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Advances in Mathematics 178 (2003) 311–374

ADVANCES IN Mathematics

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Some explicit solutions of the additive Deligne–Simpson problem and their applications

Oleg Gleizer

University of California, Los Angeles CA, USA

Received 8 February 2002; accepted 22 May 2002

Communicated by Pavel Etingof

Abstract

In this paper we construct three infinite series and two extra triples (E_8 and $\hat{E}_8$) of complex matrices **B**, **C**, and $\mathbf{A} = \mathbf{B} + \mathbf{C}$ of special spectral types associated to Simpson's classification in Amer. Math. Soc. Proc. 1 (1992) 157 and Magyar et al. classification in Adv. Math. 141 (1999) 97. This enables us to construct Fuchsian systems of differential equations which generalize the hypergeometric equation of Gauss–Riemann. In a sense, they are the closest relatives of the famous equation, because their triples of spectral flags have finitely many orbits for the diagonal action of the general linear group in the space of solutions. In all the cases except for E_8, we also explicitly construct scalar products such that **A**, **B**, and **C** are self-adjoint with respect to them. In the context of Fuchsian systems, these scalar products become monodromy invariant complex symmetric bilinear forms in the spaces of solutions.

When the eigenvalues of **A**, **B**, and **C** are real, the matrices and the scalar products become real as well. We find inequalities on the eigenvalues of **A**, **B**, and **C** which make the scalar products positive-definite.

As proved by Klyachko, spectra of three hermitian (or real symmetric) matrices **B**, **C**, and $\mathbf{A} = \mathbf{B} + \mathbf{C}$ form a polyhedral convex cone in the space of triple spectra. He also gave a recursive algorithm to generate inequalities describing the cone. The inequalities we obtain describe non-recursively some faces of the Klyachko cone.

MSC: 15; 13

Keywords: Fuchsian systems; Rigid local systems; Klyachko inequalities

E-mail address: oleg@math.ucla.edu.

0001-8708/03/$ - see front matter
doi:10.1016/S0001-8708(02)00077-4

1. Introduction

Let V be a vector space over complex numbers such that $\dim V = n$ where $1<n<\infty$. Let $\mathbf{B}, \mathbf{C}$ be linear operators in V and let $\mathbf{A} = \mathbf{B} + \mathbf{C}$. We call the pair $\mathbf{B}, \mathbf{C}$ *irreducible* if the operators do not preserve simultaneously any proper subspace of V.

Let $O_{\mathbf{A}}$ be the adjoint orbit of $\mathbf{A}$ in End V under the GL(V) action. We call the triple $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *rigid*, if any other triple $\mathbf{B}'$, $\mathbf{C}'$, $\mathbf{A}' = \mathbf{B}' + \mathbf{C}'$ such that $\mathbf{A}' \in O_{\mathbf{A}}$, $\mathbf{B}' \in O_{\mathbf{B}}$, and $\mathbf{C}' \in O_{\mathbf{C}}$ is conjugate to the triple $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$.

For a linear operator $\mathbf{A} \in \operatorname{End} V$, we call the multiset of its eigenvalues the *spectrum of* $\mathbf{A}$. This means that each eigenvalue λ_i is taken with its multiplicity m_i. Any ordering $\lambda_1, \lambda_2, \ldots, \lambda_k$ of distinct eigenvalues of $\mathbf{A}$ allows us to represent the spectrum of $\mathbf{A}$ by a vector $s(\mathbf{A}) = (\underbrace{\lambda_1 \cdots \lambda_1}_{m_1 \text{ times}}, \underbrace{\lambda_2 \cdots \lambda_2}_{m_2 \text{ times}}, \ldots, \underbrace{\lambda_k \cdots \lambda_k}_{m_k \text{ times}}) \in \mathbb{C}^n$. For a diagonalizable operator $\mathbf{A}$, we call the partition $(m_1, m_2, \ldots, m_k)$ of n the *spectral type* of $\mathbf{A}$. With slight abuse of terminology, we also call the spectral type of $\mathbf{A}$ any composition obtained by some ordering of $\lambda_1, \ldots, \lambda_k$. We say that a vector $(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n) \in (\mathbb{C}^n)^3$ satisfies the *trace condition* if $\sum_{i=1}^n x_i = \sum_{i=1}^n (y_i + z_i)$. Then $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ belongs to the hyperplane in $(\mathbb{C}^n)^3$ defined by the trace condition. We call this hyperplane the *space of triple spectra*. Let $\alpha = (m_1, m_2, \ldots, m_p)$, $\beta = (n_1, n_2, \ldots, n_q)$, and $\gamma = (k_1, k_2, \ldots, k_r)$ (compositions of n) be the spectral types of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Then $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ lies in the part $S(\alpha, \beta, \gamma) \subset \mathbb{C}^{3n}$ defined as follows. A vector $(x, y, z) \in \mathbb{C}^{3n}$ is in $S(\alpha, \beta, \gamma)$ if $x_1 = x_2 = \cdots = x_{m_1} \neq x_{m_1+1} = \cdots = x_{m_1+m_2} \neq \cdots$ and the same for y and z.

Consider the following table of triples of spectral types.

hypergeometric family	$(1, m-1), (1^m), (1^m)$	$m \geqslant 2$
even family	$(m, m), (1, m-1, m), (1^{2m})$	$m \geqslant 2$
odd family	$(m+1, m), (1, m, m), (1^{2m+1})$	$m \geqslant 2$
extra case	$(4, 2), (2, 2, 2), (1^6)$	

(1.1)

Here and later (1^n) is a shorthand for $(\underbrace{1, 1, \ldots, 1}_{n \text{ times}})$.

Theorem 1.1 (Simpson, Kostov). *Let (α, β, γ) be a triple of spectral types such that at least one of them is (1^n). The following conditions are equivalent*:

1. *for a generic point* $(x, y, z) \in S(\alpha, \beta, \gamma)$ *there exists a rigid irreducible triple* $(\mathbf{A} = \mathbf{B} + \mathbf{C}, \mathbf{B}, \mathbf{C})$ *of diagonalizable operators such that* $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C})) = (x, y, z)$;
2. (α, β, γ) *is one of the triples in* (1.1).

Remark 1.1. This theorem is an additive version of Theorem 4 of [29]. This version easily follows from Simpson's results. A more elementary proof of Theorem 4 of [29] and a proof of Theorem 1.1 were given by Kostov [17].

The first main result of this paper is that for each triple (α, β, γ) of spectral types from (1.1) and a generic vector from $S(\alpha, \beta, \gamma)$ we explicitly construct the corresponding triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

Recently, there appeared algorithms to produce all rigid irreducible r-tuples of matrices $M_1, \ldots, M_r$ such that $M_1 + \cdots + M_r = 0$, see [4,5]. We use a different (less general, but more powerful for our particular purposes) tool: Magyar et al. [26] constructed all indecomposable triple partial flag varieties with finitely many orbits for the diagonal action of the general linear group. Their list (4.41) is strikingly similar to list (1.1) of Simpson. It has just one more family: the E_8-family. A triple of *spectral flags* (for the definition, see Section 4) of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ constructed from Simpson's list (1.1) gives a representative of the open orbit of the corresponding triple flag variety from (4.41).

Our $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ have the following common features. $\mathbf{B}$ is block upper-triangular, $\mathbf{C}$ is block lower-triangular. The block sizes of $\mathbf{B}$ and $\mathbf{C}$ are given by the compositions β and γ, respectively. Each entry of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ is a ratio of products of linear forms in the eigenvalues of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. The coordinates of all eigenvectors of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are also ratios of products of linear forms in the eigenvalues. The linear forms are remarkably simple: all the coefficients are equal to either 1 or -1. As a corollary of our construction, we obtain the following.

Theorem 1.2. *For every composition* (α, β, γ) *from Simpson's list* (1.1), *there exist open subsets* $S''(\alpha, \beta, \gamma) \subset S'(\alpha, \beta, \gamma) \subset S(\alpha, \beta, \gamma)$ *with the following properties.*

1. *Each of* $S'(\alpha, \beta, \gamma)$ *and* $S''(\alpha, \beta, \gamma)$ *is obtained from* $S(\alpha, \beta, \gamma)$ *by removing finitely many hyperplanes.*
2. *If* $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C})) \in S'(\alpha, \beta, \gamma)$, *then there exists a non-zero symmetric bilinear form on* V *such that* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are self-adjoint with respect to it.*
3. *If* $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C})) \in S''(\alpha, \beta, \gamma)$, *then there exists a non-degenerate symmetric bilinear form on* V *such that* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are self-adjoint with respect to it.*

This theorem is proved case by case in Theorems 2.2, 2.5, 2.8, 2.11 for the bilinear forms given by formulas (2.4), (2.14), (2.22), (2.25) correspondingly.

Remark 1.2. The main virtue of this theorem is not the proof of existence of the objects, but an explicit construction of all of them.

In Simpson's list (1.1), the last composition γ is always (1^n). Thus, the matrix $\mathbf{C}$ has all eigenvalues distinct. Let $\mathbf{v}_i$ be the eigenvector of $\mathbf{C}$ corresponding to the eigenvalue c_i. If $\mathbf{C}$ is self-adjoint with respect to a scalar product $\langle *, * \rangle$ on V, then $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = l_i \delta_{ij}$. If we manage to find l_i such that the matrix $\mathbf{B}$ becomes self-adjoint with respect to $\langle *, * \rangle$ as well, then $\mathbf{A}$ is also self-adjoint with respect to $\langle *, * \rangle$ as $\mathbf{A} = \mathbf{B} + \mathbf{C}$. We find the l_i and it turns out that they are also ratios of products of linear forms in the eigenvalues of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. And again all the coefficients of the forms are equal to either 1 or -1. The set of linear forms that appear in the l_i includes the set of linear forms that appear in the matrix elements of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ and in the coordinates of their eigenvectors. The hyperplanes one has to remove from $S(\alpha, \beta, \gamma)$ to obtain $S''(\alpha, \beta, \gamma)$ of Theorem 1.2 are exactly the zero levels of the linear forms that appear in the l_i. The explicit formulas we find for the l_i give explicit description of these hyperplanes.

Probably the most important applications of our explicit construction is to *Fuchsian systems* (see Section 6 for the definition). Let z_1, z_2, z_3 be distinct points of $\mathbb{CP}^1$. Consider the following system of differential equations

$$\frac{df}{dz} = \left[\frac{\mathbf{B}}{z - z_2} + \frac{\mathbf{C}}{z - z_3} - \frac{\mathbf{A}}{z - z_1} \right] f(z), \tag{1.2}$$

where $\mathbf{A} = \mathbf{B} + \mathbf{C}$, $z \in \mathbb{CP}^1 \backslash \{z_1, z_2, z_3\}$ and f takes values in V. The matrices A, B, and C are called the *residue matrices* of (1.2). Their eigenvalues are called *local exponents*. Real parts of the local exponents are the rates of growth of solutions of (1.2) expanded analytically towards the corresponding singularities (and restricted to sectors centered at the singularities). Thus, at each singularity the space of solutions stratifies into a flag. Local basis changes near each singularity turn this flag into a flag variety. The triple of flag varieties of the Gauss–Riemann equation has finitely many orbits for the action of the general linear group in the space of solutions. The Fuchsian systems constructed by means of our matrices exhaust the list of Fuchsian systems (with more than two singularities) having this property. In this sense, they are the simplest Fuchsian systems possible and we expect their solutions to be interesting functions.

It was known to Klein that if the hypergeometric equation of Gauss–Riemann had real local exponents, then there existed a monodromy invariant hermitian form in the space of solutions. If the local exponents were generic, then the form was non-degenerate and unique up to a real constant multiple. We prove the same for all the Fuchsian systems constructed from (1.1). Indeed, when all the eigenvalues of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are real, the form $\langle *, * \rangle$ becomes real as well. So do the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ themselves. Thus, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ become matrices of real operators acting on the real space $V_{\mathbb{R}}$ and self-adjoint with respect to the real symmetric bilinear form $\langle *, * \rangle_{\mathbb{R}}$. Let us extend the form $\langle *, * \rangle_{\mathbb{R}}$ to the hermitian form $(*, *)$ on V. This form gives rise to the monodromy invariant hermitian form in the space of solutions of (1.2). Once again, the forms are constructed explicitly. For the hypergeometric family, this result is not new. The Fuchsian systems from the hypergeometric family are equivalent to the generalized hypergeometric equations studied by Beukers and Heckman [2]. Among other things, they construct the hermitian form. Also the generalized hypergeometric equations were studied in what became later known as the *Okubo normal form* by Okubo [27]. For the generalized hypergeometric equations in the Okubo normal form, the monodromy invariant hermitian form was constructed by Haraoka [12].

As proved by Klyachko [14], if a hermitian form is positive definite, then the spectra of hermitian matrices **B**, **C** and $\mathbf{A} = \mathbf{B} + \mathbf{C}$ form a polyhedral convex cone in the space of triple spectra. His proof contains a recursive algorithm to compute the inequalities describing the cone. We call this cone the *Klyachko cone* and we call the inequalities the *Klyachko inequalities*. Beukers and Heckman [2] give explicitly the inequalities on the local exponents of the generalized hypergeometric equation which make the monodromy invariant hermitian form in the space of solutions positive definite. Thus, they describe non-recursively a non-trivial face of the Klyachko cone. We do the same for all the Fuchsian systems constructed from (1.1). Hence the second important application of our results is an explicit description of some interesting faces of the Klyachko cone. Beukers and Heckman [2] use their criterion of positivity of the hermitian form to see when solutions of the generalized hypergeometric equations are algebraic functions. It is also needed to know the signature of the form for applications to number theory, see [2,5]. Our construction provides tools to answer similar questions about the solutions of our Fuchsian systems.

Let λ, μ, and ν be highest weights of $\mathrm{GL}(V)$. Let V_λ, V_μ, and V_ν be the corresponding rational irreducible representations of $\mathrm{GL}(V)$. Let $V_\lambda \otimes V_\mu = \sum_\nu c^\nu_{\lambda\mu} V_\nu$ be the decomposition of the tensor product of V_λ and V_μ into the sum of irreducible representations. It follows from the results of Klyachko [14] combined with a refinement by Knutson and Tao [15] that the lattice points of the Klyachko cone are precisely the triples of weights λ, μ, ν with non-zero Littlewood–Richardson coefficient $c^\nu_{\lambda\mu}$. Thus, our techniques allow us to explicitly describe some triples of highest weights with non-zero Littlewood–Richardson coefficients. In fact, for all the cases we consider, the Littlewood–Richardson coefficients are equal to 1.

The paper is organized as follows. In Section 2, we formulate main results of the paper. Namely, we list the triples $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, the scalar products, and the Klyachko inequalities for all the partitions from Simpson's list (1.1). In Section 3, theorems of Section 2 are proved and elaborated.

Although the actual construction of the matrices relied heavily on the explicit description of representatives of the open orbits [26], it turned out that once the answers were known, it was much simpler to prove them by inspection. We start using the results of [26] directly only in Section 4. In the section, we construct the matrices **A**, **B**, and **C** which give rise to the E_8-family of Magyar, Weyman, and Zelevinsky. We also prove the following theorem.

Theorem 1.3. *Let* (α, β, γ) *be a triple of spectral types from Theorem* 1.1. *If* $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ *is a point of* $S''(\alpha, \beta, \gamma)$ *then the triple* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *is irreducible.*

The proof of the Theorem for the hypergeometric family is on page 22, for all other families—on page 40.

Remark 1.3. If we take a different ordering of the eigenvalues of **A**, **B**, and **C**, then the scalar products $\langle \mathbf{v}_i, \mathbf{v}_i \rangle$ change. So does the set S', but the set S'' does not.

Remark 1.4. The genericity condition of Theorem 1.1 is somewhat a mysterious one. A theorem of Katz (see [13]) excludes the coexistence of irreducible and reducible triples in rigid cases. If one deals with a reducible triple, then except for the "big" trace condition one also has a "small" trace condition coming from the reduced submatrices. These are the trace conditions resulting from the diagonal blocks of a block upper-triangular triple of matrices. Thus, people call generic spectra that stay away from all the "small trace condition" hyperplanes possible (see Kostov's papers [17–25]). In our cases however, Theorem 1.3 gives an explicit meaning to the genericity condition: "generic" means "not in S''".

Let **A**, **B**, and **C** be self-adjoint with respect to a non-zero symmetric bilinear form on V. Let (α, β, γ) be a triple of spectral types from Theorem 1.1. Then the following corollary of Theorem 1.3 strengthens the third statement of Theorem 1.2.

Corollary 1.1. *If* $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ *is a point of* $S''(\alpha, \beta, \gamma)$, *then the form is unique up to a constant multiple.*

Proof. If $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ is a point of $S''(\alpha, \beta, \gamma)$, then it follows from Theorem 1.3 that the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is irreducible. If a triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is irreducible, then uniqueness of the form follows from Schur's lemma. □

In Section 5, we introduce the *Berenstein–Zelevinsky triangles* which provide a geometric version of the celebrated Littlewood–Richardson rule. For the E_8 family, we do not have formulas for the hermitian form as nice as we have for other families. However, the Berenstein–Zelevinsky triangles enable us to compute the Klyachko inequalities for the E_8-family as well.

Section 6 contains no new results. In the section, we provide (very) basic facts about Fuchsian systems and raise questions we plan to answer in subsequent publications. In particular, we quote some results from [11,12,31], which are very similar to (but different from) ours.

Most of the proofs of the paper boil down to proofs of certain rational identities. These identities are collected in Section 7 (the appendix).

2. Main results

2.1. Hypergeometric family

Let us pick a vector $(a_1, \underbrace{a_2, \ldots, a_2}_{m-1 \text{ times}}, b_1, \ldots, b_m, c_1, \ldots, c_m)$ from $S((1, m-1), (1^m), (1^m))$. Recall that this means $a_1 \neq a_2$, all b_i are distinct, all c_i are distinct, and the trace condition holds: $a_1 + (m-1)a_2 = \sum_{i=1}^{m}(b_i + c_i)$. Define the matrix elements of $\mathbf{B}$ and $\mathbf{C}$ as follows:

$$B_{ij} = \begin{cases} 0 & \text{if } i<j \\ b_i & \text{if } i=j, \\ b_i + c_{m+1-i} - a_2 & \text{if } i>j \end{cases} \quad C_{ij} = \begin{cases} b_i + c_{m+1-i} - a_2 & \text{if } i<j, \\ c_{m+1-i} & \text{if } i=j, \\ 0 & \text{if } i>j. \end{cases} \tag{2.3}$$

Here is an example with $m = 5$.

Example 2.1.

$$\mathbf{B} = \begin{bmatrix} b_1 & b_1 + c_5 - a_2 & b_1 + c_5 - a_2 & b_1 + c_5 - a_2 & b_1 + c_5 - a_2 \\ 0 & b_2 & b_2 + c_4 - a_2 & b_2 + c_4 - a_2 & b_2 + c_4 - a_2 \\ 0 & 0 & b_3 & b_3 + c_3 - a_2 & b_3 + c_3 - a_2 \\ 0 & 0 & 0 & b_4 & b_4 + c_2 - a_2 \\ 0 & 0 & 0 & 0 & b_5 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} c_5 & 0 & 0 & 0 & 0 \\ b_2 + c_4 - a_2 & c_4 & 0 & 0 & 0 \\ b_3 + c_3 - a_2 & b_3 + c_3 - a_2 & c_3 & 0 & 0 \\ b_4 + c_2 - a_2 & b_4 + c_2 - a_2 & b_4 + c_2 - a_2 & c_2 & 0 \\ b_5 + c_1 - a_2 & b_5 + c_1 - a_2 & b_5 + c_1 - a_2 & b_5 + c_1 - a_2 & c_1 \end{bmatrix}.$$

It is clear that $s(\mathbf{B}) = \{b_1, \ldots, b_m\}$ and $s(\mathbf{C}) = \{c_1, \ldots, c_m\}$. Since all the b_i and all the c_i are distinct, $\mathbf{B}$ and $\mathbf{C}$ are diagonalizable.

Theorem 2.1. *If* $\mathbf{B}$ *and* $\mathbf{C}$ *are given by* (2.3), *then* $\mathbf{A} = \mathbf{B} + \mathbf{C}$ *is diagonalizable and* $s(\mathbf{A}) = \{a_1, \underbrace{a_2, a_2, \ldots, a_2}_{m-1 \text{ times}}\}$.

For $i = 1, \ldots, m$, let $\mathbf{v}_i = (v_i^1, \ldots, v_i^{i-1}, 1, 0, \ldots, 0)$ be the eigenvector of $\mathbf{B}$ with the eigenvalue b_i.

Lemma 2.1. *For* $1 \leqslant j < i \leqslant m$, *we have*

$$v_i^j = \frac{b_j + c_{m+1-j} - a_2}{b_i - b_j} \prod_{k=1}^{i-j-1} \frac{b_i + c_{m+1-j-k} - a_2}{b_i - b_{j+k}}.$$

Here and in the sequel, all empty products are understood to be equal to 1.

Here is an example with $m = 5$.

Example 2.2.

$$\mathbf{v}_1 = (1, 0, 0, 0, 0),$$

$$\mathbf{v}_2 = \left(\frac{b_1 + c_5 - a_2}{b_2 - b_1}, 1, 0, 0, 0\right),$$

$$\mathbf{v}_3 = \left(\frac{(b_1 + c_5 - a_2)(b_3 + c_4 - a_2)}{(b_3 - b_1)(b_3 - b_2)}, \frac{b_2 + c_4 - a_2}{b_3 - b_2}, 1, 0, 0\right),$$

$$\mathbf{v}_4 = \left(\frac{(b_1 + c_5 - a_2)(b_4 + c_3 - a_2)(b_4 + c_4 - a_2)}{(b_4 - b_1)(b_4 - b_2)(b_4 - b_3)}, \frac{(b_2 + c_4 - a_2)(b_4 + c_3 - a_2)}{(b_4 - b_2)(b_4 - b_3)},\right.$$
$$\left.\frac{b_3 + c_3 - a_2}{b_4 - c_3}, 1, 0\right),$$

$$\mathbf{v}_5 = \left(\frac{(b_1 + c_5 - a_2)(b_5 + c_2 - a_2)(b_5 + c_3 - a_2)(b_5 + c_4 - a_2)}{(b_5 - b_1)(b_5 - b_2)(b_5 - b_3)(b_5 - b_4)},\right.$$
$$\frac{(b_2 + c_4 - a_2)(b_5 + c_2 - a_2)(b_5 + c_3 - a_2)}{(b_5 - b_2)(b_5 - b_3)(b_5 - b_4)}, \frac{(b_3 + c_3 - a_2)(b_5 + c_2 - a_2)}{(b_5 - b_3)(b_5 - b_4)},$$
$$\left.\frac{b_4 + c_2 - a_2}{b_5 - b_4}, 1\right).$$

We define a scalar product on V by setting

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij} \frac{\prod_{k=i+1}^{m} (b_i - b_k)}{\prod_{k=1}^{i-1} (b_i - b_k)} \frac{\prod_{k=m+2-i}^{m} (b_i + c_k - a_2)}{\prod_{k=1}^{m+1-i} (b_i + c_k - a_2)}. \tag{2.4}$$

Here is an example of the form with $m = 5$.

Example 2.3.

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \frac{(b_1 - b_2)(b_1 - b_3)(b_1 - b_4)(b_1 - b_5)}{(b_1 + c_1 - a_2)(b_1 + c_2 - a_2)(b_1 + c_3 - a_2)(b_1 + c_4 - a_2)(b_1 + c_5 - a_2)},$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{(b_2 - b_3)(b_2 - b_4)(b_2 - b_5)(b_2 + c_5 - a_2)}{(b_2 - b_1)(b_2 + c_1 - a_2)(b_2 + c_2 - a_2)(b_2 + c_3 - a_2)(b_2 + c_4 - a_2)},$$

$$\langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \frac{(b_3 - b_4)(b_3 - b_5)(b_3 + c_4 - a_2)(b_3 + c_5 - a_2)}{(b_3 - b_1)(b_3 - b_2)(b_3 + c_1 - a_2)(b_3 + c_2 - a_2)(b_3 + c_3 - a_2)},$$

$$\langle \mathbf{v}_4, \mathbf{v}_4 \rangle = \frac{(b_4 - b_5)(b_4 + c_3 - a_2)(b_4 + c_4 - a_2)(b_4 + c_5 - a_2)}{(b_4 - b_1)(b_4 - b_2)(b_4 - b_3)(b_4 + c_1 - a_2)(b_4 + c_2 - a_2)},$$

$$\langle \mathbf{v}_5, \mathbf{v}_5 \rangle = \frac{(b_5 + c_2 - a_2)(b_5 + c_3 - a_2)(b_5 + c_4 - a_2)(b_5 + c_5 - a_2)}{(b_5 - b_1)(b_5 - b_2)(b_5 - b_3)(b_5 - b_4)(b_5 + c_1 - a_2)}.$$

Let $S'((1, m-1), (1^m), (1^m))$ be obtained from $S((1, m-1), (1^m), (1^m))$ by removing the hyperplanes which are zero levels of the linear forms in the denominators of $\langle \mathbf{v}_i, \mathbf{v}_i \rangle$. Let $S''((1, m-1), (1^m), (1^m))$ be obtained from $S'((1, m-1), (1^m), (1^m))$ by removing the hyperplanes which are zero levels of the linear forms in the numerators of $\langle \mathbf{v}_i, \mathbf{v}_i \rangle$. It is clear that if $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ lies in $S'((1, m-1), (1^m), (1^m))$, then form (2.4) is well-defined. It is clear that if $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ lies in $S''((1, m-1), (1^m), (1^m))$, then the form (2.4) is non-degenerate.

