Duality on locally convex cones

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Abstract

The theory of locally convex cones as a branch of functional analysis was presented by K. Keimel and W. Roth in [K. Keimel, W. Roth, Ordered Cones and Approximation, Lecture Notes in Math., vol. 1517, Springer-Verlag, Heidelberg, 1992]. We study some more results about dual cones and adjoint operators on locally convex cones. Moreover we introduce the concept of the uniformly precompact sets and discuss their relations with $\sigma$-bounded sets. Some results obtained about inductive limit, projective limit, metrizability and quotients of locally convex cones.

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1. Introduction

An ordered cone is a set $\mathcal{P}$ together with two operations; addition and scalar multiplication for non-negative real numbers $\lambda \geq 0$. The addition is associative and commutative, and there is an element $0 \in \mathcal{P}$ such that $a + 0 = a$ for all $a \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, $1a = a$ and $0a = 0$ for every $a \in \mathcal{P}$. In addition, the cone $\mathcal{P}$ carries a preorder, i.e., a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\lambda a \leq \lambda b$ for all $a, b, c \in \mathcal{P}$ and $\lambda \geq 0$. As equality in $\mathcal{P}$ is obviously such a relation, all results about ordered cones apply to cones without order structures as well. Also $\mathbb{R} = \mathbb{R} \cup \{+\infty\}$ is a preordered cone with respect to usual addition, multiplication and order on $\mathbb{R}$.

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Considering a preorder relation on a cone always gives rise an (abstract) 0-neighborhood system say; \( V \), that consists of some elements of \( P \) with the following properties:

\((v_1)\) \( 0 < v \) for all \( v \in V \).
\((v_2)\) For all \( u, v \in V \) there is \( w \in V \) with \( w \leq u \) and \( w \leq v \).
\((v_3)\) \( u + v \in V \) and \( \lambda v \in V \) whenever \( u, v \in V \) and \( \lambda > 0 \).

Every (abstract) 0-neighborhood system on \( P \) corresponds to three topologies called upper, lower, and symmetric topologies. The neighborhood of an element \( a \in P \) with respect to \( v \in V \) is defined to be

\[ v(a) = \{ b \in P : b \leq a + v \}, \quad (a)v = \{ b \in P : a \leq b + v \}, \]

and \( v(a)v = v(a) \cap (a)v \) in these topologies, respectively. Observe that the symmetric topology is the common refinement of the upper and lower topologies.

If we assume that all elements of \( P \) are bounded below, that is for every \( a \in P \) and \( v \in V \) we have \( 0 \leq a + \rho v \) for some \( \rho > 0 \), then the pair \( (P, V) \) is called a full locally convex cone. A locally convex cone \( (P, V) \) is a subcone of a full locally convex cone, not necessarily containing the (abstract) 0-neighborhood system \( V \).

There is another equivalent useful construction to topologize cones. Let \( P \) be a cone, A collection \( U \) of convex subsets of \( P \times P \) is called a convex quasi-uniform structure on \( P \), if the following hold:

\((u_1)\) \( \Delta \subset U \) for all \( U \in U \), \( \Delta = \{(a, a) : a \in P \} \).
\((u_2)\) For all \( U, V \in U \) there is \( W \in U \) such that \( W \subseteq U \cap V \).
\((u_3)\) \( \lambda U \circ \mu U \subseteq (\lambda + \mu)U \) for all \( \lambda, \mu > 0 \) and \( U \in U \), where \( \lambda U \circ \mu U = \{(a, b) \in P^2 : \exists c \in P \text{ with } (c, a) \in \lambda U \text{ and } (c, b) \in \mu U \} \).
\((u_4)\) \( \lambda U \in U \) for all \( U \in U \) and \( \lambda > 0 \).

We order the cone \( P \) via convex quasi-uniform structure by defining the preorder \( a \leq b \) if and only if \( (a, b) \in U \) for all \( U \in U \). The neighborhood bases for an element \( a \in P \) in the upper and lower topologies are given, respectively, by the sets

\[ U(a) = \{ b \in P : (b, a) \in U \}, \quad (a)U = \{ b \in P : (a, b) \in U \}. \]

Note that the topology induced by the uniform structure \( U_1 = \{ U \cap U^{-1} : U \in U \} \) is the common refinement of the upper and lower topologies, where \( U^{-1} = \{(b, a) : (a, b) \in U \} \).

The notions of an (abstract) 0-neighborhood system \( V \) and a convex quasi-uniform structure \( U \) on a cone \( P \) are equivalent in the following sense:

For a locally convex cone \( (P, V) \) and each \( v \in V \), we put

\[ \tilde{v} = \{(a, b) \in P \times P : a \leq b + v \}. \]

The collection \( \tilde{V} = \{ \tilde{v} : v \in V \} \) is a convex quasi-uniform structure on \( P \), which induces the same upper, lower and symmetric topologies. On the other hand, if \( P \) is a cone with a convex quasi-uniform structure \( U \), then one can find a preorder and an (abstract) 0-neighborhood system \( V \) such that the convex quasi-uniform structure \( \tilde{V} \) is equivalent to \( U \) [1, Chapter I, 5.5].

If \( (P, V) \) is a locally convex cone, the condition that every element \( a \in P \) has to be bounded below translates into, for each \( \tilde{v} \in \tilde{V} \) there is some \( \rho > 0 \) such that \( (0, a) \in \rho \tilde{v} \). On the other hand, if a quasi-uniform structure \( U \) on a cone \( P \) has the extra property...
For locally convex cones $\mathcal{P}$ and $\mathcal{Q}$, with convex quasi-uniform structures $\mathcal{U}$ and $\mathcal{V}$, respectively, a linear mapping $t: \mathcal{P} \to \mathcal{Q}$ is called uniformly continuous ($u$-continuous) if for every $V \in \mathcal{V}$, there is some $U \in \mathcal{U}$ such that $(a, b) \in U$ implies $(t(a), t(b)) \in V$, i.e., $T(U) \subseteq V$, $T = t \times t$. If $\mathcal{V}$ and $\mathcal{W}$ are (abstract) 0-neighborhood systems on $\mathcal{P}$ and $\mathcal{Q}$, $t$ is $u$-continuous if and only if for every $w \in \mathcal{W}$ there is some $v \in \mathcal{V}$, such that $(a, b) \in \tilde{v}$ implies $(t(a), t(b)) \in \tilde{w}$ or equivalently; $t(a) \leq t(b) + w$ whenever $a \leq b + v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$. The set of all $u$-continuous linear functionals $\mu : \mathcal{P} \to \mathbb{R}$ is a cone called the dual cone of $\mathcal{P}$ and denoted by $\mathcal{P}^\ast$. In a locally convex cone $(\mathcal{P}, \mathcal{V})$ the polar $v^\circ$ of $v \in \mathcal{V}$ is defined by $v^\circ = \{ \mu \in \mathcal{P}^\ast : a \leq b + v \implies \mu(a) \leq \mu(b) + 1 \}$. Obviously we have $\mathcal{P}^\ast = \bigcup_{v \in \mathcal{V}} v^\circ$.

W. Roth has extended the theory of locally convex cones in several papers after [1]. We have specially used some notions and definitions from [4] and [6].

In Section 2, we define dual cones, $X$-topology and study their properties via convex quasi-uniform structures. Also we introduce the concept of uniformly precompact sets and study their relations with $\sigma$-bounded subsets. Metrizability of a locally convex cone is discussed and some results obtained.

We define adjoint operator in Section 3 and obtain some results including results related to inductive limits, projective limits and locally convex quotient cones.

2. Dual pairs and dual cones

In [1], dual pair and $X$-topology are defined as following.

**Definition 2.1.** A dual pair $(\mathcal{P}, \mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping

$$(a, x) \mapsto \langle a, x \rangle : \mathcal{P} \times \mathcal{Q} \to \mathbb{R}.$$ 

**Definition 2.2.** Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair and $X$ be a collection of subsets of $\mathcal{Q}$ such that:

(P0) $\inf \{ \langle a, x \rangle : x \in A \} > -\infty$ for all $a \in \mathcal{P}$ and $A \in X$.

