Some Properties of Convex Fuzzy Sets

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1. INTRODUCTION

In the basic and classical paper [10], where the important concept of fuzzy set was first introduced, Zadeh developed a basic framework to treat mathematically the fuzzy phenomena or systems which, due to intrinsic indefiniteness, cannot themselves be characterized precisely. He pays special attention to the investigation of the convex fuzzy sets which covers nearly the second half of the space of the paper. The results of the investigation [10] are mainly as follows: (1) The separation theorem; and (2) The theorem on the shadows of convex fuzzy sets. The revised correct version of the separation theorem has been given [9] by employing induced fuzzy topology. Using the concept of fuzzy hyperplane given by himself, Lowen has established some further separation theorems for convex fuzzy sets [6]. Concerning the theorem of shadow of convex fuzzy sets, Zadeh has made further investigation [11]. But in this respect, there still exist some drawbacks which will be shown by a counterexample in the present paper. Perhaps the lack of fuzzy topological assumption in the above mentioned results leads to the appearance of these drawbacks. Such a situation seems to be natural in the early stage of development of fuzzy set theory. Adding some assumptions about fuzzy topology, now we yield several positive results on the shadows of convex fuzzy sets. Finally we shall give simple and direct proofs of two theorems that describe the relationships between the fuzzy convex cones and the fuzzy linear subspaces and that have already been presented by Lowen [6]. The present proofs do not appeal to his representation theorem.

For simplicity, we consider only the convex fuzzy sets defined on the Euclidean space in this paper. But it is not difficult to generalize most of the results obtained in the paper to the case that convex fuzzy sets are defined in linear space over real field or complex field.
2. Preliminaries

Throughout this paper I denotes the unit interval $[0, 1]$, $E$ the Euclidean space of dimension $n$, and $Y$ an ordinary (crisp) nonempty set. A map $\lambda$ from $Y$ to $I$ is called a fuzzy set on $Y$, denoted usually by a lower case Greek letter. The ordinary set $\{ y \in Y: \lambda(y) > 0 \}$ is called the support of $\lambda$ and denoted by supp $\lambda$. The fuzzy set $\lambda'$ defined by $\lambda'(y) = 1 - \lambda(y)$ is called the complement of $\lambda$. For any family $\mathcal{B} = \{ \lambda_j: j \in J \}$ of fuzzy sets on $Y$, we define the intersection $\inf \mathcal{B}$ and the union $\sup \mathcal{B}$, respectively, by the formulae

$$\inf \mathcal{B}(y) = \inf \{ \lambda_j(y): j \in J \},$$
$$\sup \mathcal{B}(y) = \sup \{ \lambda_j(y): j \in J \}.$$ 

For real $a$, $\lambda - a$ denotes the map defined by $(\lambda - a)(y) = \lambda(y) - a$. We define a fuzzy topological space as a pair $(Y, \mathcal{F})$, where $\mathcal{F} \subseteq I^Y$ (all maps from $X$ to $I$) and $\mathcal{F}$ is closed under arbitrary union and finite intersection. In general, the fuzzy topology $\mathcal{F}$ does not include all constant maps, so it is different from the one given in $[S]$. A set is called open if it is in $\mathcal{F}$ and closed if its complement is in $\mathcal{F}$. Unless otherwise stated, the fuzzy topology on the Euclidean space $E$ will refer to induced fuzzy topology $[9]$, i.e., the family of all lower semicontinuous function in $E$. For a more detailed account of the concepts outlined above, the reader is referred to $[7, 10]$.

DEFINITION 1. The fuzzy set $\lambda$ on $E$ is said to be convex fuzzy set iff for all $x, y \in E$ and $a \in I$,

$$\lambda(ax + (1-a)y) \geq \lambda(x) \wedge \lambda(y).$$

It is easy to see that $\lambda$ is a convex fuzzy set iff there exists a dense subset $D$ of $I$ and for each $a \in D$, $\lambda^{-1}[a, 1]$ (or $\lambda^{-1}(a, 1)$) is convex. (These equivalence of fuzzy convexity was essentially shown in $[10]$, see also $[6$, Proposition 6.1$]$.)