Theorem 2.2. *The operators* **A**, **B**, *and* **C** *are self-adjoint with respect to the scalar product* (2.4).

Now suppose that the eigenvalues of the matrices **A**, **B**, and **C** are real numbers. Let $b_1 > b_2 > \cdots > b_m$ and $c_1 > c_2 > \cdots > c_m$. We call a real symmetric bilinear (or hermitian) form *sign-definite* if it is either positive-definite or negative-definite.

Theorem 2.3. *Let the form* $\langle *, * \rangle$ *be defined by* (2.4). *Then it is sign-definite precisely in the following two situations*:

$$
\left|\begin{array}{lclcl}
b_m + c_1 & > & a_2 & > & b_m + c_2 \\
b_{m-1} + c_2 & > & a_2 & > & b_{m-1} + c_3 \\
\vdots & & \vdots & & \vdots \\
b_2 + c_{m-1} & > & a_2 & > & b_2 + c_m \\
b_1 + c_m & > & a_2 & &
\end{array}\right|
\begin{array}{lclcl}
b_1 + c_{m-1} & > & a_2 & > & b_1 + c_m \\
b_2 + c_{m-2} & > & a_2 & > & b_2 + c_{m-1} \\
\vdots & & \vdots & & \vdots \\
b_{m-1} + c_1 & > & a_2 & > & b_{m-1} + c_2 \\
 & & a_2 & > & b_m + c_1
\end{array}\Bigg| . \tag{2.5}
$$

If the form $\varepsilon\langle *, * \rangle$ is positive-definite for $\varepsilon = \pm 1$, then $\varepsilon = \operatorname{sign}(a_1 - a_2)$. If the inequalities of the first column hold, then $a_1 > a_2$. If the inequalities of the second column hold, then $a_1 < a_2$.

2.2. Even family

Let us pick a vector $(\underbrace{a_1 \cdots, a_1}_{m \text{ times}}, \underbrace{a_2, \ldots, a_2}_{m \text{ times}}, b_1, \underbrace{b_2, \ldots, b_2}_{m-1 \text{ times}}, \underbrace{b_3, \ldots, b_3}_{m \text{ times}}, c_1, \ldots, c_{2m})$ from $S((m, m), (1, m-1, m), (1^{2m}))$. Recall that this means $a_1 \neq a_2$, all b_i are distinct, all c_j are distinct, and the trace condition holds: $ma_1 + ma_2 = b_1 + (m-1)b_2 + mb_3 + \sum_{i=1}^{2m} c_i$. Let us set up the following notation:

$$
p_i^{jk} = c_i + b_j - a_k, \quad q_{ij} = c_i + c_j + b_2 + b_3 - a_1 - a_2. \tag{2.6}
$$

We now define the matrices **B** and **C** by setting

$$
\mathbf{B} = \begin{array}{c|c|c|c|}
 & 1 & \text{m-1} & \text{m} \\
\hline
1 & b_1 & B_{1,1+j} & B_{1,m+j} \\
\hline
\text{m-1} & 0 & b_2\, Id_{m-1} & B_{1+i,m+j} \\
\hline
\text{m} & 0 & 0 & b_3\, Id_m \\
\hline
\end{array} ,
$$

where

$$
B_{1,1+j} = (-1)^{m+1-j} \frac{\prod_{k=j+1}^{m} q_{k,2m-j}}{\prod_{k=m+1}^{2m-1-j} (c_k - c_{2m-j})} \quad (1 \leqslant j \leqslant m-1), \tag{2.7}
$$

$$B_{1,m+j} = (-1)^{m-j} p^{31}_{m+1-j} \frac{\prod_{k=m+j}^{2m-1} q_{m+1-j,k}}{\prod_{k=1}^{m-j}(c_k - c_{m+1-j})} \quad (1 \leqslant j \leqslant m), \tag{2.8}$$

$$B_{1+i,m+j}$$

$$= (-1)^{m-j} p^{31}_{m+1-j} \frac{\prod_{\substack{k=1 \\ k \neq m+1-j}}^{i} q_{k,2m-i} \prod_{\substack{k=m+j \\ k \neq 2m-i}}^{2m-1} q_{m+1-j,k}}{\prod_{k=2m+1-i}^{2m-1} (c_{2m-i} - c_k) \prod_{k=1}^{m-j} (c_k - c_{m+1-j})}$$
$$(1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant m), \tag{2.9}$$

$$\mathbf{C} = \begin{array}{r|c|c|c|} & 1 & m-1 & m \\ \hline 1 & c_{2m} & 0 & 0 \\ \hline m-1 & C_{1+i,1} & \begin{matrix} c_{2m-1} & & 0 \\ & \ddots & \\ 0 & & c_{m+1} \end{matrix} & 0 \\ \hline m & C_{m+i,1} & C_{m+i,1+j} & \begin{matrix} c_m & & 0 \\ & \ddots & \\ 0 & & c_1 \end{matrix} \\ \hline \end{array},$$

where

$$C_{1+i,1} = -\frac{\prod_{k=1}^{i} q_{k,2m-i}}{\prod_{k=2m+1-i}^{2m-1} (c_{2m-i} - c_k)} \quad (1 \leqslant i \leqslant m-1), \tag{2.10}$$

$$C_{m+i,1} = -p^{32}_{m+1-i} \frac{\prod_{k=m+1}^{m-1+i} q_{m+1-i,k}}{\prod_{k=m+2-i}^{m} (c_{m+1-i} - c_k)} \quad (1 \leqslant i \leqslant m), \tag{2.11}$$

$$C_{m+i,1+j}$$

$$= (-1)^{m+1-j} p^{32}_{m+1-i} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-j}}^{m-1+i} q_{m+1-i,k} \prod_{\substack{k=j+1 \\ k \neq m+1-i}}^{m} q_{k,2m-j}}{\prod_{k=m+2-i}^{m} (c_{m+1-i} - c_k) \prod_{k=m+1}^{2m-1-j} (c_k - c_{2m-j})}$$
$$(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m-1). \tag{2.12}$$

Here is an example with $m = 3$.

Example 2.4.

$$\mathbf{B}=\left[\begin{array}{c|cc|ccc} b_1 & -\frac{q_{25}\,q_{35}}{c_4-c_5} & q_{34} & \frac{p_3^{31}\,q_{34}\,q_{35}}{(c_1-c_3)(c_2-c_3)} & -\frac{p_2^{31}\,q_{25}}{c_1-c_2} & p_1^{31} \\ \hline 0 & b_2 & 0 & \frac{p_3^{31}\,q_{15}\,q_{34}}{(c_1-c_3)(c_2-c_3)} & -\frac{p_2^{31}\,q_{15}}{c_1-c_2} & p_1^{31} \\ 0 & 0 & b_2 & \frac{p_3^{31}\,q_{14}\,q_{24}\,q_{35}}{(c_1-c_3)(c_2-c_3)(c_4-c_5)} & -\frac{p_2^{31}\,q_{14}\,q_{25}}{(c_1-c_2)(c_4-c_5)} & \frac{p_1^{31}\,q_{24}}{c_4-c_5} \\ \hline 0 & 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3 \end{array}\right]$$

$$\mathbf{C}=\left[\begin{array}{c|cc|ccc} c_6 & 0 & 0 & 0 & 0 & 0 \\ \hline -q_{15} & c_5 & 0 & 0 & 0 & 0 \\ -\frac{q_{14}\,q_{24}}{c_4-c_5} & 0 & c_4 & 0 & 0 & 0 \\ \hline -p_3^{32} & -\frac{p_3^{32}\,q_{25}}{c_4-c_5} & p_3^{32} & c_3 & 0 & 0 \\ -\frac{p_2^{32}\,q_{24}}{c_2-c_3} & -\frac{p_2^{32}\,q_{24}\,q_{35}}{(c_2-c_3)(c_4-c_5)} & \frac{p_2^{32}\,q_{34}}{c_2-c_3} & 0 & c_2 & 0 \\ -\frac{p_1^{32}\,q_{14}\,q_{15}}{(c_1-c_2)(c_1-c_3)} & -\frac{p_1^{32}\,q_{14}\,q_{25}\,q_{35}}{(c_1-c_2)(c_1-c_3)(c_4-c_5)} & \frac{p_1^{32}\,q_{15}\,q_{34}}{(c_1-c_2)(c_1-c_3)} & 0 & 0 & c_1 \end{array}\right]$$

It is clear that $\mathbf{B}$ and $\mathbf{C}$ are diagonalizable and that their spectra are $s(\mathbf{B})=\{b_1,\underbrace{b_2,b_2,\dots,b_2}_{m-1\ \text{times}},\underbrace{b_3,b_3,\dots,b_3,b_3}_{m\ \text{times}}\}$, $s(\mathbf{C})=\{c_1,\dots,c_{2m}\}$.

Theorem 2.4. *If* $\mathbf{B}$ *and* $\mathbf{C}$ *are as above, then* $\mathbf{A}=\mathbf{B}+\mathbf{C}$ *is diagonalizable and* $s(\mathbf{A})=\{\underbrace{a_1,\dots,a_1}_{m\ \text{times}},\underbrace{a_2,\dots,a_2}_{m\ \text{times}}\}$.

For $i=1,\dots,2m$, let $\mathbf{v}_i=(0,\dots,0,1,v_i^{2m+2-i},\dots,v_i^{2m})$ be the eigenvector of the matrix $\mathbf{C}$ with the eigenvalue c_i.

Lemma 2.2. 1. *For* $1 \leqslant i \leqslant m$, *we have* $v_i^j = 0$ *for all* $2m+1-i < j$.

2. *For* $1 \leqslant i \leqslant m-1$, *we have* $v_{m+i}^j = 0$ *when* $m+1-i < j \leqslant m$ *and*

$$v_{m+i}^{m+j} = (-1)^i p_{m+1-j}^{32} \frac{\prod_{\substack{k=m+1 \\ k \neq m+i}}^{m-1+j} q_{m+1-j,k} \prod_{\substack{k=m+1-i \\ k \neq m+1-j}}^{m} q_{k,m+i}}{(c_{m+1-j} - c_{m+i}) \prod_{k=m+2-j}^{m} (c_{m+1-j} - c_k) \prod_{k=m+1}^{m-1+i} (c_k - c_{m+i})}$$

for $1 \leqslant j \leqslant m$.

3. *For* $1 \leqslant j \leqslant m-1$ *we have*

$$v_{2m}^{1+j} = \frac{\prod_{k=1}^{j} q_{k,2m-j}}{\prod_{k=2m+1-j}^{2m} (c_{2m-j} - c_k)}$$

and for $1 \leqslant j \leqslant m$ *we have*

$$v_{2m}^{m+j} = (-1)^{m+1} \frac{p_{m+1-j}^{32}}{c_{m+1-j} - c_{2m}} \times \frac{\prod_{\substack{k=1 \\ k \neq m+1-j}}^{m} q_{k,2m} \prod_{k=m+1}^{m-1+j} q_{m+1-j,k}}{\prod_{k=m+1}^{2m-1} (c_k - c_{2m}) \prod_{k=m+2-j}^{m} (c_{m+1-j} - c_k)}. \tag{2.13}$$

Here is an example with $m = 3$ ($\mathbf{e}_1, \ldots, \mathbf{e}_{2m}$ is the standard basis of V).

Example 2.5

$$\mathbf{v}_1 = \mathbf{e}_6, \quad \mathbf{v}_2 = \mathbf{e}_5, \quad \mathbf{v}_3 = \mathbf{e}_4,$$

$$\mathbf{v}_4 = \left(0, 0, 1, -\frac{p_3^{32}}{c_3 - c_4}, -\frac{p_2^{32} q_{34}}{(c_2 - c_3)(c_2 - c_4)}, -\frac{p_1^{32} q_{15} q_{34}}{(c_1 - c_2)(c_1 - c_3)(c_1 - c_4)}\right),$$

$$\mathbf{v}_5 = \left(0, 1, 0, \frac{p_3^{32} q_{25}}{(c_3 - c_5)(c_4 - c_5)}, \frac{p_2^{32} q_{24} q_{35}}{(c_2 - c_3)(c_2 - c_5)(c_4 - c_5)}, \frac{p_1^{32} q_{14} q_{25} q_{35}}{(c_1 - c_2)(c_1 - c_3)(c_1 - c_5)(c_4 - c_5)}\right),$$

$$\mathbf{v}_6 = \left(1, \frac{q_{15}}{c_5 - c_6}, \frac{q_{14}q_{24}}{(c_4 - c_5)(c_4 - c_6)}, \frac{p_3^{32}q_{16}q_{26}}{(c_3 - c_6)(c_4 - c_6)(c_5 - c_6)},\right.$$

$$\frac{p_2^{32}q_{24}q_{16}q_{36}}{(c_2 - c_3)(c_2 - c_6)(c_4 - c_6)(c_5 - c_6)},$$

$$\left.\frac{p_1^{32}q_{14}q_{15}q_{26}q_{36}}{(c_1 - c_2)(c_1 - c_3)(c_1 - c_6)(c_4 - c_6)(c_5 - c_6)}\right).$$

We define a scalar product on V by setting

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij} \frac{\prod_{k=i+1}^{2m} (c_i - c_k)}{\prod_{k=1}^{i-1}(c_i - c_k)} \frac{\prod_{\substack{k=2m+1-i \\ k \neq i}}^{2m} q_{ik}m}{\prod_{\substack{k=1 \\ k\neq i}}^{2m-i} q_{ik}} \times \begin{cases} \dfrac{p_i^{31}}{p_i^{32}} & \text{if } i \leqslant m, \\ p_i^{31}\, p_i^{32} & \text{if } i > m. \end{cases} \tag{2.14}$$

Here is an example with $m = 3$.

Example 2.6.

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = (c_1 - c_2)(c_1 - c_3)(c_1 - c_4)(c_1 - c_5)(c_1 - c_6) \times \frac{p_1^{31}}{p_1^{32}} \times \frac{q_{16}}{q_{12}q_{13}q_{14}q_{15}},$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{(c_2 - c_3)(c_2 - c_4)(c_2 - c_5)(c_2 - c_6)}{c_2 - c_1} \times \frac{p_2^{31}}{p_2^{32}} \times \frac{q_{25}q_{26}}{q_{12}q_{23}q_{24}},$$

$$\langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \frac{(c_3 - c_4)(c_3 - c_5)(c_3 - c_6)}{(c_3 - c_1)(c_3 - c_2)} \times \frac{p_3^{31}}{p_3^{32}} \times \frac{q_{34}q_{35}q_{36}}{q_{13}q_{23}},$$

$$\langle \mathbf{v}_4, \mathbf{v}_4 \rangle = \frac{(c_4 - c_5)(c_4 - c_6)}{(c_4 - c_1)(c_4 - c_2)(c_4 - c_3)} \times p_4^{31}p_4^{32} \times \frac{q_{34}q_{45}q_{46}}{q_{14}q_{24}},$$

$$\langle \mathbf{v}_5, \mathbf{v}_5 \rangle = \frac{c_5 - c_6}{(c_5 - c_1)(c_5 - c_2)(c_5 - c_3)(c_5 - c_4)} \times p_5^{31}p_5^{32} \times \frac{q_{25}q_{35}q_{45}q_{56}}{q_{15}},$$

$$\langle \mathbf{v}_6, \mathbf{v}_6 \rangle = \frac{1}{(c_6 - c_1)(c_6 - c_2)(c_6 - c_3)(c_6 - c_4)(c_6 - c_5)} \times p_6^{31} p_6^{32} \times q_{16} q_{26} q_{36} q_{46} q_{56}.$$

The sets $S'((m,m),(1,m-1,m),(1^{2m}))$ and $S''((m,m),(1,m-1,m),(1^{2m}))$ are constructed from (2.14) similarly to the hypergeometric case (see page 7) and have the same properties.

Theorem 2.5. *The operators* **A**, **B**, *and* **C** *are self-adjoint with respect to the scalar product* (2.14).

Now suppose that the eigenvalues of the matrices **A**, **B**, and **C** are real numbers. Let $a_1 > a_2$ and $c_1 > c_2 > \cdots > c_{2m}$.

Theorem 2.6. *The form* $\langle *, * \rangle$ *defined by* (2.14) *is sign-definite precisely in the following six situations*:

$b_1 > b_3 > b_2$ $p^{31}_{m-1} > 0 > p^{31}_m$ $p^{32}_{2m-1} > 0 > p^{32}_{2m}$ $q_{1,2m-2} > 0 > q_{1,2m-1}$ $q_{2,2m-3} > 0 > q_{2,2m-2}$ $q_{3,2m-4} > 0 > q_{3,2m-3}$ $\vdots$ $q_{m-1,m} > 0 > q_{m-1,m+1}$	$b_1 > b_2 > b_3$ $0 > p^{31}_1$ $p^{32}_m > 0 > p^{32}_{m+1}$ $q_{1,2m-1} > 0 > q_{1,2m}$ $q_{2,2m-2} > 0 > q_{2,2m-1}$ $\vdots$ $q_{m-1,m+1} > 0 > q_{m-1,m+2}$ $0 > q_{m,m+1}$	$b_2 > b_1 > b_3$ $0 > p^{31}_1$ $p^{32}_m > 0 > p^{32}_{m+1}$ $q_{1,2m} > 0 > q_{2,2m}$ $q_{2,2m-1} > 0 > q_{3,2m-1}$ $\vdots$ $q_{m-1,m+2} > 0 > q_{m,m+2}$ $q_{m,m+1} > 0$
$b_2 > b_3 > b_1$ $p^{31}_1 > 0 > p^{31}_2$ $p^{32}_{m+1} > 0 > p^{32}_{m+2}$ $q_{2,2m} > 0 > q_{3,2m}$ $q_{3,2m-1} > 0 > q_{4,2m-1}$ $q_{4,2m-2} > 0 > q_{5,2m-2}$ $\vdots$ $q_{m,m+2} > 0 > q_{m+1,m+2}$	$b_3 > b_2 > b_1$ $p^{31}_m > 0 > p^{31}_{m+1}$ $p^{32}_{2m} > 0$ $q_{1,2m} > 0 > q_{2,2m}$ $q_{2,2m-1} > 0 > q_{3,2m-1}$ $\vdots$ $q_{m-1,m+2} > 0 > q_{m,m+2}$ $q_{m,m+1} > 0$	$b_3 > b_1 > b_2$ $p^{31}_m > 0 > p^{31}_{m+1}$ $p^{32}_{2m} > 0$ $q_{1,2m-1} > 0 > q_{1,2m}$ $q_{2,2m-2} > 0 > q_{2,2m-1}$ $\vdots$ $q_{m-1,m+1} > 0 > q_{m-1,m+2}$ $0 > q_{m,m+1}$

.

If the form $\varepsilon\langle *, * \rangle$ *is positive-definite for* $\varepsilon = \pm 1$, *then* $\varepsilon = \operatorname{sign}((b_1 - b_2)(b_1 - b_3))$. *In each case, the inequalities between* b_1, b_2, *and* b_3 *are implied by other inequalities.*

2.3. Odd family

Let us pick a vector $(\underbrace{a_1, \dots, a_1}_{m+1 \text{ times}}, \underbrace{a_2, \dots, a_2}_{m \text{ times}}, b_1, \underbrace{b_2, \dots, b_2}_{m \text{ times}}, \underbrace{b_3, \dots, b_3}_{m \text{ times}}, c_1, \dots, c_{2m+1})$ from $S((m+1, m), (1, m, m), (1^{2m+1}))$. Recall that this means $a_1 \neq a_2$, all b_i are distinct, all c_j are distinct, and the trace condition holds: $(m+1)a_1 + ma_2 = b_1 + mb_2 + mb_3 + \sum_{i=1}^{2m+1} c_i$.

We now define the matrices **B** and **C** by setting

$$\mathbf{B} = \begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{\mathrm{m}} & \multicolumn{1}{c}{\mathrm{m}} \\ \cline{2-4} 1 & b_1 & B_{1,1+j} & B_{1,m+1+j} \\ \cline{2-4} \mathrm{m} & 0 & b_2\, Id_m & B_{1+i,m+1+j} \\ \cline{2-4} \mathrm{m} & 0 & 0 & b_3\, Id_m \\ \cline{2-4} \end{array},$$

where

$$B_{1,1+j} = (-1)^{m-j} p^{21}_{2m+1-j} \frac{\prod_{k=j+1}^{m} q_{k,2m+1-j}}{\prod_{k=m+1}^{2m-j} (c_k - c_{2m+1-j})} \quad (1 \leqslant j \leqslant m), \tag{2.15}$$

$$B_{1,m+1+j} = (-1)^{m-j} p^{31}_{m+1-j} \frac{\prod_{k=m+1+j}^{2m} q_{m+1-j,k}}{\prod_{k=1}^{m-j} (c_k - c_{m+1-j})} \quad (1 \leqslant j \leqslant m), \tag{2.16}$$

$$\begin{aligned} B_{1+i,m+1+j} &= (-1)^{m-j} p^{31}_{m+1-j} \\ &\quad \times \frac{\prod_{\substack{k=1 \\ k \neq m+1-j}}^{i} q_{k,2m+1-i} \prod_{\substack{k=m+1+j \\ k \neq 2m+1-i}}^{2m} q_{m+1-j,k}}{\prod_{k=2m+2-i}^{2m} (c_{2m+1-i} - c_k) \prod_{k=1}^{m-j} (c_k - c_{m+1-j})} \\ &\quad (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m), \end{aligned} \tag{2.17}$$

$$
\mathbf{C} = \begin{array}{c|c|c|c|}
 & 1 & \mathrm{m} & \mathrm{m} \\
\hline
1 & c_{2m+1} & 0 & 0 \\
\hline
\mathrm{m} & C_{1+i,1} & \begin{matrix} c_{2m} & & 0 \\ & \ddots & \\ 0 & & c_{m+1} \end{matrix} & 0 \\
\hline
\mathrm{m} & C_{m+1+i,1} & C_{m+1+i,1+j} & \begin{matrix} c_{m} & & 0 \\ & \ddots & \\ 0 & & c_{1} \end{matrix} \\
\hline
\end{array},
$$

where

$$C_{1+i,1} = -\frac{\prod_{k=1}^{i} q_{k,2m+1-i}}{\prod_{k=2m+2-i}^{2m}(c_{2m+1-i} - c_k)} \quad (1 \leqslant i \leqslant m), \tag{2.18}$$

$$C_{m+1+i,1} = -\frac{\prod_{k=m+1}^{m+i} q_{m+1-i,k}}{\prod_{k=m+2-i}^{m} (c_{m+1-i} - c_k)} \quad (1 \leqslant i \leqslant m), \tag{2.19}$$

$$C_{m+1+i,1+j} = (-1)^{m-j} p^{21}_{2m+1-j} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m+1-j}}^{m+i} q_{m+1-i,k} \prod_{\substack{k=j+1 \\ k \neq m+1-i}}^{m} q_{k,2m+1-j}}{\prod_{k=m+2-i}^{m} (c_{m+1-i} - c_k) \prod_{k=m+1}^{2m-j} (c_k - c_{2m+1-j})}$$

$$(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m). \tag{2.20}$$

Here is an example with $m = 3$.

Example 2.7.

$$\mathbf{B} = \left[\begin{array}{c|ccc|ccc} b_1 & \frac{p_6^{21} q_{26} q_{36}}{(c_4-c_6)(c_5-c_6)} & -\frac{p_5^{21} q_{35}}{c_4-c_5} & p_4^{21} & \frac{p_3^{31} q_{35} q_{36}}{(c_1-c_3)(c_2-c_3)} & -\frac{p_2^{31} q_{26}}{c_1-c_2} & p_1^{31} \\ \hline 0 & b_2 & 0 & 0 & \frac{p_3^{31} q_{16} q_{35}}{(c_1-c_3)(c_2-c_3)} & -\frac{p_2^{31} q_{16}}{c_1-c_2} & p_1^{31} \\ 0 & 0 & b_2 & 0 & \frac{p_3^{31} q_{15} q_{25} q_{36}}{(c_1-c_3)(c_2-c_3)(c_5-c_6)} & -\frac{p_2^{31} q_{15} q_{26}}{(c_1-c_2)(c_5-c_6)} & \frac{p_1^{31} q_{25}}{c_5-c_6} \\ 0 & 0 & 0 & b_2 & \frac{p_3^{31} q_{14} q_{24} q_{35} q_{36}}{(c_1-c_3)(c_2-c_3)(c_4-c_5)(c_4-c_6)} & -\frac{p_2^{31} q_{14} q_{34} q_{26}}{(c_1-c_2)(c_4-c_5)(c_4-c_6)} & \frac{p_1^{31} q_{24} q_{34}}{(c_4-c_5)(c_4-c_6)} \\ \hline 0 & 0 & 0 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_3 \end{array}\right]$$

$$\mathbf{C} = \left[\begin{array}{c|ccc|ccc} c_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -q_{16} & c_6 & 0 & 0 & 0 & 0 & 0 \\ -\frac{q_{15} q_{25}}{c_5-c_6} & 0 & c_5 & 0 & 0 & 0 & 0 \\ -\frac{q_{14} q_{24} q_{34}}{(c_4-c_5)(c_4-c_6)} & 0 & 0 & c_4 & 0 & 0 & 0 \\ \hline -q_{34} & \frac{p_6^{21} q_{34} q_{26}}{(c_4-c_6)(c_5-c_6)} & -\frac{p_5^{21} q_{34}}{c_4-c_5} & p_4^{21} & c_3 & 0 & 0 \\ -\frac{q_{24} q_{25}}{c_2-c_3} & \frac{p_6^{21} q_{24} q_{25} q_{36}}{(c_2-c_3)(c_4-c_6)(c_5-c_6)} & -\frac{p_5^{21} q_{24} q_{35}}{(c_2-c_3)(c_4-c_5)} & \frac{p_4^{21} q_{25}}{c_2-c_3} & 0 & c_2 & 0 \\ -\frac{q_{14} q_{15} q_{16}}{(c_1-c_2)(c_1-c_3)} & \frac{p_6^{21} q_{14} q_{15} q_{26} q_{36}}{(c_1-c_2)(c_1-c_3)(c_4-c_6)(c_5-c_6)} & -\frac{p_5^{21} q_{14} q_{16} q_{35}}{(c_1-c_2)(c_1-c_3)(c_4-c_5)} & \frac{p_4^{21} q_{15} q_{16}}{(c_1-c_2)(c_1-c_3)} & 0 & 0 & c_1 \end{array}\right].$$

It is clear that $\mathbf{B}$ and $\mathbf{C}$ are diagonalizable and that their spectra are $s(\mathbf{B}) = \{b_1, \underbrace{b_2, b_2, \dots, b_2,}_{m \text{ times}} \underbrace{b_3, b_3, \dots, b_3}_{m \text{ times}}\}$, $s(\mathbf{C}) = \{c_1, c_2, \dots, c_{2m+1}\}$.