(P1) $\lambda A \in X$ for all $A \in X$ and $\lambda > 0$.

(P2) For all $A, B \in X$ there is some $C \in X$ such that $A \cup B \subseteq C$.

For each $A \in X$ we define

$$U_A = \{ (a, b) \in \mathcal{P} \times \mathcal{P} : \langle a, x \rangle \leq \langle b, x \rangle + 1 \text{ for all } x \in A \}. \quad (2.1)$$

The set of all $U_A, A \in X$ is a convex quasi-uniform structure with property (u5) and defines a locally convex structure on $\mathcal{P}$. This is called the $X$-topology on $\mathcal{P}$. For each $A \in X$ we denote by $v_A$ the (abstract) 0-neighborhood induced on $\mathcal{P}$ by $U_A$. Therefore $(a, b) \in U_A$ if and only if $a \leq b + v_A$. Obviously an $X$-topology on $\mathcal{P}$ defines at the same time upper, lower and symmetric topologies on $\mathcal{P}$.

We give some examples.
Example 2.3. (i) Let \((P, Q)\) be a dual pair. For each \(a \in Q\) the collection \(X_a\) of all sets \(\{\lambda_i a : i = 1, \ldots, n\}\), \(n \in N\), \(\lambda_i > 0\), satisfies the properties \((P_0)\)–\((P_2)\). The resulting \(X_a\)-topologies are the coarsest ones.

For the set \(X = \{\lambda_i a : i = 1, 2, \ldots, n\}\), \(\lambda_i > 0\), \(n \in N\), \(a_i \in P\), which is in fact the set of all finite subsets of \(Q\), \(X\)-topology is finer than each \(X_a\)-topology, and is denoted by \(\sigma(P, Q)\).

Note that the dual cone of \(P\) under the \(\sigma(P, Q)\)-topology is not necessary to be \(Q\). For example, if \(P = Q = \mathbb{R}\), considering the dual pair \((P, Q)\) with bilinear form

\[
\langle a, x \rangle = \begin{cases} 
ax, & a, x \neq +\infty; \\
+\infty, & a = +\infty \text{ or } x = +\infty,
\end{cases}
\]

define the linear mapping \(\mu: \mathbb{R} \to \mathbb{R}\) as

\[
\mu(a) = \begin{cases} 
0, & a \neq +\infty; \\
+\infty, & a = +\infty.
\end{cases}
\]

If we put \(A = \{1\}\), then for all \(a, b \in \mathbb{R}\) with \(a \leq b + v_A\), we have \(\mu(a) \leq \mu(b) + 1\) hence \(\mu \in \mathbb{R}^*_a\) but \(\mu \notin \mathbb{R}\).

(ii) Let \((P, V)\) be a locally convex cone. \(A \subset P\) is called bounded if for every \(v \in V\) there exists \(\lambda > 0\) such that

\[a \leq \lambda v \quad \text{and} \quad 0 \leq a + \lambda v \quad \text{for all } a \in A.\]

Also \(A \subset P\) is called internally bounded if for every \(v \in V\) there exists \(\lambda > 0\) such that \(a \leq b + \lambda v\) for all \(a, b \in A\).

The set of all bounded (internally bounded) sets of \(P\) satisfies the properties \((P_0)\)–\((P_2)\) and defines an \(X\)-topology on \(P^*\). For \((P_0)\), let \(A \subset P\) be bounded and \(\mu \in P^*\). Since \(P^* = \bigcup_{v \in V} v^o\), there is \(v \in V\) such that \(\mu \in v^o\). Thus there is some \(\lambda > 0\) such that \(b \leq \lambda v\) and \(0 \leq b + \lambda v\) for all \(b \in A\), hence

\[
\inf \{\mu(b) : b \in A\} \geq -\lambda > -\infty.
\]

It is easy to see that finite unions and positive scalar multiples of (internally) bounded sets are (internally) bounded, hence \((P_1)\) and \((P_2)\) are satisfied. If \(A \subset P\) is internally bounded and \(\mu \in P^*\), there is \(v \in V\) such that \(\mu \in v^o\). For this \(v\), there is some \(\lambda_1 > 0\) such that \(a \leq b + \lambda_1 v\) for all \(a, b \in A\). Now fix \(a \in A\), since \(a\) is lower bounded, there is some \(\lambda_2 > 0\) such that \(0 \leq a + \lambda_2 v\). Hence \(0 \leq b + (\lambda_1 + \lambda_2) v\) for all \(b \in A\) which implies that

\[
\inf \{\mu(b) : b \in A\} \geq -(\lambda_1 + \lambda_2) > -\infty.
\]

(iii) \(A \subset (P, V)\) is called uniformly precompact (u-precompact) if for each \(v \in V\) there exist subsets \(A_1, \ldots, A_n \subset P\) such that \(A \subset \bigcup_{i=1}^n A_i\) and \(A_i \times A_j \subset \tilde{v}\), \(i = 1, 2, \ldots, n\). The set of all u-precompact subsets of \(P\) satisfies the properties \((P_0)\)–\((P_2)\) and defines an \(X\)-topology on \(P^*\). For \((P_0)\), let \(\mu \in P^*\) and choose \(v \in V\) such that \(\mu \in v^o\). Choose \(a_i \in A_i, i = 1, 2, \ldots, n\), and let \(k = \min\{\mu(a_i) - 1 : i = 1, 2, \ldots, n\}\). Then for each \(a \in A\) we have \(\mu(a) \geq k\), hence \(\inf\{\mu(a) : a \in A\} \geq k > -\infty\). \((P_1)\) and \((P_2)\) are clear.

In the same way it is easy to see that the set of all compact subsets of \(P\) with respect to the lower topology (or with respect to the symmetric topology) satisfies the properties \((P_0)\)–\((P_2)\).

Remark 2.4. A locally convex cone \((P, V)\) is said to have strict separation property, in short (SP), if for all \(a, b \in P\) and \(v \in V\) with \(a \notin b + \rho v\) for some \(\rho > 1\), there is \(\mu \in v^o\) such that \(\mu(a) > \mu(b) + 1\).
Every cone $P$ with an $X$-topology has (SP), also if $(P, V)$ is a locally convex cone with (SP), the original topology on $P$ is equivalent with the $X$-topology where $X = \{v^\circ: v \in V\}$ (cf. [1, Chapter II, 3.2 and 3.3]) in the dual pair $(P, P^*)$. This means that, if $(P, V)$ has (SP), then the (abstract) 0-neighborhood systems $V$ and $V' = \{v_{v^\circ}: v \in V\}$ are equivalent. Moreover for each $v \in V$ we have $v^\circ = v_{v^\circ}^\circ$, where $v_{v^\circ} = (v_{v^\circ})^\circ$, i.e., although may be $v \neq v_{v^\circ}$ but their polars are equal. For, if $\mu \in v^\circ$ and $a, b \in P$ with $a \leq b + v_{v^\circ}$, i.e., $(a, b) \in U_{v^\circ}$, we have $(a, \mu) \leq (b, \mu) + 1$ or $\mu(a) \leq \mu(b) + 1$, thus $\mu \in v_{v^\circ}^\circ$. On the other hand, if $\mu \in v_{v^\circ}$ and $a, b \in P$ with $a \leq b + v$, then for each $\lambda \in v^\circ$, we have $\lambda(a) \leq \lambda(b) + 1$, i.e., $(a, b) \in U_{v^\circ}$ or $a \leq b + v_{v^\circ}$, thus $\mu(a) \leq \mu(b) + 1$ which implies that $\mu \in v_{v^\circ}^\circ$.

In general, we have:

**Proposition 2.5.** Let $(P, Q)$ be a dual pair and $X$ be a collection of subsets of $Q$ that satisfies the properties (P0)–(P2). Then for every $A \in X$, $A \subseteq v_A^\circ = v_{v_A}^\circ$.

The proof is an immediate consequence of the strict separation property of $X$-topology.