DEFINITION 2. Suppose $\lambda$ is a fuzzy set on $E$. The fuzzy convex hull of $\lambda$ is defined by

$$\text{conv } \lambda = \inf \{ v \geq \lambda: v \text{ convex fuzzy} \} = \text{smallest convex fuzzy set containing } \lambda.$$

Since the intersection of some convex fuzzy sets is still convex fuzzy, it is obvious that for each $\lambda$, its fuzzy convex hull $\text{conv } \lambda$ always exists. Further-
more, as shown in [6], if for any \( x \in E \) and \( p \in \mathbb{N} \) (\( \mathbb{N} \) denotes the set of positive integer) put

\[
C(x, p) = \left\{ \{x_1, \ldots, x_p\} \subseteq E: \text{there exist } a_i \in I, \sum_{i=1}^{p} a_i = 1, x = \sum_{i=1}^{p} a_i x_i \right\}
\]

then

\[
\text{conv} \lambda(x) = \sup_{p \in \mathbb{N}} \sup_{A \in C(x, p)} \inf \{\lambda(y): y \in A\}.
\]

**DEFINITION 3.** A fuzzy set \( \lambda \) on \( E \) is a fuzzy subspace iff for all \( x, y \in E \) and reals \( a, b \),

\[
\lambda(ax + by) \geq \lambda(x) \land \lambda(y).
\]

**LEMMA 1.** \( \lambda \) is fuzzy subspace iff \( \lambda \) satisfies the following three conditions:

1. \( \lambda(0) = \sup\{\lambda(x): x \in E\} \).
2. For each \( x \in E \) and real \( a \neq 0 \), \( \lambda(ax) = \lambda(x) \).
3. \( \lambda \) is fuzzy convex set.

The proof is trivial, see also Proposition 3.3 of [6].

**DEFINITION 4.** A fuzzy set \( \lambda \) on \( E \) is a fuzzy convex cone iff it is convex and for each \( x \in E \) and real \( a > 0 \), \( \lambda(ax) = \lambda(x) \).

We can easily verify that \( \lambda \) is a fuzzy convex cone iff there exists a dense subset \( D \) of \( I \) and for each \( a \in D \), \( \lambda^{-1}[a, 1] \) (\( \lambda^{-1}(a, 1] \), respectively) is the ordinary convex cone in \( E \). (See also Proposition 6.4 of [6].)

**DEFINITION 5.** Let \( H \) be the ordinary hyperplane of the Euclidean space \( E \). The orthogonal projection \( p: E \rightarrow H \) induces a correspondence \( S_H \) from \( I^E \) (all maps from \( E \) to \( I \)) into \( I^H \). Then for each fuzzy set \( \lambda \) on \( E \), the image \( S_H(\lambda) \) is called the shadow of \( \lambda \) on \( H \).

**Remark.** The shadow \( S_H(\lambda) \) can be expressed as \( S_H(\lambda)(y) = \sup\{\lambda(x): x \in E \text{ and } p(x) = y\} \) (cf. Definition 1.1 of [8]).

**DEFINITION 6.** Suppose that \( (Y, \mathcal{F}) \) is a fuzzy topological space. The fuzzy set \( \lambda \) on \( Y \) is said to be fuzzy compact iff for all family \( \mathcal{B} \subseteq \mathcal{F} \) satisfying \( \sup \mathcal{B} \geq \lambda \) and for all \( \varepsilon > 0 \), there exists a finite subfamily \( \mathcal{B}_0 \) such that \( \sup \mathcal{B}_0 \geq \lambda - \varepsilon \).
LEMMA 2. If $\lambda$ is fuzzy compact set on $E$ ($E$ equipped with induced fuzzy topology), then for each $a > 0$, $\lambda^{-1}[a, 1]$ is compact.