Theorem 2.7. *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be as above and let* $\mathbf{A} = \mathbf{B} + \mathbf{C}$. *Then* $\mathbf{A}$ *is diagonalizable and* $s(\mathbf{A}) = \{\underbrace{a_1, \dots, a_1}_{m+1 \text{ times}} \underbrace{a_2, \dots, a_2}_{m \text{ times}}\}$.

For $i = 1, \dots, 2m+1$, let $\mathbf{v}_i = (0, \dots, 0, 1, v_i^{2m+3-i}, \dots, v_i^{2m+1})$ be the eigenvector of the matrix $\mathbf{C}$ with the eigenvalue c_i.

Lemma 2.3. 1. *For* $1 \leqslant i \leqslant m$, *we have* $v_i^j = 0$ *for all* $2m + 2 - i < j$.

2. *For* $1 \leqslant i \leqslant m$, *we have* $v^j_{m+i} = 0$ *when* $2m+2-i < j \leqslant m+1$ *and*

$$v^{m+1+j}_{m+i} = \frac{(-1)^i\, p^{21}_{m+i}}{c_{m+1-j} - c_{m+i}} \frac{\prod_{\substack{k=m+1\\ k \neq m+i}}^{m+j} q_{m+1-j,k} \prod_{\substack{k=m+2-i\\ k\neq m+1-j}}^{m} q_{k,m+i}}{\prod_{k=m+2-j}^{m} (c_{m+1-j} - c_k) \prod_{k=m+1}^{m-1+i} (c_k - c_{m+i})}$$

for $1 \leqslant j \leqslant m$.

3. *For* $1 \leqslant i \leqslant m$, *we have*

$$v^{1+i}_{2m+1} = \frac{\prod_{k=1}^{i} q_{k,2m+1-i}}{\prod_{k=2m+2-i}^{2m+1} (c_{2m+1-i} - c_k)}$$

and for $1 \leqslant j \leqslant m$, *we have*

$$v^{m+1+j}_{2m+1} = (-1)^m \frac{p^{21}_{2m+1}}{c_{m+1-j} - c_{2m+1}} \times \frac{\prod_{\substack{k=1\\ k\neq m+1-j}}^{m} q_{k,2m+1} \prod_{k=m+1}^{m+j} q_{m+1-j,k}}{\prod_{k=m+1}^{2m} (c_k - c_{2m+1}) \prod_{k=m+2-j}^{m} (c_{m+1-j} - c_k)}. \tag{2.21}$$

Here is an example with $m = 3$.

Example 2.8.

$$\mathbf{v}_1 = \mathbf{e}_7, \quad \mathbf{v}_2 = \mathbf{e}_6, \quad \mathbf{v}_3 = \mathbf{e}_5,$$

$$\mathbf{v}_4 = \left(0,0,0,1,-\frac{p^{21}_4}{c_3 - c_4}, -\frac{p^{21}_4 q_{25}}{(c_2-c_3)(c_2-c_4)}, -\frac{p^{21}_4 q_{15} q_{16}}{(c_1-c_2)(c_1-c_3)(c_1-c_4)}\right),$$

$$\mathbf{v}_5 = \left(0,0,1,0,\frac{p^{21}_5 q_{34}}{(c_3-c_5)(c_4-c_5)}, \frac{p^{21}_5 q_{24} q_{35}}{(c_2-c_3)(c_2-c_5)(c_4-c_5)}, \frac{p^{21}_5 q_{14} q_{16} q_{35}}{(c_1-c_2)(c_1-c_3)(c_1-c_5)(c_4-c_5)}\right),$$

$$\mathbf{v}_6 = \left(0,1,0,0,-\frac{p^{21}_6 q_{26} q_{34}}{(c_3-c_6)(c_4-c_6)(c_5-c_6)}, -\frac{p^{21}_6 q_{24} q_{25} q_{36}}{(c_2-c_3)(c_2-c_6)(c_4-c_6)(c_5-c_6)}, -\frac{p^{21}_6 q_{14} q_{15} q_{26} q_{36}}{(c_1-c_2)(c_1-c_3)(c_1-c_6)(c_4-c_6)(c_5-c_6)}\right),$$

$$\mathbf{v}_7 = \left(1, \frac{q_{16}}{c_6 - c_7}, \frac{q_{15}q_{25}}{(c_5 - c_6)(c_5 - c_7)}, \frac{q_{14}q_{24}q_{34}}{(c_4 - c_5)(c_4 - c_6)(c_4 - c_7)}, \right.$$

$$- \frac{p_7^{21} q_{17}q_{27}q_{34}}{(c_3 - c_7)(c_4 - c_7)(c_5 - c_7)(c_6 - c_7)},$$

$$- \frac{p_7^{21} q_{17}q_{37}q_{24}q_{25}}{(c_2 - c_3)(c_2 - c_7)(c_4 - c_7)(c_5 - c_7)(c_6 - c_7)},$$

$$\left. - \frac{p_7^{21} q_{27}q_{37}q_{14}q_{15}q_{16}}{(c_1 - c_2)(c_1 - c_3)(c_1 - c_7)(c_4 - c_7)(c_5 - c_7)(c_6 - c_7)}\right).$$

We define a scalar product on V by setting

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij} \frac{\prod_{k=1+i}^{2m+1}(c_i - c_k)}{\prod_{k=1}^{i-1}(c_i - c_k)} \frac{\prod_{\substack{k=2m+2-i \\ k \neq i}}^{2m+1} q_{ik}}{\prod_{\substack{k=1 \\ k \neq i}}^{2m+1-i} q_{ik}} \times \begin{cases} \dfrac{p_i^{31}}{p_i^{21}} & \text{if } i \leqslant m, \\ p_i^{31} p_i^{21} & \text{if } i > m. \end{cases} \tag{2.22}$$

Here is an example with $m = 3$.

Example 2.9.

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = (c_1 - c_2)(c_1 - c_3)(c_1 - c_4)(c_1 - c_5)(c_1 - c_6)(c_1 - c_7) \times \frac{p_1^{31}}{p_1^{21}} \times \frac{q_{17}}{q_{12}q_{13}q_{14}q_{15}q_{16}},$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{(c_2 - c_3)(c_2 - c_4)(c_2 - c_5)(c_2 - c_6)(c_2 - c_7)}{c_2 - c_1} \times \frac{p_2^{31}}{p_2^{21}} \times \frac{q_{26}q_{27}}{q_{12}q_{23}q_{24}q_{25}},$$

$$\langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \frac{(c_3 - c_4)(c_3 - c_5)(c_3 - c_6)(c_3 - c_7)}{(c_3 - c_1)(c_3 - c_2)} \times \frac{p_3^{31}}{p_3^{21}} \times \frac{q_{35}q_{36}q_{37}}{q_{13}q_{23}q_{34}},$$

$$\langle \mathbf{v}_4, \mathbf{v}_4 \rangle = \frac{(c_4 - c_5)(c_4 - c_6)(c_4 - c_7)}{(c_4 - c_1)(c_4 - c_2)(c_4 - c_3)} \times p_4^{31} p_4^{21} \times \frac{q_{45}q_{46}q_{47}}{q_{14}q_{24}q_{34}},$$

$$\langle \mathbf{v}_5, \mathbf{v}_5 \rangle = \frac{(c_5 - c_6)(c_5 - c_7)}{(c_5 - c_1)(c_5 - c_2)(c_5 - c_3)(c_5 - c_4)} \times p_5^{31} p_5^{21} \times \frac{q_{35}q_{45}q_{56}q_{57}}{q_{15}q_{25}},$$

$$\langle \mathbf{v}_6, \mathbf{v}_6 \rangle = \frac{c_6 - c_7}{(c_6 - c_1)(c_6 - c_2)(c_6 - c_3)(c_6 - c_4)(c_6 - c_5)} \times p_6^{31} p_6^{21} \times \frac{q_{26} q_{36} q_{46} q_{56} q_{67}}{q_{16}},$$

$$\langle \mathbf{v}_7, \mathbf{v}_7 \rangle = \frac{1}{(c_7 - c_1)(c_7 - c_2)(c_7 - c_3)(c_7 - c_4)(c_7 - c_5)(c_7 - c_6)} \times p_7^{31} p_7^{21} \times q_{17} q_{27} q_{37} q_{47} q_{57} q_{67}.$$

The sets $S'((m+1,m),(1,m,m),(1^{2m+1}))$ and $S''((m+1,m),(1,m,m),(1^{2m+1}))$ are constructed from (2.22) similarly to the hypergeometric case (see page 7) and have the same properties.

Theorem 2.8. *The operators* **A**, **B**, *and* **C** *are self-adjoint with respect to the scalar product* (2.22).

Now suppose that the eigenvalues of the matrices **A**, **B**, and **C** are real numbers. Let $b_2 > b_3$ and $c_1 > c_2 > \cdots > c_{2m+1}$.

Theorem 2.9. *Under the condition* $b_2 > b_3$, *the form* $\langle *, * \rangle$ *defined by* (2.22) *is sign-definite precisely in the following three situations*:

$b_1 > b_2 > b_3$	$b_2 > b_1 > b_3$	$b_2 > b_3 > b_1$
$0 > p_1^{31}$ $p_m^{21} > 0 > p_{m+1}^{21}$	$0 > p_1^{31}$ $p_{m+1}^{21} > 0 > p_{m+2}^{21}$	$p_1^{31} > 0 > p_2^{31}$ $r_{m+1}^{21} > 0 > r_{m+2}^{21}$
$q_{1,2m} > 0 > q_{1,2m+1}$ $q_{2,2m-1} > 0 > q_{2,2m}$ $q_{3,2m-2} > 0 > q_{3,2m-1}$ $\vdots$ $q_{m,m+1} > 0 > q_{m,m+2}$	$q_{1,2m+1} > 0 > q_{2,2m+1}$ $q_{2,2m} > 0 > q_{3,2m}$ $\vdots$ $q_{m-1,m+3} > 0 > q_{m,m+3}$ $q_{m,m+2} > 0 > q_{m+1,m+2}$	$q_{2,2m+1} > 0 > q_{3,2m+1}$ $q_{3,2m} > 0 > q_{4,2m}$ $\vdots$ $q_{m,m+3} > 0 > q_{m+1,m+3}$ $q_{m+1,m+2} > 0$

If the form $\varepsilon \langle *, * \rangle$ *is positive-definite for* $\varepsilon = \pm 1$, *then* $\varepsilon = \mathrm{sign}((b_1 - b_2)(b_1 - b_3))$. *In each case, the inequalities between* b_1 *and* b_2 *or* b_3 *are implied by other inequalities.*

2.4. Extra case

Let us pick a vector $(a_1, a_1, a_1, a_1, a_2, a_2, b_1, b_1, b_2, b_2, b_3, b_3, c_1, c_2, c_3, c_4, c_5, c_6)$ from $S((4,2),(2,2,2),(1^6))$. Recall that this means $a_1 \neq a_2$, all b_i are distinct, all c_j are distinct, and the trace condition holds: $4a_1 + 2a_2 = 2b_1 + 2b_2 + 2b_3 + \sum_{i=1}^{6} c_i$. Let us set up the following notation:

$$p_{ij} = b_i + c_j - a_1, \quad q_{ijk} = 2c_i + 2c_j + 2c_k - \frac{1}{2}\sum_{l=1}^{6} c_l. \tag{2.23}$$

We now define the matrices **B** and **C** by setting

$$\mathbf{B} = \left[\begin{array}{cc|cc|cc} b_1 & 0 & -\frac{p_{16}\,q_{245}}{c_3-c_4} & p_{16} & \frac{p_{16}\,q_{245}}{c_1-c_2} & p_{16} \\ 0 & b_1 & \frac{p_{15}\,q_{235}\,q_{246}}{(c_3-c_4)(c_5-c_6)} & -\frac{p_{15}\,q_{236}}{c_5-c_6} & -\frac{p_{15}\,q_{236}\,q_{246}}{(c_1-c_2)(c_5-c_6)} & -\frac{p_{15}\,q_{235}}{c_5-c_6} \\ \hline 0 & 0 & b_2 & 0 & -\frac{p_{24}\,q_{236}}{c_1-c_2} & p_{24} \\ 0 & 0 & 0 & b_2 & -\frac{p_{23}\,q_{245}\,q_{246}}{(c_1-c_2)(c_3-c_4)} & \frac{p_{23}\,q_{235}}{c_3-c_4} \\ \hline 0 & 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3 \end{array}\right], \tag{2.24}$$

$$\mathbf{C} = \left[\begin{array}{cc|cc|cc} c_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_5 & 0 & 0 & 0 & 0 \\ \hline -\frac{p_{24}\,q_{236}}{c_5-c_6} & -p_{24} & c_4 & 0 & 0 & 0 \\ -\frac{p_{23}\,q_{235}\,q_{246}}{(c_3-c_4)(c_5-c_6)} & -\frac{p_{23}\,q_{245}}{c_3-c_4} & 0 & c_3 & 0 & 0 \\ \hline -\frac{p_{32}\,q_{235}}{c_5-c_6} & -p_{32} & \frac{p_{32}\,q_{235}}{c_3-c_4} & -p_{32} & c_2 & 0 \\ \frac{p_{31}\,q_{236}\,q_{246}}{(c_1-c_2)(c_5-c_6)} & \frac{p_{31}\,q_{245}}{c_1-c_2} & \frac{p_{31}\,q_{245}\,q_{246}}{(c_1-c_2)(c_3-c_4)} & -\frac{p_{31}\,q_{236}}{c_1-c_2} & 0 & c_1 \end{array}\right].$$

It is clear that **B** and **C** are diagonalizable and that their spectra are $s(\mathbf{B}) = \{b_1, b_1, b_2, b_2, b_3, b_3\}$, $s(\mathbf{C}) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$.

Theorem 2.10. *For* $\mathbf{B}$ *and* $\mathbf{C}$ *as above, let* $\mathbf{A} = \mathbf{B} + \mathbf{C}$. *Then* $\mathbf{A}$ *is diagonalizable and* $s(\mathbf{A}) = \{a_1, a_1, a_1, a_1, a_2, a_2\}$.

Lemma 2.4. *The following are the eigenvectors of the matrix* $\mathbf{C}$ ($\mathbf{v}_i$ *corresponds to the eigenvalue* c_i):

$$\mathbf{v}_1 = (0, 0, 0, 0, 0, 1),$$

$$\mathbf{v}_2 = (0, 0, 0, 0, 1, 0),$$

$$\mathbf{v}_3 = \left(0, 0, 0, 1, \frac{p_{32}}{c_2 - c_3}, \frac{p_{31}q_{236}}{(c_1 - c_2)(c_1 - c_3)}\right),$$

$$\mathbf{v}_4 = \left(0, 0, 1, 0, -\frac{p_{32}q_{235}}{(c_2 - c_4)(c_3 - c_4)}, -\frac{p_{31}q_{245}q_{246}}{(c_1 - c_2)(c_1 - c_4)(c_3 - c_4)}\right),$$

$$\mathbf{v}_5 = \left(0, 1, \frac{p_{24}}{c_4 - c_5}, \frac{p_{23}q_{245}}{(c_3 - c_4)(c_3 - c_5)}, -\frac{p_{25}p_{32}q_{234}}{(c_2 - c_5)(c_3 - c_5)(c_4 - c_5)}, -\frac{p_{25}p_{31}q_{245}q_{256}}{(c_1 - c_2)(c_1 - c_5)(c_3 - c_5)(c_4 - c_5)}\right),$$

$$\mathbf{v}_6 = \left(1, 0, \frac{p_{24}q_{236}}{(c_4 - c_6)(c_5 - c_6)}, \frac{p_{23}q_{235}q_{246}}{(c_3 - c_4)(c_3 - c_6)(c_5 - c_6)}, - \frac{p_{26}p_{32}q_{234}q_{235}}{(c_2 - c_6)(c_3 - c_6)(c_4 - c_6)(c_5 - c_6)}, - \frac{p_{26}p_{31}q_{236}q_{246}q_{256}}{(c_1 - c_2)(c_1 - c_6)(c_3 - c_6)(c_4 - c_6)(c_5 - c_6)}\right).$$

Let us define a scalar product on V by setting $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$ and setting

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = -\frac{(c_1 - c_2)(c_1 - c_3)(c_1 - c_4)(c_1 - c_5)(c_1 - c_6)}{p_{11}p_{21}p_{31}},$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{(c_2 - c_3)(c_2 - c_4)(c_2 - c_5)(c_2 - c_6)}{(c_2 - c_1)p_{12}p_{22}p_{32}} \times \frac{q_{134}q_{135}q_{136}q_{145}}{q_{146}q_{156}},$$

$$\langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \frac{(c_3 - c_4)(c_3 - c_5)(c_3 - c_6)p_{33}}{(c_3 - c_1)(c_3 - c_2)p_{13}p_{23}} \times \frac{q_{124}q_{125}q_{126}q_{145}}{q_{146}q_{156}},$$

$$\langle \mathbf{v}_4, \mathbf{v}_4 \rangle = \frac{(c_4 - c_5)(c_4 - c_6)p_{34}}{(c_4 - c_1)(c_4 - c_2)(c_4 - c_3)p_{14}p_{24}} \times \frac{q_{123}q_{125}q_{126}q_{135}q_{136}}{q_{156}},$$

$$\langle \mathbf{v}_5, \mathbf{v}_5 \rangle = \frac{(c_5 - c_6)p_{25}p_{35}}{(c_5 - c_1)(c_5 - c_2)(c_5 - c_3)(c_5 - c_4)p_{15}} \times \frac{q_{123}q_{124}q_{126}q_{134}q_{136}}{q_{146}},$$

$$\langle \mathbf{v}_6, \mathbf{v}_6 \rangle = \frac{p_{26}p_{36}}{(c_6 - c_1)(c_6 - c_2)(c_6 - c_3)(c_6 - c_4)(c_6 - c_5)p_{16}} \times q_{123}q_{124}q_{125}q_{134}q_{135}q_{145}. \tag{2.25}$$

The sets $S'((4,2),(2,2,2),(1^6))$ and $S''((4,2),(2,2,2),(1^6))$ are constructed from (2.25) similarly to the hypergeometric case (see page 7) and have the same properties.

Theorem 2.11. *The operators* **A**, **B**, *and* **C** *are self-adjoint with respect to the scalar product* (2.25).

Now suppose that the eigenvalues of the matrices **A**, **B**, and **C** are real numbers. Let $b_1 > b_2 > b_3$ and $c_1 > c_2 > \cdots > c_6$.

Theorem 2.12. *The form* $\langle *, * \rangle$ *defined by* (2.25) *is sign-definite precisely in the following two situations*:

$b_1 + c_4 > a_1 > b_1 + c_5$ $b_2 + c_2 > a_1 > b_2 + c_3$ $a_1 > b_3 + c_1$	$b_3 + c_2 > a_1 > b_3 + c_3$ $b_2 + c_4 > a_1 > b_2 + c_5$ $b_1 + c_6 > a_1$
$c_1 + c_4 + c_5 > c_2 + c_3 + c_6$ $c_1 + c_3 + c_6 > c_2 + c_4 + c_5$ $c_2 + c_3 + c_5 > c_1 + c_4 + c_6$	$c_1 + c_4 + c_5 > c_2 + c_3 + c_6$ $c_1 + c_3 + c_6 > c_2 + c_4 + c_5$ $c_2 + c_3 + c_5 > c_1 + c_4 + c_6$

If the form $\varepsilon\langle *, * \rangle$ *is positive-definite for* $\varepsilon = \pm 1$, *then* $\varepsilon = \operatorname{sign}(a_1 - a_2)$. *If the inequalities of the first column hold, then* $a_1 > a_2$. *If the inequalities of the second column hold, then* $a_1 < a_2$.

3. Proofs and more results

In this section we prove theorems from Section 2. In the process, some new results are obtained.

The following simple observation is helpful in this section. If we replace a triple $(\mathbf{A} = \mathbf{B} + \mathbf{C}, \mathbf{B}, \mathbf{C})$ by the triple

$$\tilde{\mathbf{A}} = k\mathbf{A} + \theta\mathbf{Id},$$

$$\tilde{\mathbf{B}} = k\mathbf{B} + \phi\mathbf{Id},$$

$$\tilde{\mathbf{C}} = k\mathbf{C} + (\theta - \phi)\mathbf{Id},$$

$$k, \theta, \phi \in \mathbb{C} \text{ and } k \neq 0, \tag{3.26}$$

then we still have $\tilde{\mathbf{A}} = \tilde{\mathbf{B}} + \tilde{\mathbf{C}}$. This transformation changes neither irreducibility nor rigidity of the triple. If, say, $\mathbf{v}$ is an eigenvector of $\mathbf{B}$ with the eigenvalue b, then $\mathbf{v}$ is an eigenvector of $\tilde{\mathbf{B}}$ with the eigenvalue $kb + \phi$. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ were self-adjoint with respect to a symmetric bilinear form , then $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, and $\tilde{\mathbf{C}}$ are self-adjoint with respect to the form as well.

3.1. Hypergeometric family

An affine transformation (3.26) with $k = 1$, $\theta = -a_2$, and $\phi = -a_2/2$ normalizes $\mathbf{A}$ to $\tilde{\mathbf{A}}$ such that the eigenvalue of $\tilde{\mathbf{A}}$ of multiplicity $m - 1$ is 0. So, without loss of generality, we can assume that $a_2 = 0$. Now let us prove Theorem 2.1.

Proof of Theorem 2.1. Consider the matrix $\mathbf{A}$ (with $a_2 = 0$). Here is an example with $m = 5$.

$$\mathbf{A} = \begin{bmatrix} b_1 + c_5 & b_1 + c_5 & b_1 + c_5 & b_1 + c_5 & b_1 + c_5 \\ b_2 + c_4 & b_2 + c_4 & b_2 + c_4 & b_2 + c_4 & b_2 + c_4 \\ b_3 + c_3 & b_3 + c_3 & b_3 + c_3 & b_3 + c_3 & b_3 + c_3 \\ b_4 + c_2 & b_4 + c_2 & b_4 + c_2 & b_4 + c_2 & b_4 + c_2 \\ b_5 + c_1 & b_5 + c_1 & b_5 + c_1 & b_5 + c_1 & b_5 + c_1 \end{bmatrix}.$$

Now $\mathbf{A}$ has rank 1, and its image is the linear span of the vector $\mathbf{i} = (b_1 + c_m, b_2 + c_{m-1}, \dots, b_m + c_1)$. $\mathbf{A}\mathbf{i} = a_1\mathbf{i} \neq 0$. Thus, $\mathbf{A}$ is diagonalizable, and $s(\mathbf{A}) = \{\sum_{i=1}^m (b_i + c_i), 0, 0, \dots, 0\}$. Thus, before the normalizing affine transformation we had $s(\mathbf{A}) = \{a_1, \underbrace{a_2, \dots, a_2}_{m-1 \text{ times}}\}$. $\square$

In our normalized version,

$$B_{ij} = \begin{cases} 0 & \text{if } i<j \\ b_i & \text{if } i=j, \\ b_i + c_{m+1-i} & \text{if } i>j \end{cases} \quad C_{ij} = \begin{cases} 0 & \text{if } i>j, \\ c_{m+1-i} & \text{if } i=j, \\ b_i + c_{m+1-j} & \text{if } i<j. \end{cases} \tag{3.27}$$

We are ready to prove Lemma 2.1, that is to show that for every $i = 1, \ldots, m$ the vector $\mathbf{v}_i = (v_i^1, \ldots, v_i^{i-1}, 1, 0, \ldots, 0)$ with

$$v_i^j = \frac{b_j + c_{m+1-j}}{b_i - b_j} \prod_{k=1}^{i-j-1} \frac{b_i + c_{m+1-j-k}}{b_i - b_{j+k}} i \quad (j = 1, \ldots, i-1) \tag{3.28}$$

is an eigenvector of $\mathbf{B}$ with the eigenvalue b_i.