**Definition 2.6.** Let $(P, V)$ be a locally convex cone. The subset $A \subseteq P^*$ is called uniformly equicontinuous (u-equicontinuous), if there is some $v \in V$ such that for all $a, b \in P$ and $\mu \in A$,

$$a \leq b + v \quad \text{implies} \quad \mu(a) \leq \mu(b) + 1.$$

In other words, the subset $A \subseteq P^*$ is u-equicontinuous if and only if there is some $v \in V$ such that $A \subseteq v^\circ$. Thus for every $v \in V$, $v^\circ$ is a u-equicontinuous subset of $P^*$.

Remark that considering the dual pair $(P, Q)$ and the $X$-topology on $P$, since $A \subseteq v_A^\circ$ for every $A \in X$, the members of $X$ are u-equicontinuous subsets in the dual cone $P^*_X$, where $P^*_X = \bigcup_{A \in X} v_A^\circ$ means the dual cone of $P$ where its (abstract) 0-neighborhood system is defined by $X$.

Considering the dual pair $(P, P^*)$, the set of all u-equicontinuous subsets of the dual cone $P^*$ satisfies the properties (P0)–(P2). It is easy to verify the properties (P1) and (P2). For the property (P0), let $a \in P$ and $A \subseteq P^*$ be u-equicontinuous. By definition there is some $v \in V$ such that $A \subseteq v^\circ$. Choose $\lambda > 0$ such that $0 \leq a + \lambda v$, we have

$$\inf\{\mu(a): \mu \in A\} \geq -\lambda > -\infty.$$

**Theorem 2.7.** Let $(P, V)$ be a locally convex cone with strict separation property (SP), and $X$ be the set of all u-equicontinuous subsets of the dual cone $P^*$. Then the convex quasi-uniform structure $\tilde{V} = \{\tilde{v}: v \in V\}$ is equivalent to the $X$-topology.

**Proof.** Let $V_X$ be the (abstract) 0-neighborhood system induced by $X$, since $V' = \{v_{v^\circ}: v \in V\}$ is equivalent by $V$ (because of (SP) property), it is enough to show that $V'$ is equivalent with $X$-topology. Given $A \in X$, there is some $v \in V$ such that $A \subseteq v^\circ$. Hence $\tilde{v}_{v^\circ} = U_{v^\circ} \subseteq U_A$ which implies that $V'$ is finer than $X$-topology. Conversely, for every $v \in V$, since $v^\circ$ is u-equicontinuous, each $\tilde{v}_{v^\circ} \in V_X$, hence the $X$-topology is also finer than $V'$. $\square$

**Corollary 2.8.** Let $(P, Q)$ be a dual pair, $X$ be a collection of subsets of $Q$ that satisfies properties (P0)–(P2) and $Y = \{B \subseteq P^*_X: B$ is u-equicontinuous$\}$. Then the convex quasi-uniform structure $U_Y = \{U_B: B \in Y\}$ is equivalent to the $X$-topology.
**Proposition 2.9.** Let \((P, Q)\) be a dual pair and \(X\) be the set of all subsets of \(Q\) that satisfy property \((P_0)\). Then \(X\) also satisfies \((P_1)\) and \((P_2)\) and the resulting \(X\)-topology is the finest.

**Proof.** Let \(A \in X\), \(\lambda > 0\), and \(a \in P\), then
\[
\inf \{\langle a, \lambda x \rangle : x \in A \} = \lambda \inf \{\langle a, x \rangle : x \in A \} > -\infty,
\]
so \(\lambda a \in X\). For property \(P_2\), let \(A, B \in X\). Then for each \(a \in P\) we have
\[
\inf \{\langle a, x \rangle : x \in A \cup B \} = \min \{\inf \{\langle a, x \rangle : x \in A \}, \inf \{\langle a, x \rangle : x \in B \} \} > -\infty,
\]
that is \(A \cup B \in X\). \(\square\)

In a locally convex cone \((P, V)\) for each \(a \in P\) we define
\[
\tilde{a} = \bigcap_{v \in V} v(a), \quad \tilde{a}^- = \bigcap_{v \in V} (a)v, \quad \tilde{a}^+ = \bigcap_{v \in V} (a)v(a).
\]
These are closures of \(\{a\} \subseteq P\) with respect to the lower, upper, and symmetric topologies, respectively. For example, \(b \in \tilde{a}\) if and only if \(b \in v(a)\) for all \(v \in V\), this means that for each \(v \in V\), \(\{a\} \cap (b)v \neq \emptyset\), i.e., \(b\) is in the closure of \(\{a\}\) with respect to the lower topology. In particular, \(\tilde{a}\) is a closed subset of \(P\) with respect to the lower topology. Likewise, \(\tilde{a}^-\) and \(\tilde{a}^+\) are closed with respect to upper and symmetric topologies, respectively. \(\tilde{a}_X, \tilde{a}^-_X\) and \(\tilde{a}^+_X\) mean the closures with respect to topologies induced by \(X\). Obviously \(\tilde{a}_X \neq \tilde{a}_Y\) in general for different \(X\) and \(Y\), but we have:

**Proposition 2.10.** Let \((P, Q)\) be a dual pair and \(X\) be the collection of all finite subsets of \(Q\) and \(Y\) be any collection of subsets of \(Q\) containing \(X\) that satisfies the properties \((P_0)–(P_2)\). Then for every \(a \in P\) we have
\[
\tilde{a}_X = \tilde{a}_Y, \quad \tilde{a}^-_X = \tilde{a}^-_Y, \quad \tilde{a}^+_X = \tilde{a}^+_Y.
\]

**Proof.** First, the lower topology induced by \(Y\) is finer than the lower topology induced by \(X\), so \(\tilde{a}_Y \subseteq \tilde{a}_X\). Next, let \(a' \not\in \tilde{a}_Y\). Since \(\tilde{a}_Y = \bigcap_{b \in Y} v_B(a)\), there is some \(b \in Y\) such that \((a', a) \not\in \tilde{a}_X\). Thus \((a', a) \not\in U_B\) and there is an element \(b \in B\) such that \((a', b) > (a, b) + 1\). Put \(A = \{b\} \subseteq X\).

Then \(a' \not\in v_A(a)\) which implies that \(a' \not\in \tilde{a}_X\). Other equalities are proved similarly. \(\square\)

Let \((P, \leq)\) be a preordered cone. If \(V\) and \(V'\) are two (abstract) 0-neighborhood systems such that \(V' \subseteq V\), then the topologies induced by \(V\) on \(P\) would be finer than the topologies induced by \(V'\). So we say that \(V\) is finer than \(V'\). On the other hand, if \(\leq\) and \(\leq'\) are two preorders on \(P\) such that
\[
a \leq b \quad \text{implies} \quad a \leq' b \quad \text{for } a, b \in P,
\]
then each neighborhood system \(V\) with respect to \(\leq\) would be a neighborhood system with respect to \(\leq'\). In this case the topologies induced by \(V\) and \(\leq\) on \(P\) are finer than the topologies induced by \(V\) with respect to \(\leq'\). For example, for \(a \in P\) and \(v \in V\) we have \(v(a') \supseteq v(a)\), where \(v(a') = \{b \in P : b \leq' a + v\}\).