Proof. This lemma has been obtained in [6]. Here we modify slightly the original proof given in [6] to apply to other case. Suppose the lemma is not true for some $a > 0$. Then by the structure of the usual Euclidean topology of $E$ there exists a sequence $\{x_n\} \subseteq \lambda^{-1}[a, 1]$ ($n = 1, 2, \ldots$) such that either the subset $x_n$ of $E$ is discrete or the sequence $x_n \rightarrow x \notin \lambda^{-1}[a, 1]$ and in both cases for each $x_n$ there exists an open neighborhood $B_n$ of $x_n$ such that the family $B_n$ are pairwise disjoint and each $B_n$ does not contain $x$. If the limit point $x$ exists, since $\lambda(x) < a$, we take $b$ satisfying $\lambda(x) < b < a$; Otherwise we put $b = 0$. Define

$$v_0(y) = b \quad y \in \{x\} \cup \{x_1, x_2, \ldots\},$$

$$= 1 \quad \text{otherwise},$$

and for $n = 1, 2, \ldots$, define

$$v_n(y) = 1 \quad y \in \bigcup \{B_j: j = 1, 2, \ldots\},$$

$$- 0 \quad \text{otherwise}.$$

Then the family $\mathcal{B} = \{v_m: m = 0, 1, \ldots\}$ is a family of open fuzzy set and $\sup \mathcal{B} > \lambda$. Since $b < a$ we may choose $\varepsilon > 0$ such that $b < a - \varepsilon$. For any finite subfamily $\mathcal{B}_0 = \{v_{n_1}, \ldots, v_{n_k}\}$ of $\mathcal{B}$, put $n > \max\{n_1, \ldots, n_k\}$, it is easy to see that $\sup \mathcal{B}_0(x_n) = b < a - \varepsilon \leq \lambda(x_n) - \varepsilon$. This, however, is in contradiction with the fuzzy compactness of $\lambda$. Thus $\lambda^{-1}[a, 1]$ is compact.

Remark. Under the assumption of the lemma, in general, the set $\lambda^{-1}[a, 1]$ is not compact.

DEFINITION 7. Let $(Y, \mathcal{T})$ be a fuzzy topological space and $\lambda \in \mathcal{P}$. The subfamily $\mathcal{B}$ of $\mathcal{T}$ is said to be the open $Q$-cover of $\lambda$ iff for each $y \in \text{supp } \lambda$, $\lambda(y) + \sup \mathcal{B}(y) > 1$. The fuzzy set $\lambda$ is said to be $Q$-compact (resp. with $\mathcal{T}$) iff for any open $Q$-cover $\mathcal{B}$ of $\lambda$ there is a finite subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $\mathcal{B}_0$ is $Q$ cover of $\lambda$ [4].

3. SHADOW OF FUZZY SET

In [10; p. 350] Zadeh asserts that supposing $\lambda$ and $\mu$ are convex fuzzy set on $E$, $S_H(\lambda) = S_H(\mu)$ for each hyperplane $H$ of $E$ implies $\lambda = \mu$. Furthermore he also claims that for a pair of fuzzy sets $\lambda$ and $\mu$, $S_H(\lambda) = S_H(\mu)$ for each hyperplane $H$ of $E$ implies $\text{conv } \lambda = \text{conv } \mu$. The following counter example will show that the above assertions are incorrect.
Counterexample. For simplicity, we are concerned only with the case that $E$ is the Euclidean plane. (When dim $E \geq 3$, the counterexample may be similarly constructed. As for dim $E = 1$, the above assertion is obviously false. In fact, in the original argument presented [10], the hypothesis that dim $E \geq 2$ had been implicitly assumed.) Let $A$ be subset of plane $\{(x, y):$ either $0 \leq x < 1$ and $0 \leq y < 1$ or $x = 1$ and $0 \leq y < \frac{1}{2}\}$, the point $p = (1, \frac{1}{2})$ and $B = A \cup \{p\}$. Suppose that $\lambda$ and $\mu$ denote the characteristic functions of $A$ and $B$, respectively. Then $\lambda \neq \mu$ and both are the (fuzzy) convex sets on $E$. For any hyperplane $H$ of $E$, i.e., straight line $H$, it is obvious that any line which goes through the point $p$ and is orthogonal to $H$ will intersect with the subset $A$. Therefore for each point $y$ of $H$, the fuzzy set $S_H(\lambda)$ and $S_H(\mu)$ will take the same value (either 1 or 0). That is to say, $S_H(\lambda) = S_H(\mu)$.

Adding some hypotheses on fuzzy topology, we are able to recast the assertion in the following correct form.