Proof of Lemma 2.1. Remembering definition (3.27) of $\mathbf{B}$, we need to show the following equality for all $j < i$: $(b_i - b_j)v_i^j = (b_j + c_{m+1-j}) \sum_{k=j+1}^{i} v_i^k$, or equivalently

$$\sum_{j+1}^{i} v_i^k = \prod_{k=1}^{i-j-1} \frac{b_i + c_{m+1-j-k}}{b_i - b_{j+k}}. \tag{3.29}$$

This identity becomes obvious once we rewrite (3.28) as

$$v_i^j = \prod_{k=1}^{i-j} \frac{b_i + c_{m+1-j-k}}{b_i - b_{j+k}} - \prod_{k=1}^{i-j-1} \frac{b_i + c_{m+1-j-k}}{b_i - b_{j+k}}, \tag{3.30}$$

and use telescoping. □

Now we are ready to prove Theorem 2.2, that is to show that the operators $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are self-adjoint with respect to the scalar product (2.4).

Proof of Theorem 2.2. The operator $\mathbf{B}$ is self-adjoint with respect to the scalar product by construction. To show that $\mathbf{A}$ is self-adjoint, we have to show that $\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{v}_i, \mathbf{A}\mathbf{v}_j \rangle$. As we have seen, $\mathbf{A}$ has a one-dimensional image spent by the vector $\mathbf{i} = (b_1 + c_m, b_2 + c_{m-1}, \ldots, b_m + c_1)$. Namely, for any vector $\mathbf{k} = (k_1, k_2, \ldots, k_m)$, $\mathbf{A}\mathbf{k} = (\sum_{i=1}^m k_i)\mathbf{i}$. In particular, $\mathbf{A}\mathbf{v}_i = (\sum_{j=1}^m v_i^j)\mathbf{i}$. In view of (3.29), we have

$$\mathbf{A}\mathbf{v}_i = \left(\prod_{k=1}^{i-1} \frac{b_i + c_{m+1-k}}{b_i - b_k} \right) \mathbf{i}. \tag{3.31}$$

It will be convenient to introduce the following notation:

$$s_i = \prod_{k=1}^{i-1} \frac{b_i + c_{m+1-k}}{b_i - b_k}, \quad x_i = (b_i + c_{m+1-i}) \prod_{k=1}^{m-i} \frac{b_i + c_{m+1-i-k}}{b_i - b_{i+k}}. \tag{3.32}$$

Then $\mathbf{A}\mathbf{v}_i = s_i\mathbf{i}$ and (2.4) can be rewritten as $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = s_i / x_i$. Now the desired equality $\langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j \rangle = s_i x_j \langle \mathbf{v}_j, \mathbf{v}_j \rangle = s_i x_j s_j / x_j = s_i s_j = \langle \mathbf{v}_i, \mathbf{A}\mathbf{v}_j \rangle$ becomes a consequence of the following lemma. □

Lemma 3.1. $\sum_{i=1}^m x_i \mathbf{v}_i = \mathbf{i}$.

Proof. We have $\mathbf{v}_i = (v_i^1, \dots, v_i^{i-1}, 1, 0 \dots, 0)$. Thus, to prove the lemma we have to prove that the identity $\sum_{j=i}^m x_j v_j^i = b_i + c_{m+1-i}$ holds for $1 \leqslant i \leqslant m$. This is equivalent to $x_i = b_i + c_{m+1-i} - \sum_{k=1}^{m-i} x_{i+k} v_{i+k}^i$. This identity after minor simplification becomes

$$\sum_{j=i}^{m} \frac{\prod_{k=1}^{m-i} (b_j + c_k)}{\prod_{\substack{k=i \\ k \neq j}}^{m} (b_j - b_k)} = 1.$$

Let us set $n = m - i + 1$; $x_1 = b_i$, $x_2 = b_{i+1}, \dots, x_n = b_m$; $y_1 = -c_1$, $y_2 = -c_2, \dots, y_{n-1} = -c_{m-i}$. This change of variables transforms the last identity into identity (7.60) which we prove in the appendix. $\square$

Now we prove Theorem 1.3 for the hypergeometric family. That is, we show that if the vector $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ lies in $S''((1, m-1), (1^m), (1^m))$, then the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is irreducible.

Proof of Theorem 1.3. Suppose that the triple preserves a non-trivial subspace of V. Then this subspace is spanned by some of the eigenvectors $\mathbf{v}_j$ of $\mathbf{B}$. But $\mathbf{A}\mathbf{v}_j = \sum_{i=1}^m s_j x_i \mathbf{v}_i$. If $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ lies in $S''((1, m-1), (1^m), (1^m))$, then all the coefficients $s_j x_i$ are non-zero, so $\mathbf{A}$ does not preserve any such proper subspace. Thus, the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is irreducible. $\square$

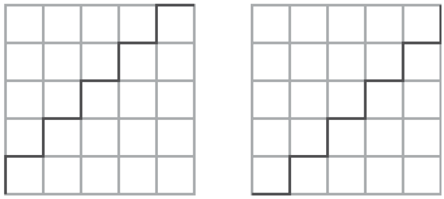
Let us prove Theorem 2.3, that is determine the inequalities on the real spectra of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ which make form (2.4) sign-definite. Recall that we are working with the normalized version $a_2 = 0$. Then the trace identity gives us $a_1 = \sum_{i=1}^m (b_i + c_i)$. Also, it is an assumption of Theorem 2.3 that $b_1 > b_2 > \dots > b_m$ and $c_1 > c_2 > \dots > c_m$.

Proof of Theorem 2.3. It immediately follows from Theorem 2.2 that

$$\operatorname{sign}(\langle \mathbf{v}_i, \mathbf{v}_i \rangle) = (-1)^{i-1} \operatorname{sign}\left(\prod_{j=1}^{m} (b_i + c_j) \right).$$

Construct an $m \times m$ matrix T where $T_{i,j} = b_i + c_j$. Notice that $T_{i,j} > T_{i,j+1}$ and $T_{i,j} > T_{i+1,j}$ for all i and j. Then $\operatorname{sign}(\langle \mathbf{v}_i, \mathbf{v}_i \rangle) = (-1)^{i-1} \times (-1)^{\#\{j:\, T_{i,j} < 0\}}$. Thus, to keep the sign constant, $\#\{j: T_{i,j} < 0\}$ must differ from $\#\{j : T_{i+1,j} < 0\}$ by an odd number for all m rows of T. This gives us only two possibilities: either $T_{i,m-i} > 0 > T_{i,m+1-i}$, or $T_{i,m+1-i} > 0 > T_{i,m+2-i}$. Here is a picture which illustrates the two situations for $m = 5$. The line separates $T_{i,j} > 0$ from

$T_{i,j} < 0$.



The sum of $T_{i,j}$ along the non-main diagonal of T equals a_1. Thus, $T_{i,m-i} > 0 > T_{i,m+1-i}$ forces $a_1 < 0$ and $T_{i,m+1-i} > 0 > T_{i,m+2-i}$ forces $a_1 > 0$. $\square$

Remark 3.1. Let the eigenvalues of **A**, **B**, and **C** be real numbers, and let form (2.4) be positive-definite. Then in the basis $\tilde{\mathbf{e}}_i = \mathbf{v}_i/\sqrt{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}$, the form becomes standard ($\langle \tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j \rangle = \delta_{ij}$). Let $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, and $\tilde{\mathbf{C}}$, be the matrices **A**, **B**, and **C** in the basis $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \ldots, \tilde{\mathbf{e}}_m$. Then for $i, j = 1, 2, \ldots, m$,

$$\tilde{A}_{ij} = \sqrt{x_i x_j s_i s_j},$$

$$\tilde{B}_{ij} = \delta_{ij} b_i,$$

$$\tilde{C}_{ij} = \tilde{A}_{ij} - \tilde{B}_{ij}.$$

3.2. *Even family*

In order to make proofs simpler, let us normalize **A**, **B**, and **C** so that they become traceless and $a_1 = 1$, $a_2 = -1$. The affine transformation (3.26) with $k = 2/(a_1 - a_2)$, $\theta = -(a_1 + a_2)/2$, and $\phi = -(b_1 + (m-1)b_2 + mb_3)/(2m)$ does the job.

Let us prove Theorem 2.4. In our normalized version, we have to prove that **A** is diagonalizable and that $s(\mathbf{A}) = \{\underbrace{1, \ldots, 1}_{m \text{ times}}, \underbrace{-1, \ldots, -1}_{m \text{ times}}\}$.

Proof of Theorem 2.4. First, let us prove that $\mathbf{A}^2 = \mathbf{Id}$. For that, we have to prove the following eleven identities.

1. The identity $\sum_{l=1}^{2m} A_{1l} A_{l1} = 1$ with the help of identity (7.60) can be reduced to the identity

$$((m-1)b_2 + mb_3 + c_1 + c_2 + \cdots + c_{2m-1})^2$$

$$= \sum_{i=1}^{m-1} \frac{\prod_{j=1}^{m} (b_2 + b_3 + c_j + c_{m+i})}{\prod_{\substack{j=1 \\ j \neq i}}^{m-1} (c_{m+i} - c_{m+j})}$$

$$+ \sum_{i=1}^{m} (b_3 + c_i)^2 \frac{\prod_{j=1}^{m-1} (b_2 + b_3 + c_i + c_{m+j})}{\prod_{\substack{j=1 \\ j \neq i}}^{m} (c_i - c_j)}.$$

For $1 \leqslant i \leqslant m-1$, let us set $x_i = c_{m+i} + b_2$. For $1 \leqslant i \leqslant m$, let us set $y_i = c_i + b_3$. Then the last identity becomes

$$\left(\sum_{i=1}^{m-1} x_i + \sum_{i=1}^{m} y_i\right)^2 = \sum_{i=1}^{m-1} \frac{\prod_{j=1}^{m} (x_i + y_j)}{\prod_{\substack{j=1 \\ j \neq i}}^{m-1} (x_i - x_j)} + \sum_{i=1}^{m} y_i^2 \frac{\prod_{j=1}^{m-1} (x_j + y_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{m} (y_i - y_j)}. \tag{3.33}$$

Introducing $x_m = x_{m+1} = 0$, we can rewrite the last term in (3.33) as

$$\sum_{i=1}^{m} y_i^2 \frac{\prod_{j=1}^{m-1} (x_j + y_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{m} (y_i - y_j)} = \sum_{i=1}^{m} \frac{\prod_{j=1}^{m+1} (y_i + x_j)}{\prod_{\substack{j=1 \\ j \neq i}}^{m} (y_i - y_j)}.$$

Now we can prove identity (3.33) with the help of identity (7.62) from the appendix.

2. For $1 \leqslant i \leqslant m-1$, the identity $\sum_{l=1}^{2m} A_{1,l} A_{l,1+i} = 0$ after some simplification becomes

$$\sum_{j=1}^{m} ((b_3 + c_j)^2 - 1) \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m-1} q_{j,k}}{\prod_{\substack{k=1 \\ k \neq j}}^{m} (c_j - c_k)} = \sum_{j=1}^{m} (c_j + b_3) + \sum_{\substack{j=m+1 \\ j \neq 2m-i}}^{2m-1} (c_j + b_2).$$

For $1 \leqslant j \leqslant m$, set $x_j = b_3 + c_j$. For $1 \leqslant j \leqslant m-1$ and $j \neq m-i$, set $y_j = -(b_2 + c_{m+j})$. The identity

$$\sum_{j=1}^{m} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m-1} q_{j,k}}{\prod_{\substack{k=1 \\ k \neq j}}^{m} (c_j - c_k)} = 0$$

is equivalent to identity (7.59) from the appendix. Now the identity to prove becomes

$$\sum_{j=1}^{m} (b_3 + c_j)^2 \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m-1} q_{j,k}}{\prod_{\substack{k=1 \\ k \neq j}}^{m} (c_j - c_k)} = \sum_{j=1}^{m} (c_j + b_3) + \sum_{\substack{j=m+1 \\ j \neq 2m-i}}^{2m-1} (c_j + b_2). \tag{3.34}$$

Introducing $y_{m-i} = y_m = 0$, we reduce the last identity to identity (7.61) from the appendix.

3. For $1 \leqslant j \leqslant m$, the identity $\sum_{l=1}^{2m} A_{1,l} A_{l,m+j} = 0$ reduces to identity (7.61) from the appendix.
4. For $1 \leqslant i \leqslant m-1$, the identity $\sum_{l=1}^{2m} A_{1+i,l} A_{l,1} = 0$ reduces to identity (7.61) from the appendix.
5. Let $1 \leqslant i, j \leqslant m-1$, $i \neq j$. The identity $\sum_{l=1}^{2m} A_{1+i,l} A_{l,1+j} = 0$ reduces to identity (7.60) from the appendix.

6. For $1\leqslant i\leqslant m-1$, the identity $\sum_{l=1}^{2m} A_{1+i,l}A_{l,1+i}=1$ after some simplification becomes

$$\sum_{j=1}^{m} p_j^{31}p_j^{32}\frac{\prod_{\substack{k=1\\k\neq j}}^{m} q_{k,i}\prod_{\substack{k=m+1\\k\neq i}}^{2m-1} q_{j,k}}{\prod_{\substack{k=1\\k\neq j}}^{m}(c_j-c_k)\prod_{\substack{k=m+1\\k\neq i}}^{2m-1}(c_i-c_k)} = 1-(b_2+c_i)^2+\frac{\prod_{k=1}^{m} q_{k,i}}{\prod_{\substack{k=m+1\\k\neq i}}^{2m-1}(c_i-c_k)}.$$

Recall that in the normalized version $p_i^{31}=c_i+b_3-1$ and $p_i^{32}=c_i+b_3+1$. For $1\leqslant i\leqslant m$, let us set $x_i=b_3+c_i$. For $1\leqslant i\leqslant m-1$, let us set $y_i=-b_2-c_{m+i}$. The above identity splits into two homogeneous identities: one of degree 0 and the other of 2 (in x_i and y_j). The first is equivalent to identity (7.63) from the appendix. The second is equivalent to identity (7.64) from the appendix.

7. For $1\leqslant i\leqslant m-1$ and $1\leqslant j\leqslant m$, the identity $\sum_{l=1}^{2m} A_{1+i,l}A_{l,m+j}=0$ reduces to the trivial identity $\frac{q_{2m-i,m+1-j}}{q_{2m-i,m+1-j}}-1=0$.

8. For $1\leqslant i\leqslant m$, the identity $\sum_{l=1}^{2m} A_{m+j,l}A_{l,1}=0$ reduces to the identity

$$\sum_{j=1}^{m-1}\frac{\prod^{m}_{\substack{k=1\\k\neq m+1-i}} q_{k,2m-j}}{\prod_{\substack{k=m+1\\k\neq 2m-j}}^{2m-1}(c_{2m-j}-c_k)}=-b_1-b_3-c_{m+1-i}-c_{2m}.$$

The latter follows from identity (7.61) of the appendix and from the fact that the normalized **B** and **C** are traceless.

9. Let $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant m-1$. The identity $\sum_{l=1}^{2m} A_{m+j,l}A_{l,1+i}=0$ reduces to the trivial identity $\frac{q_{2m-j,m+1-i}}{q_{2m-j,m+1-i}}-1=0$.

10. Let $1\leqslant i\neq j\leqslant m$. The identity $\sum_{l=1}^{2m} A_{m+j,l}A_{l,m+i}=0$ reduces to identity (7.60) from the appendix.

11. Let $1\leqslant i\leqslant m$. To prove that $\sum_{l=1}^{2m} A_{m+i,l}A_{l,m+i}=1$, we set $x_1=b_3+c_1$, $x_2=b_3+c_2,\dots,x_m=b_3+c_m$; $y_1=b_2+c_{m+1}$, $y_2=b_2+c_{m+2},\dots,y_{m-1}=b_2+c_{2m-1}$. This reduces the identity in question to identity (7.65) of the appendix.

Now we are ready to prove that if $(s(\mathbf{A}),s(\mathbf{B}),s(\mathbf{C}))$ is a point of $S''((m,m),(1,m-1,m),(1^{2m}))$, then $s(\mathbf{A})=\{\underbrace{1,\dots,1}_{m\text{ times}},\underbrace{-1,\dots,-1}_{m\text{ times}}\}$. We know that $\mathbf{A}^2=\mathbf{Id}$. Thus, $\mathbf{A}$ is diagonalizable and the eigenvalues of $\mathbf{A}$ are 1 and -1. For $1\leqslant i\leqslant m$, let us set

$$\mathbf{a}_i^+=(\mathbf{A}+\mathbf{Id})\mathbf{e}_{m+i},$$

$$\mathbf{a}_i^-=(\mathbf{A}-\mathbf{Id})\mathbf{e}_{m+i}. \tag{3.35}$$

Then $(\mathbf{A}-\mathbf{Id})\mathbf{a}_i^+ = (\mathbf{A}-\mathbf{Id})(\mathbf{A}+\mathbf{Id})\mathbf{e}_{m+i} = 0$ and $(\mathbf{A}+\mathbf{Id})\mathbf{a}_i^- = (\mathbf{A}+\mathbf{Id})(\mathbf{A}-\mathbf{Id})\mathbf{e}_{m+i} = 0$. If we take a look at the matrix $\mathbf{A}$, we see that the condition $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C})) \in S''((m,m),(1,m-1,m),(1^{2m}))$ guarantees that the vectors $\{\mathbf{a}_i^+\}_{i=1,\dots,m}$ are linearly independent as well as the vectors $\{\mathbf{a}_i^-\}_{i=1,\dots,m}$. (If this condition is violated, then $\mathbf{a}_i^+$ and $\mathbf{a}_i^-$ are not necessarily linearly independent. For example, if $p_i^{31}=0$, then $\mathbf{a}_i^- = 0$.) $\square$

Let us prove Lemma 2.2, that is compute the coordinates v_i^j of the eigenvectors $\mathbf{v}_i$ of the matrix $\mathbf{C}$.

Proof of Lemma 2.2. The only non-trivial part of the lemma is formula (2.13). A direct computation gives

$$v_{2m}^{m+i} = \frac{1}{c_{m+1-i}-c_{2m}}\left[-C_{m+i,1} + \sum_{j=1}^{m-1} \frac{C_{m+i,j+1}C_{j+1,1}}{c_{2m-j}-c_{2m}}\right],$$

where $C_{j+1,1}$ is given by (2.10), $C_{m+i,1}$ is given by (2.11), and $C_{m+i,j+1}$ is given by (2.13), see page 9. Comparing the formula for v_{2m}^{m+i} given by (2.13) to the right-hand side of the last formula, we obtain an identity which reduces to identity (7.60) from the appendix. $\square$

The following lemma expresses the vectors $\mathbf{e}_i$ of the standard basis in terms of the eigenvectors $\mathbf{v}_j$ of the matrix $\mathbf{C}$.

Lemma 3.2. 1. $\mathbf{e}_1 = \mathbf{v}_{2m} - \sum_{i=1}^{m-1} \frac{\prod_{k=1}^{m-i} q_{k,m+i}}{\prod_{k=m+1+i}^{2m}(c_{m+i}-c_k)}\mathbf{v}_{m+i} - \sum_{i=1}^{m} \frac{p_i^{32}\prod_{k=1, k\neq i}^{2m-i} q_{ik}}{\prod_{k=1+i}^{2m}(c_i-c_k)}\mathbf{v}_i$,

2. $\mathbf{e}_{1+i} = \mathbf{v}_{2m-i} + \sum_{j=1}^{m} (-1)^{m+1-i} \frac{p_j^{32}}{c_j-c_{2m-i}} \frac{\prod_{k=m+1, k\neq 2m-i}^{2m-j} q_{j,k} \prod_{k=1+i, k\neq j}^{m} q_{k,2m-i}}{\prod_{k=1+j}^{m}(c_j-c_k)\prod_{k=m+1}^{2m-1-i}(c_k-c_{2m-i})}\mathbf{v}_j$ *for* $i = 1,2,\dots,m-1$.

3. $\mathbf{e}_{m+i} = \mathbf{v}_{m+1-i}$ *for* $i=1,2,\dots,m$.

Proof. For $1 \leqslant i \leqslant m$, let

$$e_1^i = \sum_{i=1}^{m} \frac{p_i^{32}\prod_{k=1, k\neq i}^{2m-i} q_{ik}}{\prod_{k=1+i}^{2m}(c_i-c_k)} \tag{3.36}$$

be the i coordinate of $\mathbf{e}_1$ in the basis $\mathbf{v}_i$. Formula (3.36) is the only non-trivial part of the lemma. To prove it we have to show that

$$e_1^i + v_{2m}^{2m+1-i} - \sum_{k=m+1}^{2m-1} v_k^{2m+1-i}\, v_{2m}^{2m+1-k} = 0, \quad \text{where } i=1,2,\dots,m. \tag{3.37}$$

Let us use the formulas for v_i^j given in Lemma 2.2 and the following change of variables: for a fixed $1 \leqslant i \leqslant m$, set $x_0 = c_i + b_3$, $x_1 = c_{m+1} + b_3$, $x_2 = c_{m+2} + b_3, \dots, x_m = c_{2m} + b_3$, $y_1 = -b_2 - c_1$, $y_2 = -b_2 - c_2, \dots, y_{i-1} = -b_2 - c_{i-1}$, $y_i = -b_2 - c_{i+1}, \dots, y_{m-1} = -b_2 - c_m$. This reduces (3.37) to identity (7.59) from the appendix. □

To prove Theorem 2.5, we have to compute eigenvectors of the matrix **B** (see page 8 for its description). Let us recall that **B** is diagonalizable. Let us call $\mathbf{w}_1$ the eigenvector of **B** corresponding to the eigenvalue b_1, $\mathbf{w}_2, \dots, \mathbf{w}_m$ the eigenvectors of **B** corresponding to the eigenvalue b_2, and $\mathbf{w}_{m+1}, \dots, \mathbf{w}_{2m}$ the eigenvectors of **B** corresponding to the eigenvalue b_3.

Lemma 3.3. 1. *We have* $\mathbf{w}_1 = \mathbf{e}_1$.

2. *For* $1 \leqslant i \leqslant m-1$, *we have* $\mathbf{w}_{1+i} = -\frac{B_{1,1+i}}{b_1 - b_2}\mathbf{e}_1 + \mathbf{e}_{1+i}$.

3. *For* $1 \leqslant i \leqslant m$, *we have* $\mathbf{w}_{m+i} = x^{m+i}\mathbf{e}_1 - \sum_{j=1}^{m-1} \frac{B_{j+1,m+i}}{b_2 - b_3}\mathbf{e}_{j+1} + \mathbf{e}_{m+i}$, *where*

$$x^{m+i} = \frac{(-1)^{m+1-i} p^{31}_{m+1-i}(b_1 + b_2 + c_{m+1-i} + c_{2m})}{(b_1 - b_3)(b_2 - b_3)} \frac{\prod_{k=m+i}^{2m-1} q_{m+1-i,k}}{\prod_{k=1}^{m-i} (c_k - c_{m+1-i})}. \tag{3.38}$$

Here is an example with $m = 3$.

Example 3.1.

$$\mathbf{w}_1 = (1, 0, 0, 0, 0, 0),$$

$$\mathbf{w}_2 = \left(\frac{q_{25}q_{35}}{(c_4 - c_5)(b_1 - b_2)}, 1, 0, 0, 0, 0\right),$$

$$\mathbf{w}_3 = \left(-\frac{q_{34}}{b_1 - b_2}, 0, 1, 0, 0, 0\right),$$

$$\mathbf{w}_4 = \left(-\frac{(b_1 + b_2 + c_3 + c_6)p_3^{31}q_{34}q_{35}}{(b_1 - b_3)(b_2 - b_3)(c_1 - c_3)(c_2 - c_3)}, -\frac{p_3^{31}q_{34}q_{15}}{(b_2 - b_3)(c_1 - c_3)(c_2 - c_3)},\right.$$
$$\left. -\frac{p_3^{31}q_{14}q_{24}q_{35}}{(b_2 - b_3)(c_1 - c_3)(c_2 - c_3)(c_4 - c_5)}, 1, 0, 0\right),$$

$$\mathbf{w}_5 = \left(\frac{(b_1+b_2+c_2+c_6)p_2^{31}q_{25}}{(b_1-b_3)(b_2-b_3)(c_1-c_2)}, \right.$$
$$\left. \frac{p_2^{31}q_{15}}{(b_2-b_3)(c_1-c_2)}, \frac{p_2^{31}q_{14}q_{25}}{(b_2-b_3)(c_1-c_2)(c_4-c_5)}, 0, 1, 0 \right),$$

$$\mathbf{w}_6 = \left(-\frac{(b_1+b_2+c_1+c_6)p_1^{31}}{(b_1-b_3)(b_2-b_3)}, -\frac{p_1^{31}}{b_2-b_3}, -\frac{p_1^{31}q_{24}}{(b_2-b_3)(c_4-c_5)}, 0, 0, 1 \right).$$

Proof of Lemma 3.3. All the formulas in Lemma 3.3 are immediate except for (3.38). A direct computation gives

$$(b_1-b_3)x^{m+i} + \sum_{j=1}^{m-1} (-1)^{m-j} \frac{\prod_{k=1+j}^{m} q_{k,2m-j}}{\prod_{k=m+1}^{2m-1-j} (c_k - c_{2m-j})} \frac{B_{1+j,m+i}}{b_2-b_3}$$
$$+ (-1)^{m-i} p_{m+1-i}^{31} \frac{\prod_{k=m+i}^{2m-1} q_{m+1-i,k}}{\prod_{k=1}^{m-i} (c_k - c_{m+1-i})} = 0,$$

where $B_{1+j,m+i}$ is given by (2.9). To prove (3.38), we have to show that the formula we derive for x^{m+i} from the equation above equals the formula for x^{m+i} from (3.38). This boils down to a proof of identity (7.60) from the appendix. □

Lemma 3.4. *The following is the matrix of the scalar product* (2.14) *in the standard basis* $\mathbf{e}_1, \dots, \mathbf{e}_{2m}$.