In symbol, if we denote by \(T_{\leq, V}\) the topologies induced by \(\leq\) and \(V\) on \(P\) we have
\[
T_{\leq', V} \subseteq T_{\leq, V} \subseteq T_{\leq, V'}
\]
if and only if \(a \leq b\) implies \(a \leq' b\) and \(V' \subseteq V\).
Proposition 2.11. Let \((\mathcal{P}, \mathcal{W})\) be a locally convex cone with strict separation property (SP). Let \(X\) be the collection of all finite subsets of the dual cone \(\mathcal{P}^*\). Then for every (abstract) 0-neighborhood system \(\mathcal{V}\) with \(\mathcal{V}_X \subseteq \mathcal{V} \subseteq \mathcal{W}\) we have
\[
\bar{a}_X = \bar{a}_\mathcal{V} = \bar{a}_{\mathcal{W}}, \quad \bar{a}_X = \bar{a}_\mathcal{V} = \bar{a}_{\mathcal{W}}, \quad \bar{a}_X^*= \bar{a}_\mathcal{V}^* = \bar{a}_{\mathcal{W}}^* \quad (\text{for every } a \in \mathcal{P}).
\]

Proof. It is sufficient to show that \(\bar{a}_X = \bar{a}_\mathcal{V}\). First, the lower topology induced by \(\mathcal{W}\) is finer than the lower topology induced by \(X\), so \(\bar{a}_\mathcal{V} \subseteq \bar{a}_X\). Let \(a' \notin \bar{a}_\mathcal{V}\). Then there is some \(w \in \mathcal{W}\), such that \(a' \notin a + w\). For every \(\rho > 1\) we have \(\rho a' \notin \rho a + \rho w\), hence, by (SP), there is some \(\mu \in w^o\) such that \(\mu (\rho a') > \mu (\rho a) + 1\). Put \(A = \{\rho \mu\} \subseteq X\), for some \(\rho > 1\). Then \(a' \notin v_A(a)\) which implies that \(a' \notin \bar{a}_X\). \(\square\)

Remark 2.12. (i) Let \((\mathcal{P}, \mathcal{V})\) be a locally convex cone and \(A \subseteq \mathcal{P}\). The closure of \(A\) with respect to the lower topology is given by
\[
\bar{A} = \{b \in \mathcal{P}: (b)v \cap A \neq \emptyset \text{ for all } v \in \mathcal{V}\} = \bigcap_{v \in \mathcal{V}} v(A) = \bigcap_{v \in \mathcal{V}, a \in A} v(a).
\]

Equalities like \(\bar{A}_X = \bar{A}_Y\) for a convex subset \(A\) and \(X, Y\) as in Proposition 2.10, corresponding ones in locally convex topological vector spaces, need not be true in general. For example, let \(\mathcal{P} = \mathcal{Q} = \mathbb{R}_+\). Consider the dual pair \((\mathcal{P}, \mathcal{Q})\) with bilinear mapping \(\langle a, x \rangle = ax\). The collection \(Y = X \cup \{\mathbb{R}_+\}\), where \(X\) is all finite subsets of \(\mathcal{Q}\), satisfies properties \((P_0)-(P_2)\). We have
\[
U_{\mathbb{R}_+} = \{(a, b) \in \mathbb{R}_+^2: ax \leq bx + 1, \forall x \in \mathbb{R}_+\} = \{(a, b) \in \mathbb{R}_+^2: a \leq b\}.
\]
Put \(A = (0, 1) \subseteq \mathcal{P}\). For each \(x \in A\), \(v_{\mathbb{R}_+}(x) = [0, x]\) thus
\[
v_{\mathbb{R}_+}(A) = \bigcup_{x \in A} v_{\mathbb{R}_+}(x) = \bigcup_{x \in (0, 1)} [0, x] = [0, 1).
\]
Clearly, \(X\) induces the (abstract) 0-neighborhood system \(\mathcal{V} = \{\epsilon \in \mathbb{R}: \epsilon > 0\}\) on \(\mathcal{P}\), hence
\[
\bar{A}_X = \bigcap_{\epsilon > 0} \bigcup_{x \in A} [0, x + \epsilon] = \bigcap_{\epsilon > 0} [0, 1 + \epsilon] = [0, 1]
\]
but
\[
\bar{A}_Y = \bar{A}_X \cap v_{\mathbb{R}_+}(A) = [0, 1] \cap [0, 1) = [0, 1).
\]
In the same way, inequalities \(\bar{A}_X \neq \bar{A}_Y\) and \(\bar{A}_X^* \neq \bar{A}_Y^*\) are proved.

(ii) For a finite subset \(A = \{a_1, a_2, \ldots, a_n\} \subseteq \mathcal{P}\) in the conditions of Proposition 2.10 (also Proposition 2.11) we have
\[
\bar{A}_X = \bigcup_{i=1}^n [a_i]_X = \bigcup_{i=1}^n [a_i]_X^* = \bigcup_{i=1}^n \bar{a}_i = \bar{a}_Y = \bar{A}_Y.
\]
Similarly
\[ \bar{A}_X = \bar{A}_Y \quad \text{and} \quad \bar{A}_X^s = \bar{A}_Y^s. \]

(iii) Let \( (P, Q) \) be a dual pair. By Proposition 2.10 for the \( X \)-topology on \( P \) which is finer than \( \sigma(P, Q) \)-topology (i.e., \( X \) contains all finite subsets of \( Q \) and has properties \((P_0)-(P_2)\)), we have
\[ \bar{a}_\sigma = \bar{a}_X \quad \text{(also \( \bar{a}_\sigma = \bar{a}_X \) and \( \bar{a}_\sigma^s = \bar{a}_X^s \))} \]
for each \( a \in P \). But there are many \( X \)-topologies on \( P \) that are coarser than \( \sigma(P, Q) \). For example \( X_a \)-topologies introduced in Example 2.3(i). For these topologies the above equalities do not satisfy in general. To see this let \( P = Q = \mathbb{R} \), \( a \in \mathbb{R} \) and \( X = \{+\infty\} \). Then \( \bar{a}_\sigma = (-\infty, a] \) but \( \bar{a}_X = \mathbb{R} \).

Likewise, for each subset \( S \) of \( Q \) such that \( \inf\{(a, x): x \in S\} > -\infty \) for all \( a \in P \), \( X_S = \{\bigcup_{i=1}^{\infty} \lambda_i S: n \in \mathbb{N}, \lambda_i > 0\} \) is a collection of subsets of \( Q \) that satisfies the properties \((P_0)-(P_2)\). \( X_S \)-topology on \( P \) is not compatible with \( \sigma(P, Q) \)-topology, hence \( \bar{a}_\sigma \) is not equal with \( \bar{a}_X \) in general.

(iv) Using the strict separation property on a locally convex cone, we find a base for each upper, lower and symmetric topology such that the elements of the base for the upper topology are closed in lower one, and the elements of the base for the lower topology are closed in upper one, in particular, the elements of the base for the symmetric topology are closed. Indeed, \( \mathcal{V} = \{v_\phi: v \in \bar{V}\} \) is an (abstract) 0-neighborhood system equivalent with \( \mathcal{V} \) (because of \( \text{(SP)} \) property), and for each \( a \in P \), we have
\[ v_\phi(a) = \{b \in P: b \leq a + v_\phi\} \]
\[ = \left\{b \in P: \mu(b) \leq \mu(a) + 1 \text{ for all } \mu \in v_\phi \right\} \]
\[ = \bigcap_{\mu \in v_\phi} \mu^{-1}((\mathbb{R} \setminus \mathbb{R}^+), \mu(a) + 1)) \]
which is a closed subset of \( P \) with respect to the lower topology. Similarly, \( (a)v_\phi \) is closed with respect to the upper topology and \( v_\phi(a)v_\phi \) is closed with respect to the symmetric topology.

In any case, with \( \text{(SP)} \) or not, we have
\[ \bar{a} \subseteq \bigcap_{v \in \mathcal{V}} v_\phi(a), \quad \bar{a} \subseteq \bigcap_{v \in \mathcal{V}} (a)v_\phi, \quad \bar{a}^s \subseteq \bigcap_{v \in \mathcal{V}} v_\phi(a)v_\phi. \]
To see this let \( s \in \bar{a} \), then \( s \leq a + v \) for all \( v \in \mathcal{V} \), so \( \mu(s) \leq \mu(a) + 1 \) for all \( \mu \in v_\phi \), which implies that \( (s, a) \in U_{v_\phi} \) or \( s \in v_\phi \).

If all elements of the locally convex cone \( (P, \mathcal{V}) \) are bounded, i.e., for every \( a \in P \) and \( v \in \mathcal{V} \) there is some \( \lambda > 0 \) such that \( a \leq \lambda v \), then the symmetric convex quasi-uniform structure satisfies the property \((us)\) so that defines a locally convex structure on \( P \) as well. Let us denote this by \( (P, \mathcal{V}^s) \); i.e., for \( a, b \in P \) and \( v \in \mathcal{V} \), we have
\[ a \leq b + v^s \quad \text{if and only if} \quad a \leq b + v \text{ and } b \leq a + v. \]
A simple verification shows that the upper, lower and symmetric topologies associated with the symmetric convex quasi-uniform structure coincide to the original symmetric topology.