**Theorem 1.** Let $\lambda$ and $\mu$ be convex fuzzy sets on $E$ with dim $E \geq 2$ and $S_H(\lambda) = S_H(\mu)$ for each hyperplane $H$. If $\lambda$ and $\mu$ are open or closed sets relative to induced fuzzy topology (maybe one is open and another is closed), then $\lambda = \mu$.

**Proof.** Suppose $\lambda \neq \mu$. There is $x \in E$ such that $\lambda(x) \neq \mu(x)$. Without loss of generality, we may assume $a = \lambda(x) > \mu(x) = b$. Put $c = (a + b)/2$. For the fuzzy open (closed, respectively) set $\mu$, $B = \mu^{-1}(c, 1]$ (or $B = \mu^{-1}[c, 1]$), respectively) is open (resp. closed) convex set in $E$ and $x \notin B$. By the convexity theory [2], there is hyperplane $F$ through $x$ such that $F \cap B = \emptyset$. Choose a hyperplane $H$ which goes through $x$ and is orthogonal to $F$ (note that dim $E \geq 2$, so such $H$ exists). Then $S_H(\lambda)(x) > a$ and, since $F \cap B = \emptyset$, $\max\{\mu(y): y \in F\} \leq c$, hence $S_H(\mu)(x) \leq c$. That is to say, $S_H(\lambda) \neq S_H(\mu)$, which contradicts the hypothesis.

**Definition 8.** Let $\lambda$ be a fuzzy set on $E$. The intersection of all the convex and open (resp. closed) fuzzy sets containing $\lambda$ is called the fuzzy open-convex (resp. closed-convex) hull of $\lambda$.

The fuzzy closed-convex hull of $\lambda$ is obviously the smallest closed convex fuzzy set containing $\lambda$ but the fuzzy open-convex hull may be not open. Both fuzzy open-convex hull and closed-convex hull of $\lambda$ contain the fuzzy set $\text{conv } \lambda$. Their constructions can be described as follows.

**Proposition 1.** Let $\lambda$ be the fuzzy set on $E$, $\lambda_a = \lambda^{-1}(a, 1]$ for each
In the Euclidean space $E$, we denote the collection of all the ordinary open and convex subsets containing $\lambda_a$ by

$$\{\lambda(a, j): j \text{ belongs to some index set } J_a\}$$

put

$$\Gamma_{a,j}(x) = \begin{cases} 1 & x \in \lambda(a, j), \\ a & \text{otherwise,} \end{cases}$$

then the fuzzy open-convex hull of $\lambda$ is the fuzzy set

$$\Gamma = \bigcap \{\Gamma_{a,j}: a \in [0, 1), j \in J_a\}.$$  

**Proof.** Because each $\Gamma_{a,j} \geq \lambda$ and is an open convex set, it is sufficient to show that if a fuzzy open convex set $\psi \geq \lambda$, then $\psi \geq \Gamma$. In fact, for each $x \in E$, put $\psi(x) = a$. If $a = 1$, obviously $\psi(x) \geq \Gamma(x)$. If $a < 1$, then $x \notin \psi^{-1}(a, 1]$. By $\psi \geq \lambda$, $\psi^{-1}(a, 1] \supseteq \lambda_a$, i.e., the open convex set $\psi^{-1}(a, 1]$ contains $\lambda_a$. Hence there is $j \in J_a$ such that $\psi^{-1}(a, 1] = \lambda(a, j)$. Since $x \notin \psi^{-1}(a, 1]$, so $\Gamma_{a,j}(x) = a$ and $\Gamma(x) \leq a = \psi(x)$.

**Proposition 2.** Let $\lambda$ be the fuzzy set on $E$, $\lambda_a = \lambda^{-1}[a, 1]$ for each $a \in I$. In the Euclidean space $E$, we denote the smallest closed convex ordinary subset containing $\lambda_a$ by $\lambda_a$. Put

$$\Omega_a(x) = \begin{cases} 1 & x \in \lambda_a, \\ a & \text{otherwise,} \end{cases}$$