1. $\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = (b_1-b_2)(b_1-b_3)$,
2. $\langle \mathbf{e}_1, \mathbf{e}_{1+i} \rangle = (b_1-b_3) \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i}-c_k)}$ *for* $i = 1, 2, \dots, m-1$,
3. $\langle \mathbf{e}_1, \mathbf{e}_{m+i} \rangle = -p_{m+1-i}^{31} \frac{\prod_{k=m+i}^{2m} q_{m+1-i,k}}{\prod_{k=1}^{m-i} (c_{m+1-i}-c_k)}$ *for* $i = 1, 2, \dots, m$,
4.

$$\langle \mathbf{e}_{1+i}, \mathbf{e}_{1+j} \rangle$$
$$= \frac{\prod_{k=1+i}^{m} q_{2m-i,k} \prod_{k=1+j}^{m} q_{2m-j,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i}-c_k) \prod_{k=m+1}^{2m-1-j} (c_{2m-j}-c_k)}$$
$$\times \left\{ 1 - (b_2-b_3) \sum_{r=1}^{m} c_r \frac{\prod_{\substack{k=m+1 \\ k\neq 2m-i \\ k \neq 2m-j}}^{2m} q_{r,k} \prod_{\substack{k=m+1 \\ k\neq 2m-i \\ k \neq 2m-j}}^{2m} (c_r - c_k)}{\prod_{\substack{k=1 \\ k\neq r}}^{m} q_{r,k} \prod_{\substack{k=1 \\ k\neq r}}^{m} (c_r - c_k)} \right\}$$

for $i,j = 1, 2, \dots, m-1$, $i \neq j$, 5.

$$\langle \mathbf{e}_{1+i}, \mathbf{e}_{1+i} \rangle = \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} \times \left\{ \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} - (b_2 - b_3) c_{2m-i} \frac{\prod_{k=2m+1-i}^{2m} (c_{2m-i} - c_k) \prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} q_{2m-i,k}}{\prod_{k=1}^{m} (c_{2m-i} - c_k) \prod_{k=1}^{i} q_{2m-i,k}} - (b_2 - b_3) \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} \times \sum_{r=1}^{m} \frac{c_r}{(c_r - c_{2m-i}) q_{2m-i,r}} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} q_{r,k} \prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} (c_r - c_k)}{\prod_{\substack{k=1 \\ k \neq r}}^{m} q_{r,k} \prod_{\substack{k=1 \\ k \neq r}}^{m} (c_r - c_k)} \right\}.$$

6.

$$\langle \mathbf{e}_{1+i}, \mathbf{e}_{m+j} \rangle = \frac{p_{m+1-j}^{31}}{q_{m+1-j,2m-i}} \frac{\prod_{k=1+i}^{m} q_{k,2m-i}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} \times \frac{\prod_{k=m+j}^{2m} q_{m+1-j,k}}{\prod_{k=1}^{m-j} (c_{m+1-j} - c_k)} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} (c_{m+1-j} - c_k)}{\prod_{\substack{k=1 \\ k \neq m+1-j}}^{m} q_{m+1-j,k}}$$

for $i = 1, 2, \dots, m-1$ *and* $j = 1, 2, \dots, m$

7.

$$\langle \mathbf{e}_{m+j}, \mathbf{e}_{m+i} \rangle = \delta_{ij} \frac{p_{m+1-i}^{31}}{p_{m+1-i}^{32}} \frac{\prod_{k=m+2-i}^{2m} (c_{m+1-i} - c_k)}{\prod_{k=1}^{m-i} (c_{m+1-i} - c_k)} \frac{\prod_{k=m+i}^{2m} q_{m+1-i,k}}{\prod_{\substack{k=1 \\ k \neq m+1-i}}^{m-1+i} q_{m+1-i,k}}$$

for $i, j = 1, 2, \dots, m$.

Proof.

1. We obtain by direct computation

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \sum_{i=1}^{2m} p_i^{31} p_i^{32} \frac{\prod_{\substack{k=1 \\ k \neq i}}^{2m} q_{ik}}{\prod_{\substack{k=1 \\ k \neq i}}^{2m} (c_i - c_k)}.$$

For $1 \leqslant i \leqslant 2m$ let us set $x_i = c_i + (b_2 + b_3)/2$. Identities (7.66), (7.67), and (7.68) from the appendix finish the proof.

2. A direct computation gives

$$\langle \mathbf{e}_1, \mathbf{e}_{1+i} \rangle = -p^{31}_{2m-i} p^{32}_{2m-i} \frac{\prod_{\substack{k=1+i \\ k \neq 2m-i}}^{2m} q_{2m-i,k}}{\prod_{k=1}^{2m-1-i} (c_{2m-i} - c_k)}$$

$$- \sum_{r=1}^{m} \frac{p^{31}_r p^{32}_r}{c_r - c_{2m-i}} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} q_{rk} \prod_{k=1+i}^{m} q_{k,2m-i}}{\prod_{\substack{k=1 \\ k \neq r}}^{m} (c_r - c_k) \prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)}.$$

After cancelling out common multiples, we have to prove that

$$\sum_{r=1}^{m} \frac{p^{31}_r p^{32}_r}{c_r - c_{2m-i}} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} q_{rk}}{\prod_{\substack{k=1 \\ k \neq r}}^{m} (c_r - c_k)} = -p^{31}_{2m-i} p^{32}_{2m-i} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} q_{2m-i,k}}{\prod_{k=1}^{m} (c_{2m-i} - c_k)}$$

$$- (b_1 - b_3).$$

Let us set $x_r = c_r + (b_2 + b_3)/2$, $y_r = c_{m+r} + (b_2 + b_3)/2$. Now identities (7.63), (7.69), and (7.70) from the appendix finish the proof.

3. This formula is proved by direct computation.
4. A direct computation gives

$$\langle \mathbf{e}_{1+i}, \mathbf{e}_{1+j} \rangle = \frac{\prod_{k=1+i}^{m} q_{2m-i,k} \prod_{k=1+j}^{m} q_{2m-j,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k) \prod_{k=m+1}^{2m-1-j} (c_{2m-j} - c_k)}$$

$$\times \sum_{r=1}^{m} p^{31}_r p^{32}_r \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i \\ k \neq 2m-j}}^{2m} q_{r,k} \prod_{\substack{k=m+1 \\ k \neq 2m-i \\ k \neq 2m-j}}^{2m} (c_r - c_k)}{\prod_{\substack{k=1 \\ k \neq r}}^{m} q_{r,k} \prod_{\substack{k=1 \\ k \neq r}}^{m} (c_r - c_k)}.$$

Let us set $x_1 = c_1 + (b_2 + b_3)/2, x_2 = c_2 + (b_2 + b_3)/2, \dots, x_m = c_m + (b_2 + b_3)/2$; $y_1 = c_{m+1} + (b_2 + b_3)/2, \quad y_2 = c_{m+2} + (b_2 + b_3)/2, \dots, y_m = c_{m+m} + (b_2 + b_3)/2$. Identities (7.59) and (7.60) from the appendix finish the proof.

5. A direct computation gives

$$\langle \mathbf{e}_{1+i}, \mathbf{e}_{1+i} \rangle = \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i}-c_k)} \left\{ \frac{\prod_{k=1+i}^{m} q_{2m-i,k}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i}-c_k)} \right.$$

$$\times \sum_{r=1}^{m} \frac{p_r^{31} p_r^{32}}{(c_r - c_{2m-i}) q_{2m-i,r}} \frac{\prod_{\substack{k=m+1\\k\neq 2m-i}}^{2m} q_{r,k} \prod_{\substack{k=m+1\\k\neq 2m-i}}^{2m} (c_r - c_k)}{\prod_{\substack{k=1\\k\neq r}}^{m} q_{r,k} \prod_{\substack{k=1\\k\neq r}}^{m} (c_r - c_k)}$$

$$\left. + \, p_{2m-i}^{31} p_{2m-i}^{32} \frac{\prod_{k=2m+1-i}^{2m} (c_{2m-i}-c_k)}{\prod_{k=1}^{m} (c_{2m-i}-c_k)} \frac{\prod_{\substack{k=m+1\\k\neq 2m-i}}^{2m} q_{2m-i,k}}{\prod_{k=1}^{i} q_{2m-i,k}} \right\}.$$

Let us set $x_r = c_r + (b_2 + b_3)/2$, $y_r = c_{m+r} + (b_2 + b_3)/2$ for $r = 1, 2, \ldots, m$. Identities (7.60) and (7.63) from the appendix finish the proof.

6. Proved by direct computation.
7. Proved by direct computation. □

Now it is time to prove Theorem 2.5, that is prove that the matrices **A**, **B**, and **C** are self-adjoint with respect to the scalar product (2.14). **C** is self-adjoint with respect to the scalar product by construction. The space V splits into the direct sum $V = V_{b_1} \oplus V_{b_2} \oplus V_{b_3}$ of the spectral subspaces of **B**. If the subspaces V_{b_1}, V_{b_2}, and V_{b_3} are mutually orthogonal with respect to the scalar product (2.14), then **B** is self-adjoint with respect to it as well. Then **A** is also self-adjoint, as $\mathbf{A} = \mathbf{B} + \mathbf{C}$. Proof of the following lemma finishes the proof of Theorem 2.5.

Lemma 3.5. *The subspaces V_{b_1}, V_{b_2}, and V_{b_3} are mutually orthogonal with respect to the scalar product* (2.14).

Proof. We use the formulas of Lemma 3.3 to express the eigenvectors $\mathbf{w}_i$ of the matrix **B** in terms of the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{2m}\}$. Then we use the formulas of Lemma 3.4 to expand $\langle \mathbf{e}_i, \mathbf{e}_j \rangle$.

1.

$$\begin{aligned}
\langle \mathbf{w}_1, \mathbf{w}_{1+i} \rangle &= \left\langle \mathbf{e}_1, \mathbf{e}_{1+i} - \frac{1}{b_1 - b_2} \frac{\prod_{k=1+i}^{m} q_{k,2m-i}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i}-c_k)} \mathbf{e}_1 \right\rangle \\
&= \langle \mathbf{e}_1, \mathbf{e}_{1+i} \rangle - \frac{1}{b_1 - b_2} \frac{\prod_{k=1+i}^{m} q_{k,2m-i}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i}-c_k)} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle \\
&= (b_1 - b_3) \frac{\prod_{k=1+i}^{m} q_{k,2m-i}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i}-c_k)} \\
&\quad - \frac{(b_1 - b_2)(b_1 - b_3)}{b_1 - b_2} \frac{\prod_{k=1+i}^{m} q_{k,2m-i}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i}-c_k)} = 0.
\end{aligned}$$

2. The identity $\langle \mathbf{w}_1, \mathbf{w}_{m+i} \rangle = 0$ $(i = 1, 2, \dots, m)$ reduces to identity (7.61) from the appendix.
3. Lemma 3.3 gives us

$$\mathbf{w}_{1+i} = \mathbf{e}_{1+i} - \frac{1}{b_1 - b_2} \frac{\prod_{k=1+i}^{m} q_{k,2m-i}}{\prod_{k=m+1}^{2m-1-i} (c_{2m-i} - c_k)} \mathbf{e}_1.$$

We know that $\mathbf{e}_1 = \mathbf{w}_1$ and that $\langle \mathbf{w}_1, \mathbf{w}_{m+i} \rangle = 0$. Thus, $\langle \mathbf{w}_{1+i}, \mathbf{w}_{m+j} \rangle = \langle \mathbf{e}_{1+i}, \mathbf{w}_{m+j} \rangle$. The identity $\langle \mathbf{e}_{1+i}, \mathbf{w}_{m+j} \rangle = 0$ after expansion and some simplification becomes the following identity:

$$\begin{aligned}
&\sum_{r=1}^{m-1} \frac{\prod_{\substack{k=1 \\ k \neq m+1-j}}^{m} q_{2m-r,k}}{\prod_{\substack{k=m+1 \\ k \neq 2m-r}}^{2m-1} (c_{2m-r} - c_k)} \sum_{s=1}^{m} \frac{c_s}{q_{2m-i,s}(c_s - c_{2m-i})} \\
&\times \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-r}}^{2m} q_{s,k} \prod_{\substack{k=m+1 \\ k \neq 2m-r}}^{2m} (c_s - c_k)}{\prod_{\substack{k=1 \\ k \neq s}}^{m} q_{s,k} \prod_{\substack{k=1 \\ k \neq s}}^{m} (c_s - c_k)} \\
&= 1 - \frac{c_{2m-i}(c_{2m-i} - c_{2m})}{q_{2m-i,m+1-j}} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} q_{2m-i,k}}{\prod_{k=1}^{m} (c_{2m-i} - c_k)} \\
&\quad - \frac{q_{m+1-j,2m}}{q_{m+1-j,2m-i}} \frac{\prod_{\substack{k=m+1 \\ k \neq 2m-i}}^{2m} (c_{m+1-j} - c_k)}{\prod_{\substack{k=1 \\ k \neq m+1-j}}^{m} q_{m+1-j,k}}.
\end{aligned}$$

For $i = 1, 2, \dots, m$, let us set $x_i = c_i + (b_2 + b_3)/2$ and $y_i = c_{m+i} + (b_2 + b_3)/2$. Let us write $c_s = x_s - y_{2m-i} + y_{2m-i} - (b_2 + b_3)/2$ and $c_{2m-i} = y_{2m-i} - (b_2 + b_3)/2$. Now identities (7.59), (7.63), and (7.71) of the appendix finish the proof. □

Let us prove Theorem 2.6, that is determine the inequalities on the real spectra of **A**, **B**, and **C** which make form (2.14) sign-definite. It is an assumption of Theorem 2.6 that $c_1 > c_2 > \dots > c_{2m}$. The assumption $a_1 > a_2$ of the theorem is satisfied automatically, because in our normalized version $a_1 = 1$ and $a_2 = -1$.

Proof of Theorem 2.6. It is immediately clear from Theorem 2.5 that

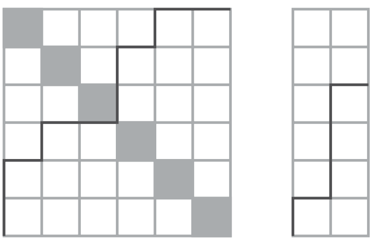
$$\operatorname{sign}(\langle \mathbf{v}_i, \mathbf{v}_i \rangle) = (-1)^{i-1} \operatorname{sign}(p_i^{31} p_i^{32}) \operatorname{sign}\left(\prod_{\substack{j=1 \\ j \neq i}}^{2m} q_{i,j} \right).$$

Let Q be a $2m \times 2m$ array such that $Q_{i,j} = q_{i,j}$ for $i \neq j$ and $Q_{i,i}$ are not defined. Then $Q_{i,j} > Q_{i,j+1}$ and $Q_{i,j} > Q_{i+1,j}$ for all i and j such that neither of the array elements involved belongs to the main diagonal. Let P be a $2m \times 2$ matrix such that $P_{i,1} = p_i^{32}$

and $P_{j,2} = p_j^{31}$. The fact $a_1 > a_2$ implies $p_i^{32} > p_i^{31}$. Then $\text{sign}(\langle \mathbf{v}_i, \mathbf{v}_i \rangle) = (-1)^{i-1+\#\{j:\, Q_{i,j}<0\}+\#\{j:\, P_{i,j}<0\}}$. In order to keep $\text{sign}(\langle \mathbf{v}_i, \mathbf{v}_i \rangle)$ constant, the number of negative elements in the i row of the arrays Q and P must differ from the number of negative elements in the $i+1$ row by an odd number. This and the fact that $Q_{i,j} = Q_{j,i}$ leaves room for the following six configurations. The first is given by the inequalities:

$$\begin{array}{ll} p_{m-1}^{31} > 0 > p_m^{31} & q_{1,2m-2} > 0 > q_{1,2m-1} \\ & q_{2,2m-3} > 0 > q_{2,2m-2} \\ p_{2m-1}^{32} > 0 > p_{2m}^{32} & q_{3,2m-4} > 0 > q_{3,2m-3} \\ & \vdots \\ & q_{m-1,m} > 0 > q_{m-1,m+1}. \end{array}$$

We have $q_{i,2m-i} < 0$ for $i = 1, 2, \ldots, m-1$. Let us sum up these inequalities with $p_{2m}^{32} < 0$ and $p_m^{31} < 0$. Recalling that $\mathbf{C}$ is traceless, we obtain $(m-1)b_2 + (m+1)b_3 < 0$. Recalling that $\mathbf{B}$ is traceless, we obtain $b_1 > b_3$. We have $p_{m-1}^{31} > 0$ and $p_{2m-1}^{32} > 0$. Thus, $-p_{m-1}^{31} - p_{2m-1}^{32} < 0$. We also have $q_{m-1,2m-1} < q_{m-1,m+1} < 0$ for $m > 2$ and $q_{m-1,2m-1} = q_{m-1,m+1} < 0$ for $m = 2$ because $c_i < c_j$ for $i > j$. Thus, we have $-p_{m-1}^{31} - p_{2m-1}^{32} + q_{m-1,2m-1} < 0$. This gives us $b_3 > b_2$. So, we have $b_1 > b_3 > b_2$. Here is a picture illustrating the case of $m = 3$. The line separates positive elements from negative.

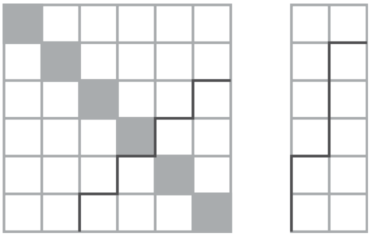


The second configuration is given by the following inequalities.

$$\begin{array}{ll} p_1^{31} > 0 > p_2^{31} & q_{2,2m} > 0 > q_{3,2m} \\ & q_{3,2m-1} > 0 > q_{4,2m-1} \\ p_{m+1}^{32} > 0 > p_{m+2}^{32} & q_{4,2m-2} > 0 > q_{5,2m-2} \\ & \vdots \\ & q_{m,m+2} > 0 > q_{m+1,m+2}. \end{array}$$

We have $p_2^{31} < 0$ and $p_{m+2}^{32} < 0$. Thus, we have $-p_2^{31} - p_{m+2}^{32} > 0$. We also have $q_{2,m+2} > q_{m,m+2} > 0$ for $m > 2$ and $q_{2,m+2} = q_{m,m+2} > 0$ for $m = 2$. Thus, $q_{2,m+2} - p_2^{31} - p_{m+2}^{32} > 0$. This implies $b_2 > b_3$. We have $q_{i,2m+2-i} > 0$ for $i = 2, 3, \ldots, m$. Summing up these inequalities with $p_1^{31} > 0$ and $p_{m+1}^{32} > 0$, we obtain $(m-1)b_2 + (m+1)b_3 > 0$.

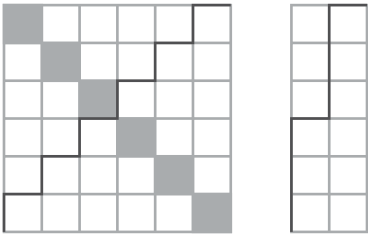
Thus, $b_3 > b_1$. So, we have $b_2 > b_3 > b_1$. Here is the picture illustrating the case of $m = 3$.



The third configuration is given by the following inequalities:

$$\begin{array}{ll} 0 > p_1^{31} & q_{1,2m-1} > 0 > q_{1,2m} \\ & q_{2,2m-2} > 0 > q_{2,2m-1} \\ p_m^{32} > 0 > p_{m+1}^{32} & \vdots \\ & q_{m-1,m+1} > 0 > q_{m-1,m+2} \\ & \qquad\qquad 0 > q_{m,m+1}. \end{array}$$

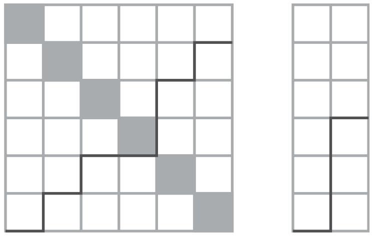
The inequalities $p_{m+1}^{32} < 0$ and $p_1^{31} < 0$ imply the inequality $2b_3 + c_1 + c_{m+1} < 0$. The inequality $q_{1,2m-1} > 0$ implies the inequality $q_{1,m+1} > 0$ because $c_{m+1} > c_{2m-1}$ for $m > 2$ and $c_{m+1} = c_{2m-1}$ for $m = 2$. Now, the inequalities $b_2 + b_3 + c_1 + c_{m+1} > 0$ and $-2b_3 - c_1 - c_{m+1} > 0$ imply the inequality $b_2 - b_3 > 0$. So, $b_2 > b_3$. Let us sum up the inequalities $q_{i,2m+1-i} < 0$ for $i = 1, 2, \ldots, m$. The sum of all the c_i is equal to zero. Thus we obtain $mb_2 + mb_3 < 0$ which is equivalent to $b_2 - b_1 < 0$. This gives us $b_1 > b_2 > b_3$. Here is the picture illustrating the case of $m = 3$.



The fourth configuration is given by the following inequalities.

$$\begin{array}{ll} p_m^{31} > 0 > p_{m+1}^{31} & q_{1,2m} > 0 > q_{2,2m} \\ & q_{2,2m-1} > 0 > q_{3,2m-1} \\ p_{2m}^{32} > 0 & \vdots \\ & q_{m-1,m+2} > 0 > q_{m,m+2} \\ & q_{m,m+1} > 0. \end{array}$$

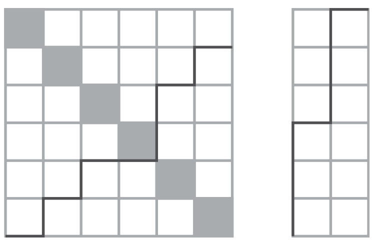
We have $q_{m,2m} < q_{m,m+2} < 0$ for $m > 2$ and $q_{m,2m} = q_{m,m+2} < 0$ for $m = 2$. We also have $-p^{32}_{2m} < 0$ and $-p^{31}_{m} < 0$. Summing up these inequalities, we obtain $b_3 > b_2$. We also have $q_{i,2m+1-i} > 0$ for $i = 1, 2, \ldots, m$. Summing up these inequalities, we obtain $b_2 > b_1$. This gives us $b_3 > b_2 > b_1$. Here is the picture illustrating the case of $m = 3$.



The fifth configuration is given by the following inequalities:

$$
\begin{array}{ll}
0 > p^{31}_1 & q_{1,2m} > 0 > q_{2,2m} \\
 & q_{2,2m-1} > 0 > q_{3,2m-1} \\
p^{32}_m > 0 > p^{32}_{m+1} & \vdots \\
 & q_{m-1,m+2} > 0 > q_{m,m+2} \\
 & q_{m,m+1} > 0.
\end{array}
$$

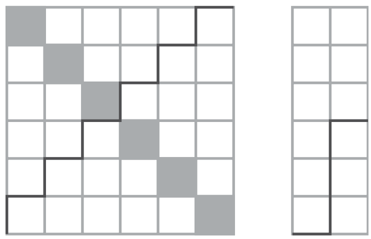
We have $q_{i,2m+2-i} < 0$ for $i = 2, 3, \ldots, m$. Summing up these inequalities with $p^{31}_1 < 0$ and $p^{32}_{m+1} < 0$, we obtain $(m-1)b_2 + (m+1)b_3 < 0$. The last is equivalent to $b_1 > b_3$. We also have $q_{i,2m+1-i} > 0$ for $i = 1, 2, \ldots, m$. Summing up these inequalities, we obtain $mb_2 + mb_3 > 0$ which is equivalent to $b_2 > b_1$. This gives us $b_2 > b_1 > b_3$. Here is the picture illustrating the case of $m = 3$.



The last configuration possible is given by the following inequalities.

$$
\begin{array}{ll}
p^{31}_m > 0 > p^{31}_{m+1} & q_{1,2m-1} > 0 > q_{1,2m} \\
 & q_{2,2m-2} > 0 > q_{2,2m-1} \\
p^{32}_{2m} > 0 & \vdots \\
 & q_{m-1,m+1} > 0 > q_{m-1,m+2} \\
 & \qquad\qquad 0 > q_{m,m+1}.
\end{array}
$$

We have $q_{i,2m+1-i}<0$ for $i=1,2,\dots,m$. Summing up these inequalities, we obtain $mb_2+mb_3<0$ which is equivalent to $b_1>b_2$. We also have $q_{i,2m-i}>0$ for $i-1,2,\dots,m-1$. Summing up these inequalities with $p^{32}_{2m}>0$ and $p^{31}_m>0$, we obtain $(m-1)b_2+(m+1)b_3>0$ which is equivalent to $b_3>b_1$. This gives us $b_3>b_1>b_2$. Here is the picture illustrating the case of $m=3$.