Now we exhibit when the symmetric topology on a locally convex cone is metrizable. We know that every convex quasi-uniform structure \( U \) gives rise to a directed family \( (d_U)_{U \in U} \) of
sublinear quasi-metrics. For details see [1,5, Chapter I]. We consider the case when this directed family defines a metric on $\mathcal{P}$ equivalent to the symmetric topology. First we recall that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is called separated if $\bar{a} = \bar{b}$ implies $a = b$ for all $a, b \in \mathcal{P}$. Also a subset $\mathcal{U}$ of $\mathcal{V}$ is called a base of $\mathcal{V}$ if for every $v \in \mathcal{V}$ there is $u \in \mathcal{U}$ and $\lambda > 0$ such that $\lambda u \leq v$.

**Proposition 2.13.** Let $(\mathcal{P}, \mathcal{V})$ be a separated locally convex cone with countable base $\mathcal{U}$ for $\mathcal{V}$, and let all elements of $\mathcal{P}$ be bounded. Then the symmetric topology on $\mathcal{P}$ is metrizable.

**Proof.** We define the directed family $(\tilde{d}_i)_{vi \in \mathcal{U}}$ of sublinear quasi-metrics by

$$d_i(a, b) = \inf \{ \rho > 0 : (a, b) \in \rho \tilde{v}_i \}, \quad v_i \in \mathcal{U},$$

and put

$$d(a, b) = \sum_{i=1}^{\infty} 2^{-n} \inf \{ d_{\tilde{v}_i}(a, b), 1 \}.$$

Then clearly $d$ is a metric and it is equivalent with the symmetric convex quasi-uniform structure. For, let $d(a, b) = 0$ for the elements $a, b \in \mathcal{P}$. Then $d_{\tilde{v}_i}(a, b) = 0$ for all $\tilde{v}_i$, and we have $(a, b) \in \rho \tilde{v}_i$, for all $\rho > 0$ and all $v_i$. This yields $\bar{a} = \bar{b}$ and by hypothesis we infer that $a = b$. □

**Example 2.14.** There are many locally convex cones with countable base. For example $\mathbb{R}$ and $\mathbb{C} = \mathbb{R} \cup \{ +\infty \}$ (with $a \leq b$ if $b = +\infty$ or $\mathcal{H}(a) \leq \mathcal{H}(b)$) with (abstract) 0-neighborhood system $\mathcal{V} = \{ \epsilon \in \mathbb{R} : \epsilon > 0 \}$. Also locally convex cones generated by an inner product [5, Section 3] have a 0-neighborhood system as $\{ \rho v : \rho > 0 \}$ for some $v$, which obviously has a countable base.

**Definition 2.15.** Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair. We say that the collection $X$ of subsets of $\mathcal{Q}$ separates the elements of $\mathcal{P}$, if for all $a, b \in \mathcal{P}$ and $a \neq b$, there is an element $x \in A$, for some $A \in X$, such that $(a, x) \neq (b, x)$. Hence $\mathcal{Q}$ separates the elements of $\mathcal{P}$, if the collection $\{ \{x\} : x \in \mathcal{Q} \}$ separates the elements of $\mathcal{P}$.

**Proposition 2.16.** Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. If $\mathcal{P}^*$ separates the elements of $\mathcal{P}$, then $(\mathcal{P}, \mathcal{V})$ is separated.

**Proof.** Let $a, b \in \mathcal{P}$ and $\bar{a} = \bar{b}$. Then $a \leq b + v$ and $b \leq a + v$ for all $v \in \mathcal{V}$. Given $\mu \in \mathcal{P}^*$ and $\lambda > 0$, there is some $v \in \mathcal{V}$ such that $(1/\lambda) \mu \in v^o$. So we have $\mu(a) \leq \mu(b) + \lambda$ and $\mu(b) \leq \mu(a) + \lambda$. Hence $\mu(a) = \mu(b)$ for all $\mu \in \mathcal{P}^*$ which implies that $a = b$, for; $\mathcal{P}^*$ separates the elements of $\mathcal{P}$. □

**Remark 2.17.** Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair, $X$ be a collection of subsets of $\mathcal{Q}$ with properties $(P_0)$–$(P_2)$, and $A \in X$. With notations introduced in Definition 2.2, it is easy to see that for each $\lambda > 0,$

$$U_{\lambda A} = \lambda^{-1} U_A \quad \text{and} \quad v_{\lambda A} = \lambda^{-1} v_A.$$ 

Also for $A, B \in X$ and $A \subseteq B$ we have $U_A \supseteq U_B$, hence $v_A \supseteq v_B$.

Now let $Y$ be a subset of $X$ such that for each $A \in X$ there exists $B \in Y$ and $\lambda > 0$ such that $A \subseteq \lambda B$. Hence for each $v_A$ in the (abstract) 0-neighborhood system for $X$-topology on $\mathcal{P}$,
there is \( v_B, B \in Y \) and \( \lambda > 0 \) such that \( \lambda^{-1}v_B \leq v_A \). That is \( \{v_B : B \in Y\} \) is a base for (abstract) 0-neighborhood system \( \{v_A : A \in X\} = \mathcal{V}_X \). So we call \( Y \) a base for \( X \). In particular, if \( Y \) is countable, say \( \{B_1, B_2, \ldots\} \), then \( \mathcal{V}_X \) would have a countable base.

### Theorem 2.18
Let \((\mathcal{P}, \mathcal{Q})\) be a dual pair, \( X \) be a collection of subsets of \( \mathcal{Q} \) with a countable base, that separates the elements of \( \mathcal{P} \) and satisfies the properties \((P_0)\)–\((P_2)\). Then if the elements of \( \mathcal{P} \) are bounded (as a linear mapping) on every \( A \in X \), the symmetric topology induced by \( X \) on \( \mathcal{P} \) is metrizable.

**Proof.** Let \( a \in \mathcal{P} \) and \( A \in X \). Since \( a \) is bounded (as a linear mapping) on \( A \), there is some \( \lambda > 0 \) such that \( \langle a, x \rangle \leq \lambda \), for every \( x \in A \), which implies that \( a \leq \lambda v_A \) hence the elements of \( \mathcal{P} \) are bounded. Let \( a, b \in \mathcal{P} \) and \( a \neq b \). For some \( A \in X \), there is an element \( x \in A \subset v_A^0 \), such that \( \langle a, x \rangle \neq \langle b, x \rangle \), hence \( \mathcal{P}^*_X \) separates the elements of \( \mathcal{P} \) and \((\mathcal{P}, \mathcal{V}_X)\) is separated by Proposition 2.16. Clearly by a countable base of \( X \) we will have an \( X\)-topology on \( \mathcal{P} \) which has a countable base. Thus by Proposition 2.13 the symmetric topology induced by \( X \) is metrizable. \( \square \)

### Example 2.19.

(i) Let \( \mathcal{P} = \mathcal{Q} = \mathbb{R}, \mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\} \) and consider the dual pair \((\mathcal{P}, \mathcal{Q})\) (Example 2.3(i)). Let \( X \) be the collection of all bounded subsets of \( \mathcal{Q} \). Let \( Y = \{B\} \), where \( B = [-1, 1] \). Clearly \( Y \subseteq X \) and for each \( A \in X \) there is \( \lambda > 0 \) such that \( A \subseteq \lambda B \). Hence \( \lambda^{-1}v_B \leq v_A \), \( \{v_B\} \) is a countable base for \( X \)-topology and the induced symmetric topology is metrizable.

(ii) Let \( \mathbb{R}^2_+ = \{(x, y) : x, y \in \mathbb{R} \text{ and } x, y \geq 0\} \). For \( \mathcal{P} = \mathcal{Q} = \mathbb{R}^2_+ \) and bilinear mapping defined by \( \langle (a, b), (x, y) \rangle = ax + by \), \((\mathcal{P}, \mathcal{Q})\) is a dual pair. For each \( r > 0 \), let \( A_r = \{(x, y) \in \mathbb{R}^2_+ : x^2 + y^2 \leq r^2\} \). Then \( X = \{A_r : r > 0\} \) satisfies the properties \((P_0)\)–\((P_2)\) and the collection \( Y = \{A_n : n \in \mathbb{N}\} \) is a basic subset of \( X \) which is countable. Hence the symmetric topology induced by \( X \) is metrizable. Note that for \( Y \) we can take \( Y = \{A_1\} \) as example (i).