then the fuzzy closed-convex hull of $\lambda$ is the fuzzy set

$$\Omega = \bigcap \{\Omega_a: a \in I\}.$$  

**Proof.** It is clear that each $\Omega_a$ is a fuzzy closed convex set containing $\lambda$. Now it is sufficient to show that if a fuzzy closed convex set $\psi \geq \lambda$, then $\psi \geq \Omega$. In fact, for each $x \in E$, put $a = \psi(x)$. If $a = 1$, naturally $\psi(x) \geq \Omega(x)$. If $a < 1$, then the set $B = \{b \in I: b > a\}$ is not empty. For each $b \in B$, $x \notin \psi^{-1}[b, 1] = \lambda_b$. Since $\lambda_b$ is convex closed set in $E$, so $\lambda_b = \lambda_b$. Hence $\Omega_a(x) = b$ by $x \notin \lambda_b$ and $\Omega(x) \leq b$. Because $b$ may be any number in $B$, so $\Omega(x) \leq a = \psi(x)$.

**Theorem 2.** Let $\lambda$ and $\mu$ be fuzzy sets on $E$ with $\dim E \geq 2$ and $S_H(\lambda) = S_H(\mu)$ for any hyperplane $H$ of $E$. If $\lambda$ and $\mu$ denote their fuzzy open-convex (closed-convex) hull, respectively, then $\lambda = \mu$. 
Proof. Suppose $\lambda \neq \mu$, that is to say, for some $x \in E$, $\lambda(x) \neq \mu(x)$. Without loss of generality, we may assume $a = \lambda(x) > \mu(x) = b$. Choose $c \in I$ such that $a > c > b$. Choose $c \in I$ such that $\lambda(x) < c$ and Proposition 1 (using also the notation there), we have $c \in I$ and $\lambda(x) < c$. Put $B = \Gamma_{e,j}^{-1}([c, 1])$. (From Proposition 2, we have $c \in I$ such that $\lambda(x) < c$ and $\Omega_{e}^{-1}(c, 1] = B$, respectively.) The subset $B$ is ordinary open (resp. closed) convex set and $x \notin B$. We say that $\sup\{ \lambda(y) : y \notin B \} > c$. In fact, on the contrary, we get a fuzzy set $v$ defined by

$$v(y) = \begin{cases} 1 & y \in B, \\ c & \text{otherwise,} \end{cases}$$

satisfying $v \geq \lambda$. $v$ is fuzzy open (resp. closed) convex set. So $v \geq \lambda$. By $x \notin B$ we have $c = v(x) = \lambda(x) = a$ which contradicts the fact: $a > c$. Now we may choose the point $z \in B$ such that $\lambda(z) < c$. From the convexity theory, in the case that $B$ is either open convex set or closed convex set, there exists a hyperplane $F$ through the point $z$ such that $F \cap B = \emptyset$. Take a hyperplane $H$ through the point $z$ and orthogonal to $F$. Then $S_{H}(\lambda)(z) > \lambda(z) > c$. On the other hand, since $\mu \leq \Gamma_{e,j}$ (resp. $\mu \leq \Omega_{e}$), hence

$$\mu^{-1}(c, 1] \subseteq \Gamma_{e,j}^{-1}(c, 1] \subseteq B(\mu^{-1}(c, 1] \subseteq B.$$

Therefore $F$ does not intersect with $\mu^{-1}(c, 1]$ (resp. $\mu^{-1}(c, 1]$) and $S_{H}(\mu)(z) \leq c$. $S_{H}(\lambda) \neq S_{H}(\mu)$, a contradiction!

Remark. The inverse statement of Proposition 2 is not true. That is to say, the fact $\lambda = \mu$ does not imply $S_{H}(\lambda) = S_{H}(\mu)$. In fact, even in the ordinary convex set theory, the corresponding counterexample is easy to construct.

In the following we shall investigate the relationship between the properties of shadow and the compactness (the fuzzy compactness and $Q$-compactness).

**Theorem 3.** Let $\lambda$ and $\mu$ be the convex fuzzy sets on $E$ with $\dim E \geq 2$ and $\lambda(x) = \lambda(x)$ for each hyperplane $H$. If $\lambda$ and $\mu$ are fuzzy compact relative to the induced topology $\mathcal{F}$, then $\lambda = \mu$.