So, in these six cases the form $\langle *,*\rangle$ is sign-definite. Lemma 3.4 gives $\langle \mathbf{e}_1,\mathbf{e}_1\rangle = (b_1-b_2)(b_1-b_3)$. Thus, $\text{sign}(\langle *,*\rangle)=\text{sign}((b_1-b_2)(b_1-b_3))$. $\square$

3.3. *Odd family*

For the hypergeometric, odd, and even family, let us call the objects $\{V;\mathbf{A}=\mathbf{B}+\mathbf{C},\mathbf{B},\mathbf{C};\langle *,*\rangle\}$ where $(\mathbf{A},\mathbf{B},\mathbf{C})$ is a rigid irreducible triple of matrices of the corresponding spectral types and $\langle *,*\rangle$ is the non-degenerate scalar product such that **A**, **B**, and **C** are self-adjoint with respect to it, *m-hypergeometric module*, *m-even module*, and *m-odd module*. Let us denote these objects as HGM_m, EM_m, and OM_m. The reason for calling these objects modules comes from the theory of quiver representations and will not be explained in this paper.

It is possible to prove Theorems 2.7, 2.8, Lemma 2.3, etc. in the same fashion as for the even family. But we choose a different approach. We show that by means of violating the "generic eigenvalues" condition it is possible to construct OM_{m-1} as a submodule of EM_m. Then all the formulas follow from the corresponding formulas for the even family.

Let V be the same $2m$-dimensional linear space as in the previous subsection and let $\mathbf{e}_1,\dots,\mathbf{e}_{2m}$ be the standard basis of V. Let **A**, **B**, and **C** be the matrices from the previous subsection, too. Fix an integer i such that $1\leqslant i\leqslant m$. Let $V^s_{\hat{i}}$ be the subspace of V spanned by the vectors $\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_{2m-i},\widehat{\mathbf{e}_{2m+1-i}},\mathbf{e}_{2m+2-i},\dots,\mathbf{e}_{2m}$. It follows from the formulas of Lemma 2.2 that $V^s_{\hat{i}}$ is spanned by $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_{i-1},\widehat{\mathbf{v}}_i,\mathbf{v}_{i+1},\dots,\mathbf{v}_{2m}$ (hence the notation). Then the following lemma follows at once from the formulas for **A**, **B**, and **C** of Section 2.2.

Lemma 3.6. *If* $p^{32}_i=0$, *then* $V^s_{\hat{i}}$ *is invariant with respect to* **A**, **B**, *and* **C**.

Thus, it makes sense to consider the restrictions of **A**, **B**, and **C** to $V^s_{\hat{i}}$ and call them $\mathbf{A}^s_{\hat{i}}$, $\mathbf{B}^s_{\hat{i}}$, and $\mathbf{C}^s_{\hat{i}}$. We will also call $\langle *, * \rangle^s_{\hat{i}}$ the form $\langle *, * \rangle$ restricted to $V^s_{\hat{i}}$. Note that $p^{32}_i = 0$ forces $q_{ij} = p^{21}_j$ and $c_i - c_j = -p^{32}_j$.

Theorem 3.1. *If* $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ *is a generic point of the intersection of* $S((m,m),(1,m-1,m),(1^{2m}))$ *with the hyperplane given by the equation* $p^{32}_i = 0$ *for a fixed* $1 \leqslant i \leqslant m$, *then* $\{V^s_{\hat{i}};\ \mathbf{A}^s_{\hat{i}}, \mathbf{B}^s_{\hat{i}}, \mathbf{C}^s_{\hat{i}}; \langle *, * \rangle^s_{\hat{i}}\}$ *is* OM_{m-1}.

Here is an example of the matrices $\mathbf{B}^s_{\hat{2}}$ and $\mathbf{C}^s_{\hat{2}}$ obtained from the matrices **B** and **C** of Example 2.4 by setting $p^{32}_2 = 0$ and restricting them to $V^s_{\hat{2}}$.

Example 3.2.

$$\mathbf{B}^s_{\hat{2}} = \left[\begin{array}{c|cc|cc} b_1 & -\frac{p_5^{21} q_{35}}{c_4-c_5} & q_{34} & -\frac{p_3^{31} q_{34} q_{35}}{p_3^{32}(c_1-c_3)} & p_1^{31} \\ \hline 0 & b_2 & 0 & -\frac{p_3^{31} q_{15} q_{34}}{p_3^{32}(c_1-c_3)} & p_1^{31} \\ 0 & 0 & b_2 & -\frac{p_3^{31} p_4^{21} q_{14} q_{35}}{p_3^{32}(c_1-c_3)(c_4-c_5)} & \frac{p_1^{31} p_4^{21}}{c_4-c_5} \\ \hline 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & b_3 \end{array}\right]$$

$$\mathbf{C}^s_{\hat{2}} = \left[\begin{array}{c|cc|cc} c_6 & 0 & 0 & 0 & 0 \\ \hline -q_{15} & c_5 & 0 & 0 & 0 \\ -\frac{p_4^{21} q_{14}}{c_4-c_5} & 0 & c_4 & 0 & 0 \\ \hline -p_3^{32} & -\frac{p_3^{32} p_5^{21}}{c_4-c_5} & p_{33} & c_3 & 0 \\ -\frac{q_{14} q_{15}}{c_1-c_3} & -\frac{p_5^{21} q_{14} q_{35}}{(c_1-c_3)(c_4-c_5)} & \frac{q_{15} q_{34}}{c_1-c_3} & 0 & c_1 \end{array}\right]$$

We first prove Theorem 3.1 and then we derive all the proofs for the odd family from what we already know about the even family. Let us prove Theorem 3.1.

Proof of Theorem 3.1. It is clear that $\mathbf{B}^s_{\hat{i}}$ is diagonalizable and that $s(\mathbf{B}^s_{\hat{i}}) = \{b_1, \underbrace{b_2, \dots, b_2}_{m-1 \text{ times}}, \underbrace{b_3, \dots, b_3}_{m-1 \text{ times}}\}$.

It is clear that $\mathbf{C}^s_{\hat{i}}$ is diagonalizable and that $s(\mathbf{C}^s_{\hat{i}}) = \{c_1, c_2, \dots, c_{i-1}, \widehat{c_i}, c_{i+1}, \dots, c_{2m}\}$. In view of Lemma 3.6 and Theorem 2.4, it is clear that $A^s_{\hat{i}}$ is diagonalizable as well. In the notations of Section 3.2, $\mathbf{A}$ has eigenvectors $\mathbf{a}^+_j$ corresponding to the eigenvalue a_1 (normalized to 1). Vectors $\mathbf{a}^-_j$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue a_2 (normalized to -1). Once we set $p^{32}_i = 0$, all the eigenvectors of $\mathbf{A}$ belong to $V^s_{\hat{i}}$ except for $\mathbf{a}^-_{m+1-i}$ and the proof follows immediately. □

To finish the rest of the proofs for the odd family, we just have to say that all the formulas for OM_m in this paper were obtained from the formulas for EM_{m+1} by setting $p^{32}_{m+1} = 0$ and renumbering the remaining $c_1, c_2, \dots, c_m, c_{m+2}, \dots, c_{2m+2}$ as $c_1, c_2, \dots, c_{2m+1}$.

Remark 3.2. In exactly the same fashion, we can construct OM_{m-1} as a factor module of EM_m by setting $p^{31}_i = 0$ for $1 \leqslant i \leqslant m$; we can construct EM_m as a factor module of OM_m by setting either $p^{31}_i = 0$ for $1 \leqslant i \leqslant m$ or $p^{21}_i = 0$ for $m+1 \leqslant i \leqslant 2m$. Also similarly, one can show that setting $b_i + c_{m+1-i} - a_2 = 0$ for $1 \leqslant i \leqslant m$ creates HGM_{m-1} as a submodule of HGM_m.

3.4. *Extra case of Simpson*

Consider the following vectors.

$$\mathbf{w}_1 = (1, 0, 0, 0, 0, 0),$$

$$\mathbf{w}_2 = (0, 1, 0, 0, 0, 0),$$

$$\mathbf{w}_3 = \left(\frac{p_{16}q_{245}}{(b_1 - b_2)(c_3 - c_4)}, -\frac{p_{15}q_{235}q_{246}}{(b_1 - b_2)(c_3 - c_4)(c_5 - c_6)}, 1, 0, 0, 0\right),$$

$$\mathbf{w}_4 = \left(-\frac{p_{16}}{b_1 - b_2}, \frac{p_{15}q_{236}}{(b_1 - b_2)(c_5 - c_6)}, 0, 1, 0, 0\right),$$

$$\mathbf{w}_5 = \left(-\frac{p_{16}q_{245}(q_{126}-p_{31})}{(b_1-b_3)(b_2-b_3)(c_1-c_2)}, \frac{p_{15}q_{236}q_{246}(q_{125}-p_{31})}{(b_1-b_3)(b_2-b_3)(c_1-c_2)(c_5-c_6)}, \right.$$
$$\left. \frac{p_{24}q_{236}}{(b_2-b_3)(c_1-c_2)}, \frac{p_{23}q_{245}q_{246}}{(b_2-b_3)(c_1-c_2)(c_3-c_4)}, 1, 0 \right),$$

$$\mathbf{w}_6 = \left(-\frac{p_{16}\,(q_{126}-p_{32})}{(b_1-b_3)(b_2-b_3)}, \frac{p_{15}q_{235}(q_{125}-p_{32})}{(b_1-b_3)(b_2-b_3)(c_5-c_6)}, -\frac{p_{24}}{b_2-b_3}, \right.$$
$$\left. -\frac{p_{23}q_{235}}{(b_2-b_3)(c_3-c_4)}, 0, 1 \right). \tag{3.39}$$

Theorem 2.10 and Lemma 2.4 are proved by direct computation as well as the following two lemmas.

Lemma 3.7. *Let* $\mathbf{B}$ *be as in* (2.24). *Then* $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *are eigenvectors of* $\mathbf{B}$ *with the eigenvalue* b_1, $\mathbf{w}_3$ *and* $\mathbf{w}_4$ *are eigenvectors of* $\mathbf{B}$ *with the eigenvalue* b_2, *and* $\mathbf{w}_5$ *and* $\mathbf{w}_6$ *are eigenvectors of* $\mathbf{B}$ *with the eigenvalue* b_3.

Lemma 3.8. *Let* $\langle *, * \rangle$ *be defined by* (2.25). *Let* V_{b_1} *be the subspace of* V *spanned by* $\mathbf{w}_1$ *and* $\mathbf{w}_2$. *Let* V_{b_2} *be the subspace of* V *spanned by* $\mathbf{w}_3$ *and* $\mathbf{w}_4$. *Let* V_{b_3} *be the subspace of* V *spanned by* $\mathbf{w}_5$ *and* $\mathbf{w}_5$. *Then* V_{b_1}, V_{b_2}, *and* V_{b_3} *are mutually orthogonal with respect to* $\langle *, * \rangle$.

Theorem 2.11 follows from Lemmas 2.4 and 3.8. Finally, Theorem 2.12 can be proved similarly to Theorems 2.3 and 2.6.

4. Indecomposable triple flag varieties with finitely many orbits

Let $i \in \{1,2,3\}$. For a triple of flags $\emptyset = V_0^i \subset V_1^i \subset V_2^i \subset \cdots \subset V_{k_i-1}^i \subset V_{k_i}^i = V$, we call the *dimension vector in the jump coordinates* the vector $((\dim(V^1/V_0^1), \dim(V_2^1/V_1^1), \ldots, \dim(V_{k_1}^1/V_{k_1-1}^1)), (\dim(V_1^2/V_0^2), \dim(V_2^2/V_1^2), \ldots, \dim(V_{k_2}^2/V_{k_2-1}^2)), (\dim(V_1^3/V_0^3), \dim(V_2^3/V_1^3), \ldots, \dim(V_{k_3}^3/V_{k_3-1}^3)))$. We say that this triple of flags is in a *standard form*, if V is given a basis $\mathbf{z}_1, \ldots, \mathbf{z}_n$ with the following property: for the flag $\emptyset = V_0^2 \subset V_1^2 \subset V_2^2 \subset \cdots \subset V_{k_2-1}^2 \subset V_{k_2}^2 = V$, the subspace V_i^2 of dimension d_i^2 is spanned by the first d_i^2 basis vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots$; for the flag $\emptyset = V_0^3 \subset V_1^3 \subset V_2^3 \subset \cdots \subset V_{k_3-1}^3 \subset V_{k_3}^3 = V$, the subspace V_j^3 of dimension d_j^3 is spanned by the last d_j^3 basis vectors $\mathbf{z}_n, \mathbf{z}_{n-1}, \ldots$.

Magyar et al. [26] classify all indecomposable triple partial flag varieties with finitely many orbits of the diagonal action of the general linear group. They work over an algebraically closed field. $\mathbb{C}$ is enough for our purposes. Among other results, they give the dimension vectors in the jump coordinates as well as explicit representatives of the open orbit in the standard form. For every element of their list, the first flag turns to be just a subspace $V_1^1 \subset V$. It also turns out that this subspace is spanned by vectors $\mathbf{a}_i$ such that all their coordinates in the standard basis $\mathbf{z}_1, \ldots, \mathbf{z}_n$ are equal to either 0 or 1. Their list is given on page 37.

Remark 4.1. Our definition of a standard form for a triple of flags is weaker than that of Magyar, Weyman, and Zelevinsky (includes more triple flags).

Recall that we proved Theorem 1.3 only for the hypergeometric family so far. Now we use the results of Magyar, Weyman and Zelevinsky to prove the counterparts of this result for all other families of Simpson. Let us begin with the even family. Recall that we work with the normalized matrices **A**, **B**, and **C**. This means that they are traceless and the eigenvalues of **A** are 1 and -1. Let **Z** be the following matrix:

$$\mathbf{Z} = \begin{array}{c|c|c|c|} & 1 & m-1 & m \\ \hline 1 & 1 & 0 & 0 \\ \hline m-1 & 0 & Z_{1+i,1+j} & 0 \\ \hline m & 0 & 0 & Z_{m+i,m+j} \\ \hline \end{array},$$

where

$$\begin{aligned} Z_{1+i,1+j} &= \begin{cases} 0 & \text{if } i<j, \\ 1 & \text{if } i=j, \\ \dfrac{\prod_{k=1+j}^{i} q_{k,2m-i}}{\prod_{k=2m+1-i}^{2m-j} (c_{2m-i}-c_k)} & \text{if } i>j, \end{cases} \\ Z_{m+i,m+j} &= \begin{cases} 0 & \text{if } i<j, \\ 1 & \text{if } i=j, \\ \dfrac{\prod_{k=m+j}^{m-1+i} q_{m+1-i,k}}{\prod_{k=m+2-i}^{m+1-j} (c_{m+1-i}-c_k)} & \text{if } i>j. \end{cases} \end{aligned} \tag{4.40}$$

Note that **Z** is lower-triangular with all the diagonal elements equal to 1.

For $1 \leqslant i \leqslant 2m$, let $\mathbf{z}_i = \mathbf{Z}\mathbf{e}_i$. The matrix $\mathbf{Z}$ is non-degenerate, so $\mathbf{z}_i$ is a basis of V. Consider the following flags: $V_1^2 \subset V_2^2 \subset V$ and $V_1^3 \subset V_2^3 \subset \cdots \subset V_{2m-1}^3 \subset V$ where V_1^2 is spanned by $\mathbf{z}_1$, V_2^2 is spanned by $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m$, and V_i^3 is spanned by $\mathbf{z}_{2m}, \mathbf{z}_{2m-1}, \ldots, \mathbf{z}_{2m+1-i}$. They are the second and the third flags of the even family with the dimension vector $((m,m),(1,m-1,m),(1^{2m}))$ in (4.41).

hypergeometric family	
$(m-1,1),(1^m),(1^m)$	$\mathbf{a}_k = \mathbf{z}_1 + \mathbf{z}_{k+1}$ $(1 \leqslant k \leqslant m-1)$
$(1,m-1),(1^m),(1^m)$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_2 + \cdots + \mathbf{z}_m$
even family	
$(m,m),(1,m-1,m),(1^{2m})$	$\mathbf{a}_k = \mathbf{z}_1 + \mathbf{z}_{k+1} + \mathbf{z}_{2m+1-k}$ $(1 \leqslant k \leqslant m-1)$, $\mathbf{a}_m = \mathbf{z}_1 + \mathbf{z}_{m+1}$
$(m,m),(1,m,m-1),(1^{2m})$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_2$, $\mathbf{a}_k = \mathbf{z}_1 + \mathbf{z}_{k+1} + \mathbf{z}_{2m+2-k}$ $(2 \leqslant k \leqslant m)$
$(m,m),(m-1,m,1),(1^{2m})$	$\mathbf{a}_k = \mathbf{z}_k + \mathbf{z}_{2m-k} + \mathbf{z}_{2m}$ $(1 \leqslant k \leqslant m-1)$, $\mathbf{a}_m = \mathbf{z}_m + \mathbf{z}_{2m}$
$(m,m),(m-1,1,m),(1^{2m})$	$\mathbf{a}_k = \mathbf{z}_k + \mathbf{z}_m + \mathbf{z}_{2m+1-k}$ $(1 \leqslant k \leqslant m-1)$, $\mathbf{a}_m = \mathbf{z}_m + \mathbf{z}_{m+1}$
$(m,m),(m,m-1,1),(1^{2m})$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_{2m}$, $\mathbf{a}_k = \mathbf{z}_k + \mathbf{z}_{2m+1-k} + \mathbf{z}_{2m}$ $(2 \leqslant k \leqslant m)$
$(m,m),(m,1,m-1),(1^{2m})$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_{m+1}$, $\mathbf{a}_k = \mathbf{z}_k + \mathbf{z}_{m+1} + \mathbf{z}_{2m+2-k}$ $(2 \leqslant k \leqslant m)$
odd family	
$(m,m+1),(1,m,m),(1^{2m+1})$ $(m+1,m),(1,m,m),(1^{2m+1})$	$\mathbf{a}_k = \mathbf{z}_1 + \mathbf{z} + k + 1 + \mathbf{z}_{2m+2-k}$ $(1 \leqslant k \leqslant m)$ $\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_2$, $\mathbf{a}_{m+1} = \mathbf{z}_1 + \mathbf{z}_{m+2}$, $\mathbf{a}_k = \mathbf{z}_1 + \mathbf{z}_{k+1} + \mathbf{z}_{2m+3-k}$ $(2 \leqslant k \leqslant m)$
$(m,m+1),(m,1,m),(1^{2m+1})$ $(m+1,m),(m,1,m),(1^{2m+1})$	$\mathbf{a}_k = \mathbf{z}_k + \mathbf{z}_{m+1} + \mathbf{z}_{2m+2-k}$ $(1 \leqslant k \leqslant m)$ $\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_{m+1}$, $\mathbf{a}_{m+1} = \mathbf{z}_{m+1} + \mathbf{z}_{m+2}$, $\mathbf{a}_k = \mathbf{z}_k + \mathbf{z}_{m+1} + \mathbf{z}_{2m+3-k}$ $(2 \leqslant k \leqslant m)$
$(m,m+1),(m,m,1),(1^{2m+1})$ $(m+1,m),(m,m,1),(1^{2m+1})$	$\mathbf{a}_k = \mathbf{z}_k + \mathbf{z}_{2m+1-k} + \mathbf{z}_{2m+1}$ $(1 \leqslant k \leqslant m)$ $\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_{2m+1}$, $\mathbf{a}_{m+1} = \mathbf{z}_{m+1} + \mathbf{z}_{2m+1}$, $\mathbf{a}_k = \mathbf{z}_k + \mathbf{z}_{2m+2-k} + \mathbf{z}_{2m+1}$ $(2 \leqslant k \leqslant m)$
$\hat{E}_8$ family	
$(2,4),(2,2,2),(1,1,1,1,1,1)$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3 + \mathbf{z}_6$, $\mathbf{a}_2 = \mathbf{z}_1 + \mathbf{z}_4 + \mathbf{z}_5$
$(4,2),(2,2,2),(1,1,1,1,1,1)$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_5$, $\mathbf{a}_2 = \mathbf{z}_2 + \mathbf{z}_3$, $\mathbf{a}_3 = \mathbf{z}_2 + \mathbf{z}_5 + \mathbf{z}_6$, $\mathbf{a}_4 = \mathbf{z}_4 + \mathbf{z}_5$
E_8 family	
$(3,3),(2,2,2),(2,1,1,1,1)$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3$, $\mathbf{a}_2 = \mathbf{z}_1 + \mathbf{z}_6$, $\mathbf{a}_3 = \mathbf{z}_2 + \mathbf{z}_4 + \mathbf{z}_5$
$(3,3),(2,2,2),(1,2,1,1,1)$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_4 + \mathbf{z}_6$, $\mathbf{a}_2 = \mathbf{z}_1 + \mathbf{z}_3$, $\mathbf{a}_3 = \mathbf{z}_1 + \mathbf{z}_5$
$(3,3),(2,2,2),(1,1,2,1,1)$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_5 + \mathbf{z}_6$, $\mathbf{a}_2 = \mathbf{z}_2 + \mathbf{z}_3 + \mathbf{z}_6$, $\mathbf{a}_3 = \mathbf{z}_4 + \mathbf{z}_5$
$(3,3),(2,2,2),(1,1,1,2,1)$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_4 + \mathbf{z}_6$, $\mathbf{a}_2 = \mathbf{z}_1 + \mathbf{z}_3$, $\mathbf{a}_3 = \mathbf{z}_1 + \mathbf{z}_5$
$(3,3),(2,2,2),(1,1,1,1,2)$	$\mathbf{a}_1 = \mathbf{z}_1 + \mathbf{z}_4 + \mathbf{z}_6$, $\mathbf{a}_2 = \mathbf{z}_2 + \mathbf{z}_4 + \mathbf{z}_5$, $\mathbf{a}_3 = \mathbf{z}_2 + \mathbf{z}_3$

(4.41)

Here is an example of the matrix $\mathbf{Z}$ with $m = 3$.

Example 4.1.

$$\mathbf{Z} = \left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{q_{24}}{c_4-c_5} & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{q_{24}}{c_2-c_3} & 1 & 0 \\ 0 & 0 & 0 & \frac{q_{14}\,q_{15}}{(c_1-c_2)(c_1-c_3)} & \frac{q_{15}}{c_1-c_2} & 1 \end{array}\right]$$

Definition 4.1. Let $\mathbf{A}$ be a diagonalizable complex linear operator with the spectrum $(\lambda_1, \lambda_2, \dots, \lambda_k)$. Let V_{λ_i} be the eigenspace of $\mathbf{A}$ corresponding to the eigenvalue λ_i. We will call the flag $V_{\lambda_1} \subset V_{\lambda_1} \oplus V_{\lambda_2} \subset \dots \subset V$ the spectral flag of $\mathbf{A}$ corresponding to the ordering $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of its spectrum.

If $(m_1, m_2, \dots, m_k)$ are the multiplicities of the spectrum of $\mathbf{A}$ from the above definition, then the dimension vector in the jump coordinates of its spectral flag is also $(m_1, m_2, \dots, m_k)$.

If we take another look at the eigenvectors of $\mathbf{B}$ (Lemma 3.3) and at the eigenvectors of $\mathbf{C}$ (Lemma 2.2), we see that the spectral flags of these matrices are exactly the second and the third flags of the Magyar, Weyman, and Zelevinsky triple $((m,m),(1,m-1,m),(1^{2m}))$.

Lemma 4.1. *The subspace* V_1^1 *spanned by the vectors* $\mathbf{a}_1, \dots, \mathbf{a}_m$ *(from the* $((m,m),(1,m-1,m),(1^{2m}))$ *entry in* (4.41)) *is the spectral subspace of the matrix* $\mathbf{A}$ *corresponding to the eigenvalue* -1.

Proof. In order to prove $(\mathbf{A} + \mathbf{Id})\mathbf{a}_i = 0$ for $1 \leqslant i \leqslant m$, we have to prove the following identities.

