}\right)\frac{(2a-1)(4a^2-a-15)}{(a+1)(2a-5)} & & \\ & & & \left(\frac{45}{128}\right)\frac{48a^4-16a^3-216a^2+58a+105}{(2a-1)(4a^2-a-15)} & \\ & & & & \left(\frac{315}{8}\right)\frac{(a+2)(a+1)a(a-1)(a-2)}{48a^4-16a^3-216a^2+58a+105} \end{pmatrix}.$$

The condition for $H_a(q/p)$ to be positive definite is equivalent to having only positive coefficients on the diagonal of D_a. First, it yields $D_a(2,2) > 0$, which means that $a \in]-1, \frac{5}{2}[$. Then, we add the condition $D_a(3,3) > 0$ which means that $a \in]\frac{1}{2}, 2, 06..[$. Then, we add the condition $D_a(4,4) > 0$ which means that $a \in]0, 9.., 2, 00..[$. And finally, we add the condition $D_a(5,5) > 0$, which means that $a \in]1, 2[$ and gives exactly the interlacing property for the polynomial $q_a(x)$.

For instance, with $a = \frac{3}{2}$ we get $p(x) = \det\left(x\mathbf{Id} - \mathrm{Td}_{\frac{3}{2}}\right)$ where:

$$\mathrm{Td}_{\frac{3}{2}} = \begin{pmatrix} 0 & \sqrt{\frac{5}{2}} & 0 & 0 & 0 \\ \sqrt{\frac{5}{2}} & 0 & \sqrt{\frac{9}{8}} & 0 & 0 \\ 0 & \sqrt{\frac{9}{8}} & 0 & \sqrt{\frac{35}{40}} & \\ 0 & 0 & \sqrt{\frac{35}{40}} & 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & \sqrt{\frac{1}{2}} & 0 \end{pmatrix}.$$

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