Proof. From the argument of Theorem 1 it is not difficult to see that if for each $a \in (0, 1]$, $v^{-1}[a, 1]$ is the closed convex subset of $E$, then the proof presented there can be continued throughout regardless of closedness or openness of $v$. Now by Lemma 2 for each $a \in (0, 1]$, $v^{-1}[a, 1]$ is compact, hence is closed set. So the present proof can be given similar to the one of Theorem 1.
In the following for the concept of fuzzy topological subspace (simply, subspace) we refer to [7].

**Proposition 3.** Let \((Y, \mathcal{F})\) be fuzzy topological space, \(\lambda\) the fuzzy set on \(Y\). If there exists an increasing positive real sequence \(\varepsilon_n \to 0\) such that each fuzzy set \((\lambda' + \varepsilon_n)\) is \(Q\)-compact in the subspace \(\text{supp} \lambda\), then \(\lambda\) is fuzzy compact, where \((\lambda' + \varepsilon_n)\) is defined by \((\lambda' + \varepsilon_n)(y) = \min\{1, 1 + \varepsilon_n - \lambda(y)\}\), \(y \in \text{supp} \lambda\).

**Proof.** Suppose that the subfamily \(\mathcal{B} \subseteq \mathcal{F}\) satisfies \(\bigcup \mathcal{B} \geq \lambda\) and \(\varepsilon > 0\). We may choose \(\varepsilon_n\) such that \(\varepsilon_n < \varepsilon\). For each fuzzy set \(\mu \in \mathcal{B}\) the restriction of \(\mu\) to the subspace \(\text{supp} \lambda\) is denoted by \(\bar{\mu}\). So we have the family

\[
\mathcal{B} = \{\bar{\mu} : \mu \in \mathcal{B}\}.
\]

We say that \(\mathcal{B}\) is an open \(Q\)-cover of \((\lambda' + \varepsilon_n)\) in the subspace \(\text{supp} \lambda\). In fact, take any \(x \in \text{supp} \lambda\). If \((\lambda' + \varepsilon_n)(x) = 1\), then by \(\bigcup \mathcal{B}(x) \geq \lambda(x) > 0\) we have \(\bigcup \mathcal{B}(x) + (\lambda' + \varepsilon_n)(x) > 1\); If \((\lambda' + \varepsilon_n)(x) < 1\), then \((\lambda' + \varepsilon_n)(x) = 1 - \lambda(x) + \varepsilon_n\). Since \(\bigcup \mathcal{B} \geq \lambda\), hence \(\bigcup \mathcal{B}(x) + (\lambda' + \varepsilon_n)(x) = \bigcup \mathcal{B}(x) - \lambda(x) + 1 + \varepsilon_n \geq 1 + \varepsilon_n > 1\). Thus \(\mathcal{B}\) indeed is an open \(Q\)-cover of \((\lambda' + \varepsilon_n)\). By the \(Q\)-compactness of \((\lambda' + \varepsilon_n)\) there exists a finite subfamily \(\mathcal{B}_0\) of \(\mathcal{B}\) such that for each \(x \in \text{supp} \lambda\), \(\bigcup \mathcal{B}_0(x) + (\lambda' + \varepsilon_n)(x) > 1\). Now when \(1 - \lambda(x) + \varepsilon_n \leq 1\) we have \((\lambda' + \varepsilon_n)(x) = 1 - \lambda(x) + \varepsilon_n\), thus \(\bigcup \mathcal{B}_0(x) > \lambda(x) - \varepsilon_n\); When \(1 - \lambda(x) + \varepsilon_n > 1\), we have \(\lambda(x) < \varepsilon_n\), hence \(\bigcup \mathcal{B}_0(x) \geq 0 \geq \lambda(x) - \varepsilon_n\). To sum up, we always have \(\bigcup \mathcal{B}_0(x) \geq \lambda(x) - \varepsilon_n\) for each \(x \in \text{supp} \lambda\). Put \(\mathcal{B}_0 = \{\mu \in \mathcal{B} : \bar{\mu} \in \mathcal{B}_0\}\). Then \(\mathcal{B}_0\) is finite subfamily of \(\mathcal{B}\) and \(\bigcup \mathcal{B}_0 \geq \lambda - \varepsilon_n > \lambda - \varepsilon\). By Definition 6, \(\lambda\) is a fuzzy compact set in \((Y, \mathcal{F})\).