1. The first identity says that the first component of $(\mathbf{A} + \mathbf{Id})\mathbf{a}_i$ is zero.

$$b_1 + c_{2m} + 1 + \sum_{j=i}^{m-1} Z_{1+j,1+i}\, B_{1,1+j} + \sum_{m+1-i}^{m} Z_{m+j,2m+1-i}\, B_{1,m+j} = 0. \tag{4.42}$$

2. The second identity says that the components 2 through m of $(\mathbf{A}+\mathbf{Id})\mathbf{a}_i$ are zero.

$$C_{1+j,1} + (c_{2m-j} + b_2 + 1)Z_{1+j,1+i} + \sum_{k=m+1-i}^{m} Z_{m+k,2m+1-i}\, B_{1+j,m+k} = 0. \qquad (4.43)$$

3. The third identity says that the components $m+1$ through $2m$ of $(\mathbf{A}+\mathbf{Id})\mathbf{a}_i$ are zero.

$$C_{m+j,1} + (c_{m+1-j} + b_3 + 1)Z_{m+j,2m+1-i} + \sum_{k=i}^{m-1} Z_{1+k,1+i}\, C_{m+j,1+k} = 0. \qquad (4.44)$$

Recall that the matrix elements of the matrices $\mathbf{B}$ and $\mathbf{C}$ are given by formulas (2.7)–(2.12) on pages 8–9. Then the first identity becomes

$$b_1 + c_{2m} + 1 + \sum_{j=i}^{m-1} \frac{\prod_{k=1+i}^{m} q_{k,2m-j}}{\prod_{\substack{k=m+1\\k\neq 2m-j}}^{2m-i} (c_{2m-j} - c_k)} + \sum_{j=m+1-i}^{m} p^{31}_{m+1-j} \frac{\prod_{k=2m+1-i}^{2m-1} q_{m+1-j,k}}{\prod_{\substack{k=1\\k\neq m+1-j}}^{i} (c_{m+1-j} - c_k)} = 0. \qquad (4.45)$$

In our normalized version, $p^{31}_{m+1-j} = c_{m+1-j} + b_3 - 1$. Thus, (4.45) splits into two identities of homogeneous degrees 0 and 1. The part of degree 0 is

$$\sum_{j=m+1-i}^{m} \frac{\prod_{k=2m+1-i}^{2m-1} q_{m+1-j,k}}{\prod_{\substack{k=1\\k\neq m+1-j}}^{i} (c_{m+1-j} - c_k)} = 1.$$

This identity is equivalent to (7.60) from the appendix. The part of degree 1 is

$$b_1 + c_{2m} + \sum_{j=i}^{m-1} \frac{\prod_{k=1+i}^{m} q_{k,2m-j}}{\prod_{\substack{k=m+1\\k\neq 2m-j}}^{2m-i} (c_{2m-j} - c_k)} + \sum_{j=m+1-i}^{m} (b_3 + c_{m+1-j}) \frac{\prod_{k=2m+1-i}^{2m-1} q_{m+1-j,k}}{\prod_{\substack{k=1\\k\neq m+1-j}}^{i} (c_{m+1-j} - c_k)} = 0.$$

This one is proved similarly to identity (3.34) on page 24 with the help of identity (7.61) from the appendix applied separately to each of the sums.

To prove the second identity (4.43), let us recall that

$$Z_{1+j,1+i} = \begin{cases} 0 & \text{if } j<i, \\ 1 & \text{if } j=i, \\ \dfrac{\prod_{k=1+i}^{j} q_{k,2m-j}}{\prod_{k=2m+1-j}^{2m-i} (c_{2m-j}-c_k)} & \text{if } j>i. \end{cases}$$

Thus, the second identity (4.43) splits into three different identities. In case $j<i$, we have

$$C_{1+j,1} + \sum_{k=m+1-i}^{m} Z_{m+k,2m+1-i}\, B_{1+j,m+k} = 0.$$

This identity, after cancelling out common multiples, splits into a sum of two identities: one of degree -1 and the other of degree 0. The first reduces to identity (7.59) and the second reduces to identity (7.60) from the appendix.

In case $i=j$, we have

$$C_{1+j,1} + (c_{2m-j}+b_2+1) + \sum_{k=m+1-i}^{m} Z_{m+k,2m+1-i}\, B_{1+j,m+k} = 0.$$

This identity splits into two parts of degree 0 and 1. The part of degree 0 reduces to identity (7.63) from the appendix. The part of degree 1 is a sum of two identities: one is equivalent to (7.60) and the other is equivalent to (7.63) from the appendix. The case $j>i$, after cancelling out common multiples, becomes equivalent to the case $j=i$.

To prove the third identity (4.44), recall that

$$Z_{m+j,2m+1-i} = \begin{cases} 0 & \text{if } i+j<m+1, \\ 1 & \text{if } i+j=1, \\ \dfrac{\prod_{k=2m+1-i}^{m-1+j} q_{m+1-j,k}}{\prod_{k=m+2-j}^{i} (c_{m+1-j}-c_k)} & \text{if } i+j>m+1. \end{cases}$$

Thus, the third identity splits into three different identities. In case $i+j<m+1$, after cancelling out common multiples, (4.44) becomes equivalent to identity (7.60) from the appendix. In cases $i+j=m+1$ and $i+j>m+1$, identity (7.65) from the appendix does the job.

Finally, it is clear from (4.41) and (4.40) that for $1 \leqslant i \leqslant m$, the vectors $\mathbf{a}_i$ are linearly independent. $\square$

Now let us get back to the proof of Theorem 1.3.

Proof of Theorem 1.3. For a triple of partitions $S''(\alpha,\beta,\gamma)$ from Simpson's list (1.1), we want to prove that if $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ is a point of $S''(\alpha,\beta,\gamma)$, then the triple

$(\mathbf{A},\mathbf{B},\mathbf{C})$ is irreducible. We have proved it already for the hypergeometric family in Section 3.1. To prove it for the even family, let us recall that if $(s(\mathbf{A}),s(\mathbf{B}),s(\mathbf{C}))\in S''(\alpha,\beta,\gamma)$, then the scalar product (2.14) is well-defined and non-degenerate. Assume that the triple of matrices $(\mathbf{A},\mathbf{B},\mathbf{C})$ is reducible. Then they all preserve a non-trivial subspace V'. This subspace is spanned by some eigenvectors of $\mathbf{C}$. Let V'' be the subspace of V spanned by the complementary eigenvectors of $\mathbf{C}$. But all eigenvectors of $\mathbf{C}$ are orthogonal to each other with respect to (2.14). Thus, the space V splits into the orthogonal direct sum $V'\oplus V''$. Thus, the matrices $\mathbf{A}$ and $\mathbf{B}$ preserve the subspace V'' as well. So, no matter how we introduce linear orders on the spectra of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, the corresponding triple of flags will decompose. However, if $(s(\mathbf{A}),s(\mathbf{B}),s(\mathbf{C}))$ is a point of $S''((m,m),(1,m-1,m),(1^{2m}))$, then as follows from Lemma 4.1 and the preceding discussion, the spectral flags of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ give the Magyar, Weyman, and Zelevinsky representative of the open orbit of the corresponding triple flag variety. According to Magyar, Weyman, and Zelevinsky, this triple of flags is indecomposable. Thus, the assumption that the triple $(\mathbf{A},\mathbf{B},\mathbf{C})$ is reducible cannot be true. This proves Theorem 1.3 in the even case.

The Z-matrix for the odd family triple $((m+1,m),(1,m,m),(1^{2m+1}))$ can be obtained from the Z-matrix for the even family triple $((m+1,m+1),(1,m,m+1),(1^{2m+2}))$ by restricting the latter to V^s_{m+1} as in the proof of Theorem 3.1 on page 34. The rest of the argument is the same. Finally, let us give the Z-matrix for the extra case of Simpson (or, more precisely, for the triple of compositions $((4,2),(2,2,2),(1^6))$ of the $\hat{E}_8$-family from (4.41)).

$$Z=\left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0\\ \frac{q_{145}}{c_5-c_6} & 1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{q_{245}}{c_3-c_4} & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & \frac{q_{236}}{c_1-c_2} & 1 \end{array}\right]. \tag{4.46}$$

The only family of Magyar, Weyman, and Zelevinsky which does not appear in the list of Simpson, is the E_8-family. We now construct the matrices $\mathbf{A}=\mathbf{B}+\mathbf{C},\mathbf{B},\mathbf{C}$ such that their spectral flags form the Magyar, Weyman, and Zelevinsky representative for the open orbit of the triple flag variety of dimension $((3,3),(2,2,2),(1,1,1,1,2))$. This time the standard basis $\mathbf{e}_i$ and the z-basis of Magyar, Weyman, and Zelevinsky $\mathbf{z}_i$ coincide ($\mathbf{z}_i=\mathbf{e}_i$).

Let $(a_1,a_1,a_1,\ a_2,a_2,a_2,b_1,b_1,b_2,b_2,b_3,b_3,c_1,c_2,c_3,c_4,c_5,c_5)\in S((3,3),\ (2,2,2),\ (1,1,1,1,2))$.

$$\text{Let}\quad \mathbf{B}=\left[\begin{array}{cc|cc|cc} b_1 & 0 & 0 & a_1+a_2-b_1-b_3-c_1-c_5 & -a_1-a_2+b_1+b_3+c_1+c_5 & -a_2+b_3+c_1 \\ 0 & b_1 & a_1-b_1-c_5 & -a_1-2a_2+b_1+b_2+b_3+c_1+c_3+c_5 & -a_1-a_2+b_2+b_3+c_2+c_4 & a_1+2a_2-b_1-b_2-b_3-c_1-c_3-c_5 \\ \hline 0 & 0 & b_2 & 0 & -a_1+b_2+c_4 & a_1+2a_2-b_1-b_2-b_3-c_1-c_3-c_5 \\ 0 & 0 & 0 & b_2 & 2a_1+a_2-b_1-2b_2-c_1-c_3-c_5 & -a_2+b_3+c_1 \\ \hline 0 & 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3 \end{array}\right], \tag{4.47}$$

$$\mathbf{C}=\left[\begin{array}{cc|cc|cc} c_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_5 & 0 & 0 & 0 & 0 \\ \hline -a_1-2a_2+b_1+b_2+b_3+c_1+c_3+c_5 & a_1-b_2-c_4 & c_4 & 0 & 0 & 0 \\ a_1+a_2-b_2-b_3-c_1-c_3 & -a_1-a_2+b_1+b_2+c_4+c_5 & a_1+a_2-b_1-b_2-c_4-c_5 & c_3 & 0 & 0 \\ \hline -a_1-2a_2+b_1+b_2+b_3+c_1+c_3+c_5 & -a_1-a_2+b_1+b_2+c_4+c_5 & a_1+a_2-b_1-b_2-c_4-c_5 & -a_1-2a_2+b_1+b_2+b_3+c_1+c_3+c_5 & c_2 & 0 \\ -a_2+b_1+c_5 & 0 & 0 & a_1+a_2-b_1-b_3-c_1-c_5 & -a_1-a_2+b_1+b_3+c_1+c_5 & c_1 \end{array}\right]. \tag{4.48}$$

It is clear that $\mathbf{B}$ and $\mathbf{C}$ are diagonalizable and that $s(\mathbf{B}) = \{b_1, b_1, b_2, b_2, b_3, b_3\}$, $s(\mathbf{C}) = \{c_1, c_2, c_3, c_4, c_5, c_5\}$. The following is proved by direct computation.

Theorem 4.1. *For* $\mathbf{B}$ *and* $\mathbf{C}$ *as in* (4.47) *and* (4.48), *let* $\mathbf{A} = \mathbf{B} + \mathbf{C}$. *Then* $\mathbf{A}$ *is diagonalizable and* $s(\mathbf{A}) = \{a_1, a_1, a_1, a_2, a_2, a_2\}$.

The existence of a scalar product on V such that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are self-adjoint with respect to it can be proved using methods of the theory of quiver representations. It follows from Schur's lemma that if $(s(\mathbf{A}), s(\mathbf{B}), s(\mathbf{C}))$ is a generic point in $S((3,3),(2,2,2),(1,1,1,1,2))$, then the form is unique up to a constant multiple. It would be interesting to find a basis in which the form is "nice" (for instance, having matrix entries as ratios of products of linear forms in the eigenvalues of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$).

5. Connections with the Littlewood–Richardson rule

An irreducible rational representation of $\mathrm{GL}(n, \mathbb{C})$ is uniquely determined by its *highest weight* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where λ_i are integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. We can decompose tensor products of irreducible representations into sums of irreducibles:

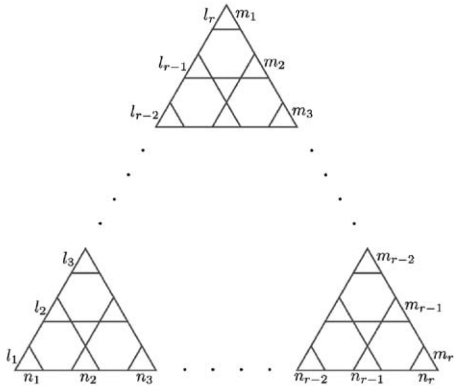
$$V_\lambda \otimes V_\mu = \sum_\nu c^\nu_{\lambda\mu} V_\nu. \tag{5.49}$$

The number $c^\nu_{\lambda\mu}$ of copies of V_ν in $V_\lambda \otimes V_\mu$ is called the *Littlewood–Richardson coefficient*. There exists a famous combinatorial algorithm to compute the Littlewood–Richardson coefficient called the *Littlewood–Richardson rule* (see [9] for more information). It follows from the results of Klyachko [14] combined with a refinement by Knutson and Tao [15], that the lattice points of the Klyachko cone are exactly the triples of highest weights with non-zero Littlewood–Richardson coefficients (see also [8] for a nice survey). The question whether all the lattice points of the Klyachko cone were such triples was raised in [32] under the name of the *saturation conjecture*. The conjecture was proved by Knutson et al. in [15]. Some of the Klyachko inequalities describing the Klyachko cone are redundant. Knutson et al. in [16] give the set of necessary inequalities for the Klyachko cone. Derksen and Weyman [6] give a proof of the saturation conjecture different from that of Knutson and Tao. They use methods of the theory of quiver representations, developing further ideas of Schofield [28]. Moreover, Derksen and Weyman [7] give description of *all* the faces of the Klyachko cone of arbitrary dimension. However, all these results involve recursive computations.

The inequalities of Theorems 2.3, 2.6, 2.9, and 2.12 give non-recursive description of some faces of the Klyachko cone. Thus, integral solutions to these inequalities give non-recursive description of some triples of highest weights with $c^\nu_{\lambda\mu} \neq 0$.

Let us show a different way to derive these inequalities and also show that the corresponding $c^\nu_{\lambda\mu} = 1$. For that, we use the *Berenstein–Zelevinsky triangle*. It was invented in [1] as a geometric version of the Littlewood–Richardson rule. A variation of the BZ-triangle was used in [15] under the name of a *honeycomb tinkertoy*. A different variation of the triangle was used in [10] under the name of a *web-function* to examine relations of the Littlewood–Richardson coefficients with a Quantum–Yang–Baxter-Type equation.

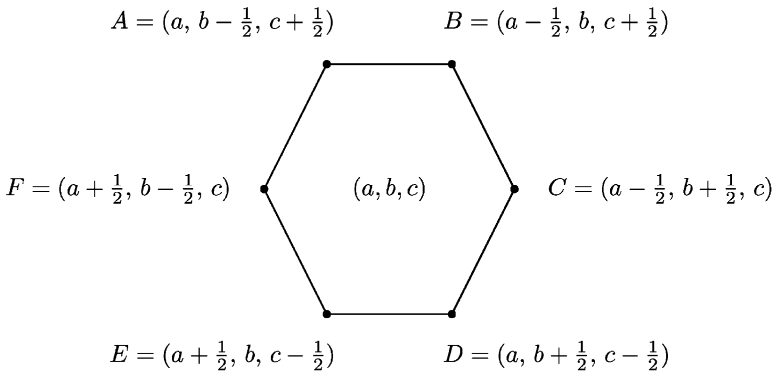
Consider the following graph.



This is the Berenstein–Zelevinsky triangle for sl_r. In order to define it formally, it is convenient to use the barycentric coordinates in $\mathbb{R}^2$. Namely, we represent a point in $\mathbb{R}^2$ by a triple (α, β, γ) such that $\alpha + \beta + \gamma = 0$. The r *Berenstein–Zelevinsky triangle* BZ_r is the set of points in $\mathbb{R}^2$ with barycentric coordinates (α, β, γ), such that

1. $0 < \beta < -\alpha < r + 1$,
2. the numbers 2α, 2β, and 2γ are integers,
3. at least one α, β, or γ is not integer.

Every integer point (a, b, c), $a + b + c = 0$, with $0 < b < -a < r + 1$ has six neighbors in BZ_r that form vertices of the following hexagon:



Definition 5.1. A function $f : BZ_r \to \{0, 1, 2, \dots\}$ is called a BZ-*filling* if for any hexagon as above we have $f(A) + f(B) = f(D) + f(E)$, $f(B) + f(C) = f(E) + f(F)$, and $f(C) + f(D) = f(F) + f(A)$ (the last condition follows from the first two). We call this the hexagon condition.

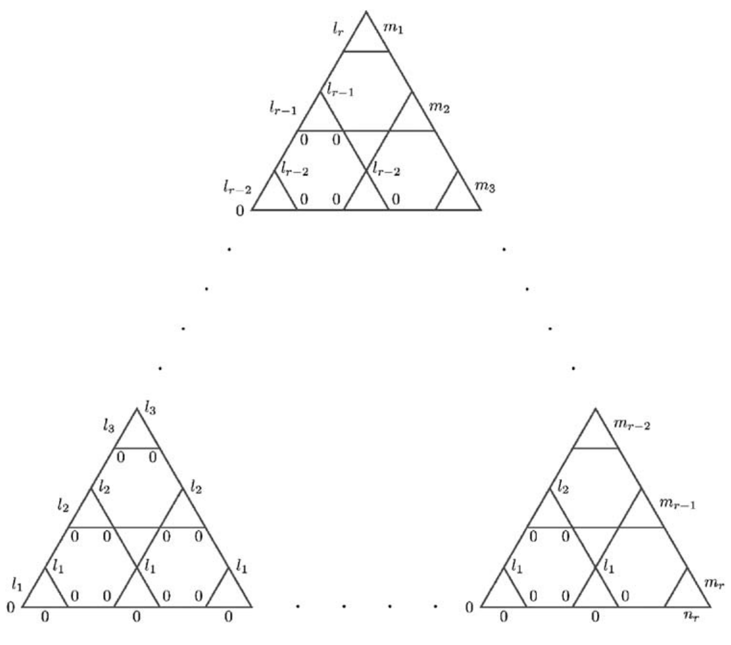
Let $\lambda = \sum_{i=1}^{r} l_i \omega_i$, $\mu = \sum_{i=1}^{r} m_j \omega_i$, and $\nu = \sum_{i=1}^{r} n_i \omega_i$, where the ω_i are the *fundamental weights* of $\mathrm{sl}(r+1,\mathbb{C})$. Let us assign l_i, m_i, and n_i to the boundary segments of the BZ_r as shown in the picture on page 43. Note that $l_i = \lambda_i - \lambda_{i+1}$, $m_i = \mu_i - \mu_{i+1}$, $n_i = \nu_i - \nu_{i+1}$.

Definition 5.2. A filling f of BZ_r satisfies boundary conditions if for any boundary segment with vertices A, B, and a non-negative integer value v assigned to the segment, $f(A) + f(B) = v$.

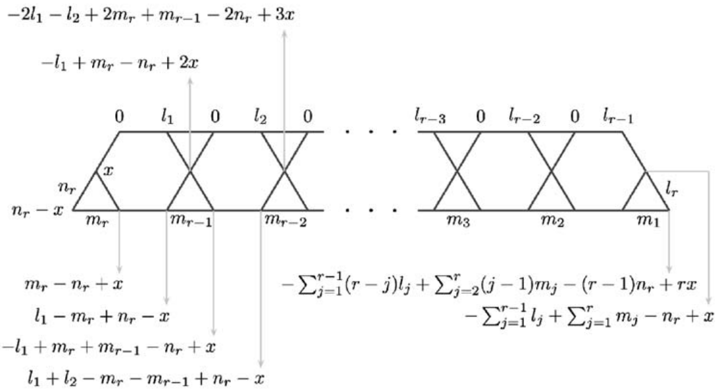
Theorem 5.1 (Berenstein, Zelevinsky). *Let* $\lambda = \sum_{i=1}^{r} l_i \omega_i$, $\mu = \sum_{i=1}^{r} m_j \omega_i$, *and* $\nu = \sum_{i=1}^{r} n_i \omega_i$ *be dominant weights of* $\mathrm{sl}(r+1,\mathbb{C})$. *Then* $c_{\lambda\mu}^{\nu} = \#\{$*of fillings of* BZ_r *satisfying the boundary conditions*$\}$.

Let us use the BZ-triangle for a different proof of Theorem 2.3, and also to show that the corresponding Littlewood–Richardson coefficient is equal to one.

Proof of Theorem 2.3. Let us assume that $a_1 < a_2$. Consider BZ_r for the hypergeometric case $(r = m-1)$. For that, we have to switch from $\mathrm{gl}(n,\mathbb{C})$ to $\mathrm{sl}(n,\mathbb{C})$. Let us set $\tilde{\mathbf{A}} = \mathbf{A} - \frac{1}{r+1}\mathrm{tr}(\mathbf{A})\mathbf{Id}$, $\tilde{\mathbf{B}} = \mathbf{B} - \frac{1}{r+1}\mathrm{tr}(\mathbf{B})\mathbf{Id}$, and $\tilde{\mathbf{C}} = \mathbf{C} - \frac{1}{r+1}\mathrm{tr}(\mathbf{C})\mathbf{Id}$. Then $s(\tilde{\mathbf{A}}) = \{\tilde{a}, \tilde{a}, \ldots, \tilde{a}, -r\,\tilde{a}\}$. Let us call the eigenvalues of $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ $\tilde{b}_i$ and $\tilde{c}_j$. We have $\sum_{j=1}^{r+1} \tilde{b}_j = \sum_{j=1}^{r+1} \tilde{c}_j = 0$. Recall that $l_i = \tilde{b}_i - \tilde{b}_{i+1}$; $m_i = \tilde{c}_i - \tilde{c}_{i+1}$; $n_1 = n_2 = \cdots = n_{r-1} = 0$, and $n_r = (r+1)a$.



If a hexagon has two zeros on a side, then the non-negativity of BZ-fillings and the hexagon condition force two zeros on the opposite side. In the hypergeometric case, this mechanism reduces the BZ-triangle to a strip. Let us put a variable x in an unfilled vertex of the strip. Then the filling is expressed in terms of x, and the boundary conditions. Also the l_r-boundary condition gives a linear equation on x.



The l_r-boundary condition gives us the equation $-\sum_{j=1}^{r-1}(r+1-j)l_j + \sum_{j=1}^{r} jm_j - rn_r + (r+1)x = l_r$. Thus,

$$x = \frac{\sum_{j=1}^{r}(r+1-j)l_j - \sum_{j=1}^{r} jm_j + rn_r}{r+1}$$

and the filling is defined uniquely. Let us list the Klyachko inequalities. First, $x = \tilde{b}_1 + \tilde{c}_n + r\tilde{a} > 0$. However, this inequality is not a generating one. If we have another look at the strip above, we see that x has a neighboring 0-vertex. So, x is the sum of the numbers at the opposite edge ($-\tilde{b}_2 - \tilde{c}_{n-1} + \tilde{a}$, and $\tilde{b}_1 + \tilde{b}_2 + \tilde{c}_{n-1} + \tilde{c}_n + (n-2)\tilde{a}$). All the numbers in the middle part of the strip except for the utmost right one ($-\tilde{b}_n - \tilde{c}_1 + \tilde{a}$) do not produce generating inequalities for the same reason. The numbers on the lower part of the strip together with the last middle number produce the following generating inequalities:

$$\begin{array}{lcccl} \tilde{b}_1 + \tilde{c}_{m-1} & > & & > & \tilde{b}_1 + \tilde{c}_m \\ \tilde{b}_2 + \tilde{c}_{m-2} & > & & > & \tilde{b}_2 + \tilde{c}_{m-1} \\ \cdots & & \tilde{a} & & \cdots \\ \tilde{b}_{m-1} + \tilde{c}_1 & > & & > & \tilde{b}_{m-1} + \tilde{c}_2 \\ & & & > & \tilde{b}_m + \tilde{c}_1. \end{array} \tag{5.50}$$

Switching back to $\mathrm{gl}(n, \mathbb{C})$ proves the theorem in this case. The case $a_1 > a_2$ is obtained from the case $a_1 < a_2$ in the following way. Let us multiply **A**, **B**, and **C** by

-1. Then let us renumber b_i and c_j so that $b_1 > b_2 > \cdots > b_m$ and $c_1 > c_2 > \cdots > c_m$ again. □

One can similarly prove Theorems 2.6, 2.9, 2.12 and also show that the corresponding Littlewood–Richardson coefficients are equal to one.