In locally convex topological vector spaces every precompact subset is bounded and these two are the same with respect to the week topology of dual pair. Here, in locally convex cones a \( u \)-precompact subset need not be bounded, also a bounded subset need not be \( u \)-precompact (see examples below). For special cases we have following results.

### Lemma 2.20
Let \((\mathcal{P}, \mathcal{W})\) be a locally convex cone and \( \mathcal{V} \subset \mathcal{W} \) such that the finite intersections of the sets \( \tilde{v}, v \subset \mathcal{V} \) form a convex quasi-uniform structure for \((\mathcal{P}, \mathcal{W})\). If the set \( A \) has a finite covering by sets say \( \{A_i\}_{i=1}^n \) with \( A_i \times A_j \subset \tilde{v} \) \( (i = 1, 2, \ldots, n) \), for each \( v \in \mathcal{V} \), then \( A \) is \( u \)-precompact.

**Proof.** First, if \( v_1, v_2 \in \mathcal{V} \) and \( \tilde{v} = \tilde{v}_1 \cap \tilde{v}_2 \), then there are sets \( B_1, \ldots, B_m \) with

\[
A \subset \bigcup_{i=1}^m B_i, \quad B_i \times B_i \subset \tilde{v}_1 \quad (i = 1, 2, \ldots, m),
\]

and sets \( C_1, \ldots, C_n \) with

\[
A \subset \bigcup_{j=1}^n C_j, \quad C_j \times C_j \subset \tilde{v}_2 \quad (j = 1, 2, \ldots, n).
\]
Then the sets $B_i \cap C_j$ form a finite covering of $A$ with

$$(B_i \cap C_j) \times (B_i \cap C_j) \subset \tilde{v} \quad (i = 1, \ldots, m, \; j = 1, \ldots, n).$$

The general case, when $\tilde{v}$ is an intersection of $k$ sets of $\tilde{v}_i$, $i = 1, \ldots, k$, can now be treated by induction on $k$.

**Theorem 2.21.** Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair. Then every $\sigma(\mathcal{P}, \mathcal{Q})$-bounded subset of $\mathcal{P}$ is u-precompact with respect to $\sigma(\mathcal{P}, \mathcal{Q})$.

**Proof.** Let $W$ be the 0-neighborhood system induced by $\sigma(\mathcal{P}, \mathcal{Q})$ and $C \subset \mathcal{P}$ be $\sigma(\mathcal{P}, \mathcal{Q})$-bounded. For every $x \in \mathcal{Q}$, let $v_x$ be the 0-neighborhood induced by $U_x \cap \mathcal{Q}$, where $U = \{v_x : x \in \mathcal{Q}\}$. If we put $V = \{v_x : x \in \mathcal{Q}\}$, then $V \subset W$ and the finite intersections of the sets $v_x$, $x \in \mathcal{Q}$, form the convex quasi-uniform structure $\tilde{W}$. For $v_x \in V$ there is some $\lambda > 0$ such that $$(C, 0) \subset \lambda v_x \quad \text{and} \quad (0, C) \subset \lambda v_x,$$ i.e., for every $c \in C$, $|\langle c, x \rangle| \leq \lambda$. Thus $(C, x)$ is a bounded subset of $\mathbb{R}$ and there are intervals $I_1, \ldots, I_n$ of diameter less than 1, such that $(C, x) \subset \bigcup_{i=1}^{n} I_i$. Thus

$$C \subset \bigcup_{i=1}^{n} v^{-1}(I_i) \quad \text{with} \quad v^{-1}(I_i) \times v^{-1}(I_i) \subset v_x.$$ 

Therefore $C$ is u-precompact, by the lemma.

**Theorem 2.22.** Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, such that every $a \in \mathcal{P}$ is bounded above. Then every u-precompact subset of $\mathcal{P}$ is bounded.

**Proof.** Let $A \subset \mathcal{P}$ be u-precompact, and $v \in \mathcal{V}$. There are subsets $A_1, \ldots, A_n$ of $\mathcal{P}$ with

$$A_i \times A_i \subset v, \quad A \subset \bigcup_{i=1}^{n} A_i \quad (i = 1, \ldots, n).$$

Choose $a_i \in A_i$ ($i = 1, \ldots, n$). There is some $\lambda > 0$, such that

$$(a_i, 0) \in \lambda v, \quad (0, a_i) \in \lambda v \quad (i = 1, \ldots, n).$$

Let $a \in A$ be arbitrary. For some $1 \leq i \leq n$, $a \in A_i$, so $(a_i, a) \in \tilde{v}$ and $(a, a_i) \in \tilde{v}$. Now we have

$$(0, a) = (0, a_i) \circ (a_i, a) \in \lambda \tilde{v} \circ v \subset (\lambda + 1) \tilde{v}$$

and

$$(a, 0) = (a, a_i) \circ (a_i, 0) \in \tilde{v} \circ \lambda v \subset (\lambda + 1) \tilde{v}.$$ 

Since the element $a \in A$ is arbitrary, we infer that

$$(A, 0) \subset (\lambda + 1) \tilde{v} \quad \text{and} \quad (0, A) \subset (\lambda + 1) \tilde{v}.$$ 

Therefore $A$ is bounded.

**Corollary 2.23.** In an ordered cone with inner product [5], if $\mathcal{B}$ is the subcone of all elements of finite norm (also, if $\mathcal{B}$ is the subcone of all bounded elements of a locally convex cone), then every u-precompact subset of $\mathcal{B}$ is bounded.
Corollary 2.24. Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair such that its bilinear form attains only finite values. Then every $u$-precompact subset of $\mathcal{P}$ with respect to $\sigma(\mathcal{P}, \mathcal{Q})$ is $\sigma(\mathcal{P}, \mathcal{Q})$-bounded.

Proof. We will show that every element of $\mathcal{P}$, is $\sigma(\mathcal{P}, \mathcal{Q})$-bounded above. Let $a \in \mathcal{P}$ be arbitrary, $A = \{x_1, \ldots, x_n\} \subset \mathcal{Q}$ and

$$U_A = \{(a, b) \in \mathcal{P} \times \mathcal{P}: \langle a, x_i \rangle \leq \langle b, x_i \rangle + 1, \ x_i \in A\}.$$  

There is some $\lambda > 0$ such that $\|a, x_i\| \leq \lambda$, $x_i \in A$. Then $(a, 0) \in \lambda U_A$, i.e., the element $a$ is $\sigma(\mathcal{P}, \mathcal{Q})$-bounded above. \qed

Example 2.25. (i) Consider $\mathbb{R} = \mathbb{R} \cup \{+\infty\}$ with $\mathcal{V} = \{\varepsilon \in \mathbb{R}: \varepsilon > 0\}$. Clearly subsets as $\{a, +\infty\}$ for all $a \in \mathbb{R}$, are $u$-precompact but they are not bounded.

(ii) Consider $\mathcal{P} = \{[0, \alpha], [0, \beta]: \alpha \geq 0, \ \beta > 0\}$ with set inclusion as order and $\mathcal{V} = \{(-\varepsilon, 0]: \varepsilon > 0\}$ as the (abstract) 0-neighborhood system. Then $\mathcal{P}$ is a locally convex cone in which each subset $A \subset \mathcal{P}$ is in form $\{[0, \alpha]\}$ or $\{[0, \beta]\}$ for some $\alpha \geq 0$ and $\beta > 0$. Hence each $A \subset \mathcal{P}$, also $a \in \mathcal{P}$ is unbounded, but it is $u$-precompact; for, $A \times A \subseteq (-\varepsilon, 0]$ for all $A = \{[0, \alpha]\}$ or $A = \{[0, \beta]\}$ and all $(-\varepsilon, 0] \in \mathcal{V}$.