From Proposition 3 and Theorem 3, we have the following

**Corollary.** Let \(\lambda\) and \(\mu\) be fuzzy convex sets on \(E\) with \(\dim E \geq 2\). If there exists an increasing positive real sequence \(\varepsilon_n \to 0\) such that each \((\lambda' + \varepsilon_n)\) and each \((\mu' + \varepsilon_n)\) are \(Q\)-compact in subspace \(\text{supp} \lambda\) and subspace \(\text{supp} \mu\), respectively, then \(S_H(\lambda) = S_H(\mu)\) for each hyperplane \(H\) implies \(\lambda = \mu\).

**Definition 9.** Let \((Y, \mathcal{F})\) be a fuzzy topological space and \(\lambda\) the fuzzy set on \(Y\). The subfamily \(\mathcal{B}\) of \(\mathcal{F}\) is said to be the complete \(Q\)-cover of \(\lambda\) iff for each \(y \in \text{supp} \lambda\), \(\sup \mathcal{B}(y) + \lambda(y) > 1\) and for each \(y \in Y \setminus \text{supp} \lambda\), there is a \(v \in \mathcal{B}\) such that \(v(y) = 1\). The fuzzy set on \(Y\) is said to be complete \(Q\)-compact iff for each complete \(Q\)-cover \(\mathcal{B}\) of \(\lambda\), there is a finite subfamily \(\mathcal{B}_0\) of \(\mathcal{B}\) such that \(\mathcal{B}_0\) is complete \(Q\)-cover of \(\lambda\).

**Remark.** When \(\lambda = \emptyset\), the complete \(Q\)-compactness of \(\lambda\) is equivalent to \(1^*\)-compactness\(^1\) of \((Y, \mathcal{F})\).
**Proposition 4.** If the fuzzy set \( \lambda' \) is complete \( Q \)-compact on \( E \) \( (E \) equipped with induced fuzzy topology), then for each \( a > 0 \), \( \lambda^{-1}[a, 1] \) is compact.

**Proof.** The argument of Lemma 2 can apply to the present case with some obvious modification.

**Theorem 4.** Let \( \lambda \) and \( \mu \) be fuzzy convex sets on \( E \) with \( \dim E \geq 2 \). If \( \lambda' \) and \( \mu' \) are complete \( Q \)-compact and for each hyperplane \( H \), \( S_{H}(\lambda) = S_{H}(\mu) \), then \( \lambda = \mu \).

**Proof.** Analogous to the proof of Theorem 3.

4. **Fuzzy Convex Cone and Fuzzy Subspace**

In the paragraph we shall give simple and direct proof of two theorems which describe the relation between the fuzzy convex cone and fuzzy subspace and are established in [6] via the so-called representation theorem presented in [6].

**Theorem 5.** Suppose that \( \lambda \) is a fuzzy convex cone and \( \lambda(0) = \sup \{ \lambda(x): x \in E \} \). Then there exists a smallest fuzzy subspace \( \bar{\lambda} \) containing \( \lambda \) and \( \bar{\lambda} \) can be defined by

\[
\bar{\lambda}(x) = \sup \{ \lambda(z) \wedge \lambda(y): z - y = x \}.
\]

**Proof.** We first point out that the fuzzy set defined by the above-mentioned formula is a subspace. In fact, by definition of \( \bar{\lambda} \) we have \( \bar{\lambda}(x) = \bar{\lambda}(-x) \). Now we need to prove the following formula holds for any reals \( a_1 \) and \( a_2 \) and any points \( x_1 \) and \( x_2 \):

\[
\bar{\lambda}(a_1 x_1 + a_2 x_2) \geq \bar{\lambda}(x_1) \wedge \bar{\lambda}(x_2). \tag{*}
\]