In the E_8 case of the Magyar, Weyman, and Zelevinsky list (4.41), we do not have an explicit criterion for positivity of the corresponding Hermitian form. However, the BZ-triangle enables us to compute the inequalities on the eigenvalues of **A**, **B**, and **C** which make the form sign-definite. In the notations of (4.47), (4.48), let $a_1 > a_2$ and $b_1 > b_2 > b_3$. Then the form is sign-definite precisely in the following situations.

$$
\begin{array}{|lcccl|}
\hline
-a_1-a_2+b_1+b_3+c_1+c_5 & > & 0 & > & -a_1-a_2+b_1+b_3+c_2+c_5 \\
-a_1-a_2+b_1+b_2+c_3+c_5 & > & 0 & > & -a_1-a_2+b_1+b_2+c_4+c_5 \\
a_1-a_2-c_1+c_2-c_3+c_4 & > & 0 & > & a_1-a_2-c_1-c_2+c_3+c_4 \\
 & & 0 & > & -a_2+b_2+c_5 \\
\hline
-a_1-a_2+b_1+b_3+c_2+c_5 & > & 0 & > & -a_1-a_2+b_1+b_3+c_3+c_5 \\
-a_2+b_2+c_5 & > & 0 & > & -a_2+b_2+c_4 \\
-a_1+a_2+c_1-c_2+c_3-c_4 & > & 0 & & \\
 & & 0 & > & -a_1-a_2+b_2+b_3+c_1+c_5 \\
 & & 0 & > & -a_1+b_1+c_4 \\
\hline
-a_1-a_2+b_2+b_3+c_1+c_5 & > & 0 & > & -a_1-a_2+b_2+b_3+c_2+c_5 \\
-a_1-a_2+b_1+b_2+c_3+c_5 & > & 0 & > & -a_1-a_2+b_1+b_2+c_4+c_5 \\
a_1-a_2-c_1+c_2-c_3+c_4 & > & 0 & & \\
-a_1+b_1+c_5 & > & 0 & & \\
 & & 0 & > & -a_2+b_3+c_5 \\
\hline
-a_1-a_2+b_1+b_3+c_2+c_5 & > & 0 & > & -a_1-a_2+b_1+b_3+c_3+c_5 \\
-a_1+b_2+c_1 & > & 0 & > & -a_1+b_2+c_5 \\
-a_1-a_2+b_1+b_2+c_4+c_5 & > & 0 & & \\
-a_2+b_3+c_1 & > & 0 & & \\
 & & 0 & > & a_1-a_2-c_1+c_2-c_3+c_4 \\
\hline
-a_1-a_2+b_1+b_3+c_3+c_5 & > & 0 & > & -a_1-a_2+b_1+b_3+c_4+c_5 \\
-a_1-a_2+b_2+b_3+c_1+c_5 & > & 0 & > & -a_1-a_2+b_2+b_3+c_2+c_5 \\
-a_1+a_2+c_1+c_2-c_3-c_4 & > & 0 & > & -a_1+a_2+c_1-c_2+c_3-c_4 \\
-a_1+b_2+c_5 & > & 0 & > & \\
\hline
\end{array}
\qquad (5.51)
$$

The first set of inequalities forces $c_1 > c_2 > c_3 > c_4 > c_5$ realizing the dimension vector $(3,3),(2,2,2),(1,1,1,1,2)$. The second set of inequalities forces $c_1 > c_2 > c_3 > c_5 > c_4$ realizing the dimension vector $(3,3),(2,2,2),(1,1,1,2,1)$. The third set of inequalities forces $c_1 > c_2 > c_5 > c_3 > c_4$ realizing the dimension vector $(3,3),(2,2,2),(1,1,2,1,1)$. The forth set of inequalities forces $c_1 > c_5 > c_2 > c_3 > c_4$ realizing the dimension vector $(3,3),(2,2,2),(1,2,1,1,1)$. The last set of inequalities forces $c_5 > c_1 > c_2 > c_3 > c_4$ realizing the dimension vector $(3,3),(2,2,2),(2,1,1,1,1)$. Thus, all the members of the E_8-family from (4.41) can be constructed this way with the help of the corresponding eigenvectors of **A**, **B**, and **C**.

6. Fuchsian systems, Fuchsian equations, Okubo normal forms, and the list of Haraoka–Yokoyama

Let us consider a system of linear differential equations on a $\mathbb{C}^n$-valued function f on $\mathbb{CP}^1$.

$$df = \omega f, \tag{6.52}$$

where ω is a $(n \times n)$ matrix-valued 1-differential form on $\mathbb{CP}^1$. Let the form be holomorphic everywhere on $\mathbb{CP}^1$ except for a finite set of points $\mathscr{D} = \{z_1, z_2, \dots, z_k\}$. Let us consider a solution of (6.52) restricted to a sectorial neighborhood centered at any $z_i \in \mathscr{D}$. If any such solution has polynomial growth when it approaches z_i within any such sector, then system (6.52) is called *linear regular*. If ω has only first-order poles at $\mathscr{D}$, then the system is called *Fuchsian*. Any Fuchsian system is linear regular, but there exist linear regular systems which are not Fuchsian (for more detailed treatment, see [3] or [30]).

An n order Fuchsian equation is a linear differential equation

$$f^n(z) + q_1(z)f^{n-1}(z) + \cdots + q_n(z)f(z) = 0 \tag{6.53}$$

such that its coefficients $q_j(z)$ have a finite set of poles $\mathscr{D} = \{z_1, z_2, \dots, z_k\}$ and in a small neighborhood of a pole z_i the coefficients of (6.53) have the form

$$q_j(z) = \frac{r_j(z)}{(z - z_i)^j}, \quad j = 1, \dots, n, \tag{6.54}$$

where the $r_j(z)$ are holomorphic functions. Solutions of Fuchsian equations have polynomial growth when continued analytically towards a pole. This distinguishes Fuchsian differential equations from all other linear differential equations on $\mathbb{CP}^1$. Thus, for linear differential equations the notions "Fuchsian" and "linear regular" coincide.

The matrix $R_i = \mathrm{Res}_{z=z_i}\omega(z)$ is called the *residue* of a linear regular system at z_i. By the Cauchy residue theorem, $\sum_{i=1}^k R_i = 0$.

Theorem 6.1 (see Bolibrukh [3]). *Any Fuchsian system has the standard form*

$$\frac{df}{dz} = \sum_{i=1}^{k} \frac{R_i}{z - z_i} f(z). \tag{6.55}$$

Theorem 6.2 (see Bolibrukh [3]). *For any Fuchsian equation on the Riemann sphere, it is possible to construct a Fuchsian system with the same singular points and the same monodromy. The converse is not true.*

Remark 6.1. Thus, the notion of a residue matrix makes sense for a Fuchsian equation as well as for a Fuchsian system.

To study Fuchsian differential equations, Okubo had invented what became later known as the *Okubo normal form* of a Fuchsian equation. In [27], he proves that any Fuchsian equation can be written in the following form:

$$(t\,\mathrm{Id} - B)\frac{dx}{dt} = Ax, \tag{6.56}$$

where t is a complex variable, $x \in \mathbb{C}^n$ is an unknown vector, Id is the identity matrix of order n, B is a constant diagonal $n \times n$ matrix, and A is a constant $n \times n$ matrix. Let

$$B = \mathrm{diag}(\underbrace{z_1, \dots, z_1}_{n_1}, \underbrace{z_2, \dots, \mathbf{z}_2}_{n_2}, \dots, \underbrace{z_k, \dots, z_k}_{n_k}), \tag{6.57}$$

such that $z_i \neq z_j$ for $i \neq j$, $n_1 + n_2 + \cdots + n_k = n$, and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. The partition $(n_1, n_2, \dots, n_k)$ of n endows A with the block decomposition $A = (A_{ij})_{1 \leqslant i,j \leqslant d}$. Let us call Λ_i the set of eigenvalues of A_{ii} and let us call Λ_∞ the set of eigenvalues of A. Then $z_1, z_2, \dots, z_k$ and ∞ are the singular points of (6.56). At the point z_i, (6.56) has n_i non-holomorphic solutions with local exponents $\lambda_j \in \Lambda_i$. At ∞, (6.56) has local exponents $\lambda_j \in \Lambda_\infty$.

Yokoyama [31] used Okubo theory to classify the spectral types of rigid irreducible Fuchsian equations. For such, all A_{ii} are diagonalizable as well as A itself. Quoting the result of Yokoyama, we will not give the spectral types of A_{ii} and A the way he does. Instead, we will list spectral types of the residue matrices (which are diagonalizable, too).

I	$(m-1,1),(1^m),(1^m)$	$m \geq 2$
II	$(m,1^m),(m,1^m),(m,m-1,1)$	$m \geq 2$
III	$(m,1^{m+1}),(m+1,1^m),(m,m,1)$	$m \geq 2$
IV	$(4,1,1),(2,1,1,1,1),(2,2,2)$	
I^*	$\underbrace{(m-1,1),\cdots,(m-1,1)}_{m\ times}$	$m \geq 2$
II^*	$(m,1^m),(m+1,1^{m-1}),(2m-1,1),(m,m)$	$m \geq 2$
III^*	$(m+1,1^m),(m+1,1^m),(2m,1),(m+1,m)$	$m \geq 2$
IV^*	$(4,1,1),(4,1,1),(4,1,1),(4,2)$	

(6.58)

Haraoka explicitly constructed the equations of the above spectral types in [11]. In [12], he explored the solutions of these equations: computed their monodromies, found monodromy invariant forms in their spaces of solutions, etc. It turns out that the solutions of these equations are important hypergeometric functions. It also turns out that the Fuchsian systems constructed in our paper are closely related to Yokoyama–Haraoka equations: sometimes the A matrices are just the same! We think it is interesting to understand the nature of this relation, find solutions to our systems, and their monodromies. We plan to do it in a subsequent publication.

Acknowledgments

I would like to thank G. Heckman who acquainted me with the problem when I was a student at the Master Class Program of the University of Utrecht. I would like to thank W. Crawley-Boevey, H. Derksen, Y. Haraoka, A. Khovansky, A. Klyachko, A. Knutson, V. Kostov, P. Magyar, A. Postnikov, and C. Simpson for answering a lot of questions.

I would like to thank the referee for very helpful suggestions which made this paper more clear and easier to read.

This paper was a basis for my Ph.D. thesis at Northeastern University of Boston. I would especially like to thank my adviser A. Zelevinsky for his tremendous support which made both the paper and the thesis possible. This paper was partially supported by NSF through research grants of A. Zelevinsky.

An extended version of this paper can be found under the same name on the Los Alamos preprint server (arXiv:math.LA/0105184).

Appendix

In this section we collect the identities needed for the proofs in the previous sections.

For $k<n-1$:

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{k}(x_i-y_j)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)} = 0, \tag{7.59}$$

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n-1}(x_i-y_j)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)} = 1, \tag{7.60}$$

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n}(x_i-y_j)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)} = \sum_{i=1}^{n}(x_i-y_i), \tag{7.61}$$

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n+1} (x_i + y_j)}{\prod_{\substack{j=1\\ j\neq i}}^{n} (x_i - x_j)} = \sum_{i=1}^{n} x_i^2 + \sum_{1\leqslant i<j\leqslant n} x_i x_j + \sum_{1\leqslant i<j\leqslant n+1} y_i y_j + \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n+1} y_i\right). \tag{7.62}$$

For $1\leqslant i\leqslant m-1$:

$$\sum_{j=1}^{m} \frac{\prod_{\substack{k=1\\ k\neq i}}^{m}(x_j - y_k)}{\prod_{\substack{k=1\\ k\neq j}}^{m}(x_j - x_k)} \frac{\prod_{\substack{k=1\\ k\neq j}}^{m} (y_i - x_k)}{\prod_{\substack{k=1\\ k\neq i}}^{m}(y_i - y_k)} = 1, \tag{7.63}$$

$$y_i^2 \frac{\prod_{\substack{k=1\\ k\neq i}}^{m-1}(y_i - y_k)}{\prod_{k=1}^{m} (y_i - x_k)} + \sum_{j=1}^{m} \frac{x_j^2}{(y_i - x_j)^2} \frac{\prod_{k=1}^{m-1} (x_j - y_k)}{\prod_{\substack{k=1\\ k\neq j}}^{m} (x_j - x_k)} = 1. \tag{7.64}$$

For $1\leqslant i\leqslant m$:

$$\frac{\prod_{\substack{k=1\\ k\neq i}}^{m} (x_i - x_k)}{\prod_{k=1}^{m-1} (x_i + y_k)} + \sum_{j=1}^{m-1} \frac{1}{x_i + y_j} \frac{\prod_{\substack{k=1\\ k\neq i}}^{m}(y_j + x_k)}{\prod_{\substack{k=1\\ k\neq j}}^{m-1} (y_j - y_k)} = 1, \tag{7.65}$$

$$\sum_{i=1}^{n} \frac{(x_i + x_1)(x_i + x_2)\cdots\widehat{(x_i + x_i)}\cdots(x_i + x_n)}{(x_i - x_1)(x_i - x_2)\cdots\widehat{(x_i - x_i)}\cdots(x_i - x_n)} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases} \tag{7.66}$$

$$\sum_{i=1}^{n} x_i \frac{(x_i + x_1)(x_i + x_2)\cdots\widehat{(x_i + x_i)}\cdots(x_i + x_n)}{(x_i - x_1)(x_i - x_2)\cdots\widehat{(x_i - x_i)}\cdots(x_i - x_n)} = x_1 + x_2 + \cdots + x_n, \tag{7.67}$$

$$\sum_{i=1}^{n} x_i^2 \frac{(x_i + x_1)(x_i + x_2)\cdots\widehat{(x_i + x_i)}\cdots(x_i + x_n)}{(x_i - x_1)(x_i - x_2)\cdots\widehat{(x_i - x_i)}\cdots(x_i - x_n)} = (x_1 + x_2 + \cdots + x_n)^2, \tag{7.68}$$

$$y_j \frac{\prod_{\substack{k=1\\ k\neq j}}^{m} (y_j + y_k)}{\prod_{k=1}^{m} (y_j - x_k)} + \sum_{r=1}^{m} \frac{x_r}{x_r - y_j} \frac{\prod_{\substack{k=1\\ k\neq j}}^{m} (x_r + y_k)}{\prod_{\substack{k=1\\ k\neq r}}^{m} (x_r - x_k)} = 1 \quad \text{for } j = 1, 2, \ldots, m-1, \tag{7.69}$$

$$y_j^2 \frac{\prod_{\substack{k=1\\k\neq j}}^{m} (y_j + y_k)}{\prod_{k=1}^{m} (y_j - x_k)} + \sum_{r=1}^{m} \frac{x_r^2}{x_r - y_j} \frac{\prod_{\substack{k=1\\k\neq j}}^{m} (x_r + y_k)}{\prod_{\substack{k=1\\k\neq r}}^{m} (x_r - x_k)} = \sum_{r=1}^{m} (x_r + y_r)$$

$$\text{for } j = 1, 2, \ldots, m-1. \tag{7.70}$$

For $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots, m$:

$$\sum_{r=1}^{m-1} \frac{\prod_{\substack{k=1\\k\neq j}}^{m} (x_k + y_r)}{\prod_{\substack{k=1\\k\neq r}}^{m-1} (y_r - y_k)} \sum_{s=1}^{m} \frac{1}{x_s + y_i} \frac{\prod_{\substack{k=1\\k\neq r}}^{m} (x_s^2 - y_k^2)}{\prod_{\substack{k=1\\k\neq s}}^{m} (x_s^2 - x_k^2)} + \frac{x_j + y_m}{x_j + y_i} \frac{\prod_{\substack{k=1\\k\neq i}}^{m} (x_j - y_k)}{\prod_{\substack{k=1\\k\neq j}}^{m} (x_j + x_k)} = 1. \tag{7.71}$$

All these identities have the following features: the left-hand side $L(x,y)$ is a rational homogeneous function in x_i and y_j. All the denominators of $L(x,y)$ are products of linear forms α of the form $(x_i \pm x_j)$, $(y_i \pm y_j)$, or $(x_i \pm y_j)$. The power of every such form in any denominator is 1. The right-hand sides $R(x,y)$ are constants or homogeneous polynomials in x_i and y_j of degree 1 or 2.

The first step to prove such an identity is to prove that $L(x,y)$ is in fact a polynomial. For that, it is enough to prove that $\alpha L(x,y)|_{\alpha=0} = 0$ for every form α from any denominator of the identity. For all the identities except for (7.71), the restriction of $\alpha L(x,y)$ to the hyperplane $\alpha = 0$ turns to be a sum of just two terms with equal absolute values and opposite signs. For example, consider identity (7.60). Let us fix p and q such that $1 \leqslant p < q \leqslant n$. Consider

$$(x_p - x_q) \sum_{i=1}^{n} \frac{\prod_{j=1}^{n-1} (x_i - y_j)}{\prod_{\substack{j=1\\j\neq i}}^{n} (x_i - x_j)}$$

restricted to the hyperplane $x_p = x_q$. The restriction equals

$$\frac{\prod_{j=1}^{n-1} (x_p - y_j)}{\prod_{\substack{j=1\\j\neq p\\j\neq q}}^{n} (x_p - x_j)} - \frac{\prod_{j=1}^{n-1} (x_q - y_j)}{\prod_{\substack{j=1\\j\neq q\\j\neq p}}^{n} (x_q - x_j)} = 0.$$

For identity (7.71), the same technique works for all the forms in the denominators except for $\alpha = x_p + x_m$ where $1 \leqslant p \leqslant m-1$. If $\alpha = x_p + x_m$, then the restriction of $\alpha L(x,y)$ to the hyperplane $\alpha = 0$ is

$$-\sum_{r=1}^{m-1} \frac{\prod_{k=1}^{m-1} (x_k + y_r)}{\prod_{\substack{k=1\\k\neq r}}^{m-1} (y_r - y_k)} \frac{\prod_{\substack{k=1\\k\neq r}}^{m} (x_p^2 - y_k^2)}{\prod_{\substack{k=1\\k\neq p}}^{m-1} (x_p^2 - x_k^2)} \frac{1}{y_1^2 - x_p^2} + \frac{y_m - x_p}{y_1 - x_p} \frac{\prod_{k=2}^{m} (x_p + y_k)}{\prod_{\substack{k=1\\k\neq p}}^{m-1} (x_p - x_k)}.$$

The fact that this restriction equals zero is equivalent to the identity

$$\sum_{r=1}^{m} \frac{\prod_{\substack{k=1\\k\neq p}}^{m} (y_r + x_k)}{\prod_{\substack{k=1\\k\neq r}}^{m} (y_r - y_k)} \frac{\prod_{\substack{k=1\\k\neq r}}^{m} (x_p - y_k)}{\prod_{\substack{k=1\\k\neq p}}^{m} (x_p + x_k)} = 1 \tag{7.72}$$

which is similar to identity (7.63), but different from it. In order to prove (7.72), it is convenient to rewrite it as

$$\sum_{r=1}^{m} \frac{\prod_{\substack{k=1\\k\neq p}}^{m} (y_r + x_k)}{\prod_{\substack{k=1\\k\neq r}}^{m} (y_r - y_k)} \frac{1}{x_p - y_r} - \frac{\prod_{\substack{k=1\\k\neq p}}^{m} (x_p + x_k)}{\prod_{k=1}^{m} (x_p - y_k)} = 0$$

and use the same technique over again.

The second step in the proofs is to show that a polynomial $L(x,y)$ equals the corresponding polynomial $R(x,y)$. Let us, for example, consider (7.68). In this case, $L(x)$ and $R(x)$ are symmetric homogeneous polynomials in x of degree 2. The space of such polynomials is two dimensional. It is spanned by $s_2 = x_1^2 + \cdots + x_n^2$ and s_1^2, where $s_1 = x_1 + \cdots + x_n$. To prove that $L(x) \equiv R(x)$, we have to find two linearly independent functionals f_1 and f_2 on this space such that $f_i(L) = f_i(R)$ for $i = 1, 2$. We will treat the cases $n = 2k$ and $n = 2k+1$ separately.

Let $n = 2k$. Let $p_1 = (-k, -k+1, \dots, -1, 1, 2, \dots, k)$ and $p_2 = (-k+1, -k+2, \dots, -1, 1, 2, \dots, k+1)$. For a symmetric homogeneous polynomial s of degree 2, let $f_i(s) = s(p_i)$, where $i = 1, 2$. Then

$$\begin{vmatrix} f_1(s_2) & f_1(s_1^2) \\ f_2(s_2) & f_2(s_1^2) \end{vmatrix} = \begin{vmatrix} \frac{(2k+1)(k+1)k}{3} & 0 \\ \frac{(2k-1)k(k-1)}{3} + k^2 + (k+1)^2 & (2k+1)^2 \end{vmatrix} = \frac{(2k+1)^3(k+1)k}{3} \neq 0.$$

Thus, f_1 and f_2 are linearly independent. We have $f_1(L) = L(p_1) = 0 = R(p_1) = f_1(R)$ and $f_2(L) = L(p_2) = (2k+1)^2 = R(p_2) = f_2(R)$. This finishes the proof for $n = 2k$. For $n = 2k+1$, take $p_1 = (-k, -k+1, \dots, k)$ and $p_2 = (-k+1, -k+2, \dots, k+1)$. The rest of the proof is analogous to the case $n = 2k$. Proofs of other identities are finished similarly.

References

[1] A. Berenstein, A. Zelevinsky, Triple multiplicities for $\mathrm{sl}(r+1)$ and the spectrum of the exterior algebra of the adjoint representation, J. Algebraic Combin. 1 (1992) 7–22.
[2] F. Beukers, G. Heckman, Monodromy for the hypergeometric function ${}_nF_{n-1}$, Invent. Math. 95 (1989) 325–354.
[3] A. Bolibrukh, The 21st Hilbert Problem for Linear Fuchsian Systems, in: Proceedings of the Steklov Institute of Mathematics, Vol. 206, American Mathematical Society, Providence, RI, 1995.
[4] W. Crawley-Boevey, On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero, preprint arXiv:math.LA/0103101, 15 March 2001.
[5] M. Dettweiler, S. Reiter, An algorithm of Katz and its applications to the inverse Galois problem, J. Symbolic Comput. 30 (2000) 761–798.
[6] H. Derksen, J. Weyman, Semi-invariants of quivers and saturation for Littlewood–Richardson coefficients, J. Amer. Math. Soc. 13 (3) (2000) 456–479.
[7] H. Derksen, J. Weyman, On the σ-stable decomposition of Quiver representations, preprint (available on Derksen's homepage).
[8] W. Fulton, Eigenvalues of sums of Hermitian matrices (after Klyachko), Séminaire Bourbaki 845, June 1998.
[9] W. Fulton, Young Tableaux with Applications to Representation Theory and Geometry, Cambridge University Press, Cambridge, 1997.
[10] O. Gleizer, A. Postnikov, Littlewood–Richardson Coefficients via Yang–Baxter Equation, IMRN 14 (2000) 740–774.
[11] Y. Haraoka, Canonical forms of differential equations free from accessory parameters, SIAM J. Math. Anal. 25 (4) (1994) 1203–1226.
[12] Y. Haraoka, Monodromy representations of systems of differential equations free from accessory parameters, SIAM J. Math. Anal. 25 (6) (1994) 1595–1621.
[13] N. Katz, Rigid local systems, in: Annals of Mathematics Studies, Vol. 139, Princeton University Press, Princeton, 1996.
[14] A. Klyachko, Stable bundles, representation theory, and hermitian operators, Selecta Math. 4 (1998) 419–445.
[15] A. Knutson, T. Tao, The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products I: proof of the saturation conjecture, J. Amer. Math. Soc. 12 (4) (1999) 1055–1090.
[16] A. Knutson, T. Tao, C. Woodward, The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products II: facets of the Littlewood–Richardson cone, preprint.
[17] V. Kostov, Monodromy groups of regular systems on Riemann's sphere, prepublication N 401 de l'Universite de Nice, 1994; Encyclopedia Math. Sci., Springer, Berlin, to appear.
[18] V. Kostov, On the existence of monodromy groups of fuchsian systems on Riemann's sphere with unipotent generators, J. Dyn. Control Systems 2 (10) (1996) 125–155.
[19] V. Kostov, On the Deligne–Simpson problem, manuscript, 47pp. Electronic preprint math.AG/0011013, Monodromy in problems of algebraic geometry and differential equations, Proceedings of the Steklov Institute, Vol. 238, 2002.
[20] V. Kostov, On some aspects of the Deligne–Simpson problem, manuscrit, 48 pp, Trudy Seminara Arnol'da, electronic preprint math.AG/0005016.
[21] V. Kostov, On the Deligne–Simpson problem C.R.A.S. Paris, t. 329, Sér. I, 1999, pp. 657–662.
[22] V. Kostov, Some examples of rigid representations, Serdica Math. J. 26 (2000) 253–276 electronic preprint math.AG/0006021.
[23] V. Kostov, Some examples related to the Deligne–Simpson problem, in: M. Mladenov, G.L. Naber (Eds.), Proceedings of Second International Conference on Geometry, Integrability and Quantization, St. Constantine and Elena, Bulgaria, June 7–15, 2000, ISBN 954-90618-2-5, Coral Press, Sofia, 2001, pp. 208–227, electronic preprint math.AG/0011015.
[24] V. Kostov, The Deligne–Simpson problem for zero index of rigidity, in: S. Dimiev, K. Sekigawa (Eds.), Perspectives in Complex Analysis, Differential Geometry and Mathematical Physics, Proceedings of the Fifth International Workshop on Complex Structures and Vector Fields, St

Constantin, Bulgaria, 3–9 September 2000, World Scientific, Singapore, 2001, pp. 1–35, electronic preprint math.AG/0011107.
[25] V. Kostov, Examples illustrating some aspects of the Deligne–Simpson problem, Serdica Math. J. 27 (2) (2001) 143–158 electronic preprint math.AG0101141.
[26] P. Magyar, J. Weyman, A. Zelevinsky, Multiple flag varieties of finite type, Adv. Math. 141 (1999) 97–118.
[27] K. Okubo, On the group of Fuchsian equations, Seminar Reports of Tokyo Metropolitan University, 1987.
[28] A. Schofield, General representations of Quivers, Proc. London Math. Soc. 65 (3) (1992) 46–64.
[29] C. Simpson, Products of matrices, Amer. Math. Soc. Proc. 1 (1992) 157–185.
[30] V. Varadarjan, Meromorphic differential equations, Expositiones Math. 9 (2) (1991) 97–188.
[31] T. Yokoyama, On an irreducibility condition for hypergeometric systems, Funk. Ekvac. 38 (1995) 11–19.
[32] A. Zelevinsky, Littlewood–Richardson Semigroups, MSRI Preprint no. 1997-044.