(iii) Let $(\mathcal{E}, \|\|)$ be a normed space and

$$\mathcal{P} = \{A \subset \mathcal{E}: \text{A is closed, convex and non-empty}\}.$$  

Define scalar multiplication in $\mathcal{P}$ as usual and for addition put $A_1 \oplus A_2 = A_1 + A_2$ for all $A_1, A_2 \in \mathcal{P}$. Also define a preorder on $\mathcal{P}$ as

$$A_1 \leq A_2 \text{ if } A_1 \subseteq A_2.$$  

Now if $B$ is the closed unit ball of $\mathcal{E}$,

$$\mathcal{V} = \{\lambda B: \lambda > 0\}$$  

is an (abstract) 0-neighborhood system making $(\mathcal{P}, \mathcal{V})$ into a full locally convex cone (see [4, 2.5]).

Via the embedding of its elements into singleton subsets, the space $\mathcal{E}$ may be considered as a subcone of $\mathcal{P}$. Thus $(\mathcal{E}, \mathcal{V})$ is a locally convex cone on which the three (upper, lower and symmetric) topologies of $\mathcal{P}$ coincide with the given norm topology on $\mathcal{E}$. It is easy to show that boundedness and precompactness in $\mathcal{E}$ are the same provided $(\mathcal{E}, \|\|)$ as a convex space or $(\mathcal{E}, \mathcal{V})$ as a locally convex cone. We prove that each precompact subset $A$ of $(\mathcal{E}, \|\|)$ is $u$-precompact in $(\mathcal{E}, \mathcal{V})$ and vice versa.

Let $A \subseteq (\mathcal{E}, \|\|)$ be precompact. Then there are subsets $A_1, A_2, \ldots, A_n$ of $\mathcal{E}$, each small of order $B$, such that $A \subseteq \bigcup_{i=1}^{n} A_i$. For each $A_i$, $i = 1, 2, \ldots, n$, and every $x, y \in A_i$ we have $x - y \in B$. This means that $\{x\} \subseteq \{y\} \oplus B = \{y\} + B$ or $(x, y) \in \tilde{B}$, i.e., $A$ is $u$-precompact. On the other hand if $A \subseteq (\mathcal{E}, \mathcal{V})$ is $u$-precompact, then there are subsets $A_1, A_2, \ldots, A_n$ of $\mathcal{E}$ such that $A \subseteq \bigcup_{i=1}^{n} A_i$ and $A_i \times A_j \subseteq \tilde{B}$ for each $i = 1, 2, \ldots, n$. Hence for every $x, y \in A_i$ we have $\{x\} \subseteq \{y\} + B$ or $\{x\} \subseteq \{y\} + B$ which implies that $x - y \in B$, that is $A_i$ is small of order $B$, hence $A$ is precompact in $(\mathcal{E}, \|\|)$.

Now if $\mathcal{E}$ is infinite dimensional, then $B$ is bounded but not precompact (otherwise $\mathcal{E}$ would be of finite dimension), hence it is not $u$-precompact.
3. Adjoint operators on locally convex cones

Definition 3.1. Let \((P, V)\) and \((Q, W)\) be locally convex cones. We denote the adjoint operator of the linear mapping \(t : P \to Q\), by \(t'\) and define as follows:

\[ t' : Q^* \to L(P, \mathbb{R}), \quad \mu \to t'\mu, \quad \mu \in Q^*, \]

\[ t'\mu : P \to \mathbb{R}, \quad t'\mu(x) = \mu(t(x)), \quad x \in P. \]

Theorem 3.2. Let \((P, V)\) and \((Q, W)\) be locally convex cones and \(t : P \to Q\) be a linear operator; then

(a) if \(t\) is u-continuous, for every \(w \in W\) there is some \(v \in V\), such that \(t'(w^o) \subset v^o\); in particular, \(t'(Q^o) \subset P^o\).

(b) if the locally convex cone \((Q, W)\) has (SP), and for every \(w \in W\) there is some \(v \in V\) such that \(t'(w^o) \subset v^o\), then \(t\) is u-continuous.

Proof. (a) Let \(w \in W\). There is some \(v \in V\) such that for all \(a, b \in P\) if \(a \leq b + v\), then \(t(a) \leq t(b) + w\). We claim that \(t'(w^o) \subset v^o\). Let \(\mu \in w^o\) and \(a, b \in P\) such that \(a \leq b + v\). Then \(\mu(t(a)) \leq \mu(t(b)) + 1\), i.e., \(t'\mu(a) \leq t'\mu(b) + 1\), i.e., \(t'\mu \in v^o\).

(b) Let \(w \in W\). There is some \(v \in V\) such that \(t'(w^o) \subset v^o\). Let \(a, b \in P\) and \(\rho > 1\) with \(a \leq b + v/\rho\), then for every \(w \in W\) we have \(t'\mu \in v^o\), hence \(t'\mu(a) \leq t'\mu(b) + 1/\rho\), i.e., \(\mu(t(a)) \leq \mu(t(b)) + 1/\rho\). This implies that \(t(a) \leq t(b) + w\); if not, by the (SP) property for \((Q, W)\) there is some \(v \in w^o\) such that \(v(t(a)) > v(t(b)) + 1/\rho\) or \(t'v(a) > t'v(b) + 1\) with \(t'v \in v^o\), which is a contradiction. \(\square\)

For locally convex spaces \(E, F\) with duals \(E', F'\) a linear mapping \(t : E \to F\) is continuous with respect to \(\sigma(E, E')\) and \(\sigma(F, F')\) if and only if \(t'(F') \subseteq E'\). In the case of locally convex cones we have stronger result:

Corollary 3.3. Let \((P, P')\) and \((Q, Q')\) be dual pairs, and \(X, Y\) be collections of subsets of \(P'\) and \(Q'\), respectively, that satisfy the properties (P0)–(P2). Let \(t : P \to Q\) be a linear mapping, then

(a) if \(t\) is u-continuous with respect to the \(X\)-topology on \(P\) and \(Y\)-topology on \(Q\), then for every \(B \in Y\), there is some \(A \in X\) such that \(t'(v^o_B) \subset v^o_A\).

(b) if for every \(B \in Y\), there is some \(A \in X\) such that \(t'(v^o_B) \subset v^o_A\), then \(t\) is u-continuous with respect to the \(X\)-topology on \(P\) and \(Y\)-topology on \(Q\).

Proof. Every \(X\)-topology has the strict separation property. \(\square\)

Let \(P\) be a cone and \(M\) be a subcone of \(P\). Consider the equivalence relation \(\sim\) on \(P\) as \(x \sim y\) if and only if \(x + M = y + M\). The equivalence class \(\hat{x}\) is a subset of \(\hat{x} = x + M\) in general, they are equal if \(M\) is a vector space. The set \(P/M = \{\hat{x} : x \in P\}\) with usual addition \((x + y) = \hat{x} + \hat{y}\) and scalar multiplication \(\lambda \hat{x} = (\lambda \hat{x})\) for \(x, y \in P\) and \(\lambda > 0\) is a cone, called quotient cone. The mapping \(k(x) = \hat{x}\) is linear which is called the canonical mapping of \(P\) onto \(P/M\).
If \( \leq \) is a preorder on \( P \), we define the preorder on \( P/M \) as \( \hat{x} \leq \hat{y} \) if for each \( s \in \hat{x} \) there is some \( t \in \hat{y} \) such that \( s \leq t \). If \( \mathcal{V} \) is an (abstract) 0-neighborhood system for \( P \), then \( \hat{\mathcal{V}} = \{ k(v) = \hat{v} : v \in \mathcal{V} \} \) is an (abstract) 0-neighborhood system for \( P/M \) which is called the quotient (abstract) 0-neighborhood system, and the pair \( (P/M, \hat{\mathcal{V}}) \) is called the locally convex quotient cone. Obviously \( k \) is a \( u \)-continuous linear mapping, (for details see [2]).

If there is a one to one linear mapping \( t \) of \((P, \mathcal{V})\) onto \((Q, \mathcal{W})\) such that both \( t \) and its inverse \( t^{-1} \) are \( u \)-continuous then these two locally convex cones called uniformly isomorphic (\( u \)-isomorphic) and that \( t \) is a \( u \)-isomorphism.