If some \( a_i = 0 \), the formula (*) follows directly from the definition of the fuzzy convex cone. If \( a_1 \cdot a_2 \neq 0 \), put \( y_i = (\text{sgn} a_i) x_i \), then \( a_i x_i = |a_i| y_i \) \((i = 1, 2)\) and

\[
\bar{\lambda}(a_1 x_1 + a_2 x_2) = \lambda(|a_1| y_1 + |a_2| y_2)
= \sup \{ \lambda(w) \wedge \lambda(z): z - w = a_1 x_1 + a_2 x_2 \}
= \sup \{ \lambda(|a_1| w_1 + |a_2| w_2)
\wedge \lambda(|a_1| z_1 + |a_2| z_2): z_1 - w_1 = y_1, z_2 - w_2 = y_2 \}.
\]

In view of the definition of the fuzzy convex cone

\[
\lambda(|a_1| w_1 + |a_2| w_2) \geq \lambda(w_1) \wedge \lambda(w_2)
\]
and

\[ \lambda(|a_1| z_1 + |a_2| z_2) \geq \lambda(z_1) \land \lambda(z_2), \]

we have

\[ \lambda(a_1 x_1 + a_2 x_2) \]

\[ \geq \sup\{ \lambda(w_1) \land \lambda(w_2) \land \lambda(z_1) \land \lambda(z_2) : z_1 - w_1 = y_1, z_2 - w_2 = y_2 \} \]

\[ = \sup\{ \lambda(w_1) \cap \lambda(z_1) : z_1 - w_1 = y_1 \} \]

\[ \land \sup\{ \lambda(w_2) \cap \lambda(z_2) : z_2 - w_2 = y_2 \} \]

\[ = \lambda(y_1) \land \lambda(y_2) = \lambda(x_1) \land \lambda(x_2), \]

i.e., the formula (*) still holds. Next, since \( \lambda(0) \) is the greatest \( \lambda(x) \), hence

\[ \lambda(x) = \sup\{ \lambda(z) \land \lambda(y) : z - y = x \} \]

\[ \geq \lambda(x) \land \lambda(0) \]

\[ = \lambda(x), \]

i.e., \( \lambda \geq \lambda \). Finally, let \( \nu \) be any fuzzy subspace containing \( \lambda \). For each \( x \in E \) we take the points \( y \) and \( z \) such that \( y - z = x \). Then

\[ \nu(x) = \nu(z - y) \geq \nu(z) \land \nu(y) \geq \lambda(z) \land \lambda(y). \]

By the definition of \( \lambda \), that is, \( \nu(x) \geq \lambda(x) \); thus \( \lambda \) is exactly the smallest fuzzy subspace which contains \( \lambda \).

**Theorem 6.** Suppose that \( \lambda \) is a fuzzy convex cone and \( \lambda(0) = \sup\{ \lambda(x) : x \in E \} \). Then there exists the largest fuzzy subspace \( \lambda \) contained in \( \lambda \) and \( \lambda \) can be defined by

\[ \lambda(x) = \lambda(x) \land \lambda(-x), \quad x \in E. \]

**Proof.** The fuzzy set \( \lambda \) defined by the above-mentioned formula is a fuzzy subspace. In fact, for any reals \( a_1 \) and \( a_2 \) and any points \( x_1 \) and \( x_2 \), put \( y_i = (\sgn a_i) x_i, \quad i = 1, 2 \), then \( a_i x_i = |a_i| y_i \). If some \( a_i = 0 \), then \( \lambda(a_1 x_1 + a_2 x_2) \) is greater than or equal to \( \lambda(x_1) \land \lambda(x_2) \). If \( a_1 \cdot a_2 \neq 0 \), we have

\[ \lambda(a_1 x_1 + a_2 x_2) = \lambda(|a_1| y_1 + |a_2| y_2) \]

\[ = \lambda(|a_1| y_1 + |a_2| y_2) \land \lambda(|a_1| (-y_2)) \]

\[ \geq \lambda(y_1) \land \lambda(y_2) \land \lambda(-y_1) \land \lambda(-y_2) \]

\[ = \lambda(y_1) \land \lambda(y_2) = \lambda(x_1) \land \lambda(x_2), \]
i.e., $\lambda$ is a fuzzy subspace. Next, it is obvious that $\lambda \leq \lambda$. Finally, let $v$ be a fuzzy subspace contained in $\lambda$. Then $v(x) = v(x) \wedge v(-x) \leq \lambda(x) \wedge \lambda(-x) = \lambda(x)$, i.e., $v \leq \lambda$.

REFERENCES