**Theorem 3.4.** Let \((P, \mathcal{V})\) be a locally convex cone, \( M \) be a subcone of \( P \) and \( X \) be a collection of subsets of \( P \) satisfying properties \((P_0)-(P_2)\). If \( \mathcal{N} = \{ v \in P^*: v(M) = 0 \} \), then

(a) \( k(X) \) has also \((P_0)-(P_2)\), where \( k : P \rightarrow P/M \) is the canonical mapping,

(b) \( \mathcal{N} \) is a subcone of \( P^* \),

(c) the dual of the locally convex quotient cone \( P/M \) is \( \mathcal{N} \).

**Proof.** (a) Let \( A \in X \) and \( v \in (P/M)^* \). We have

\[
\inf \{ \langle k(a), v \rangle: a \in A \} = \inf \{ \langle a, k'v \rangle: a \in A \} > -\infty,
\]

since \( k'v \in P^* \) by Theorem 3.2(a). The properties \((P_1)\) and \((P_2)\) are clear. Part (b) is evident.

For (c), define

\[
T : (P/M)^* \rightarrow \mathcal{N}, \quad T(\mu) = \mu \circ k.
\]

It is easy to verify that \( T \) is linear and one to one, we will show that \( T \) is onto. Let \( v \in \mathcal{N} \), define \( \mu : P/M \rightarrow \mathbb{K} \) as

\[
\mu(\hat{x}) = v(x), \quad \hat{x} \in P/M.
\]

We show that \( \mu \) is \( u \)-continuous, let \( \hat{a}, \hat{b} \in P/M \) with \( \hat{a} \leq \hat{b} + \hat{v} \). Then evidently, \( \{a\} \leq \hat{a} \leq \hat{b} + \hat{v} = b + v + M \). Choose the element \( x \in M \) such that \( a \leq b + x + v \). Since \( v \) is \( u \)-continuous and vanishes on \( M \), we have \( v(a) \leq v(b) + v(x) + 1 = v(b) + 1 \). That is \( \mu(\hat{a}) \leq \mu(\hat{b}) + 1 \), i.e., \( \mu \in (P/M)^* \). Since

\[
T(U_{k(A)}) = U_A \cap (P^* \times P^*) \quad \text{for every} \quad A \in X, \quad \text{where} \quad T = T \times T,
\]

both \( T \) and its inverse \( T^{-1} \) are \( u \)-continuous and \( T \) is a \( u \)-isomorphism. \( \square \)

**Theorem 3.5.** Let \( P \) and \( Q \) be locally convex cones, \( Y \) be a collection of subsets of \( Q^* \) satisfying properties \((P_0)-(P_2)\) and let \( t : P \rightarrow Q \) be \( u \)-continuous. Then

(a) \( t'(Y) \) has also properties \((P_0)-(P_2)\) on \( P^* \).

(b) \( U_{t'(B)} = T^{-1}(U_B) \) for every \( B \in Y \), where \( T = t \times t \).

**Proof.** (a) Let \( B \in Y \) and \( a \in P \) we have \( t'(B) \subseteq P^* \) by Theorem 3.2(a), and

\[
\inf \{ \langle t'(v), a \rangle: v \in B \} = \{ \langle v, t(a) \rangle: v \in B \} > -\infty,
\]

so \( t'(Y) \) has \((P_0)\). The properties \((P_1)\) and \((P_2)\) are clear.

(b) Let \( B \in Y \) and \( a, b \in P \). Using definitions \((2.1) \) and \((3.1)\), we have
that is Corollary 3.7. Let 

\( (a, b) \in U_{t'}(B) \),

if and only if

\[ \langle a, t'v \rangle \leq \langle b, t'v \rangle + 1 \quad \text{for every } v \in B, \]

or

\[ \{t(a), v\} \leq \{t(b), v\} + 1 \quad \text{for every } v \in B, \]

that is

\[ T(a, b) = (t(a), t(b)) \in U_B. \]

\[ \square \]

**Remark 3.6.**

(i) Under the conditions of the theorem, \( t \) is also u-continuous with respect to the \( Y \)-topology on \( Q \) and \( t'(Y) \)-topology on \( P \). Also if \( t \) is one to one , then \( (P, \mathcal{V}_{t'(Y)}) \) and \( (t(P), \mathcal{V}_Y) \) are u-isomorphic.

(ii) Let \( (P, \mathcal{V}) \) and \( (Q, \mathcal{W}) \) be locally convex cones and \( t : P \to Q \) be u-continuous, \( \mathcal{Y} = \{w^o : w \in \mathcal{W}\} \) then \( t \) is also u-continuous with respect to the \( \mathcal{Y} \)-topology on \( Q \) and \( t'(\mathcal{Y}) \)-topology on \( P \), where \( t'(\mathcal{Y}) = \{t'(w^o) : w \in \mathcal{W}\} \).

(iii) Let \( (P, P^*) \) and \( (Q, Q^*) \) be dual pairs and let \( X, Y \) be collections of subsets of \( Q' \) and \( P' \), respectively, satisfying \( (P_0)-(P_2) \). Put \( \mathcal{Y}' = \{v_B^o : B \in Y\} \) and \( t'(\mathcal{Y}) = \{t'(v_B^o) : B \in Y\} \). By the (SP) property, the \( Y \)-topology and \( \mathcal{Y} \)-topology on \( Q \) are equivalent but by Corollary 3.3(a), the \( t'(\mathcal{Y}) \)-topology is weaker than the \( \sigma \)-topology on \( P^* \) moreover by Theorem 3.5(b), \( t \) is also u-continuous with respect to the \( t'(\mathcal{Y}) \)-topology on \( P \).

**Corollary 3.7.** Let \( P \) and \( Q \) be locally convex cones and \( t : P \to Q \) be u-continuous, then \( t \) is also u-continuous with respect to the topologies \( \sigma(P, P^*) \) and \( \sigma(Q, Q^*) \) on \( P \) and \( Q \).

**Theorem 3.8.** Let \( P \) and \( Q \) be locally convex cones, \( t : P \to Q \) be u-continuous and \( X \) be a collection of subsets of \( P \) with properties \( (P_0)-(P_2) \) with respect to the duality \( (P, P^*) \). Then

(a) \( t(X) \) has also the properties \( (P_0)-(P_2) \) with respect to the duality \( (Q, Q^*) \).

(b) \( U_{t(A)} = T'^{-1}(U_{A}) \) for every \( A \in X \), where \( T' = t' \times t' \).

**Proof.** The proof is similar to the proof of Theorem 3.5. \( \square \)

**Remark 3.9.** Under the conditions of the theorem, \( t' \) is also u-continuous with respect to the \( X \)-topology on \( P^* \) and \( t(X) \)-topology on \( Q^* \). Also if \( t \) is one to one , then \( (t(P))^*, \mathcal{V}_X \) and \( (Q^*, \mathcal{V}_{t(X)}) \) are u-isomorphic.

**Corollary 3.10.** Let \( P \) and \( Q \) be locally convex cones and \( t : P \to Q \) be u-continuous, then \( t' \) is u-continuous with respect to the \( \sigma(P^*, P) \)-topology on \( P^* \) and \( \sigma(Q^*, Q) \)-topology on \( Q^* \).

**Theorem 3.11.** Let \( (P, P') \) and \( (Q, Q') \) be dual pairs, and \( t : P \to Q \) be u-continuous with respect to the topologies \( \sigma(P, P') \) and \( \sigma(Q, Q') \). Let \( X, Y \) be collections of \( \sigma(P, P') \)-bounded and \( \sigma(Q^*, Q) \)-bounded subsets of \( P \) and \( Q^* \), respectively, that satisfy the properties \( (P_0)-(P_2) \). Then the following are equivalent:
(a) For every $A \in X$, $t(A)$ is $u$-precompact under the $Y$-topology.
(b) For every $B \in Y$, $t'(B)$ is $u$-precompact under the $X$-topology.

**Proof.** Assume (b). If $A \in X$ and $B \in Y$, then there are subsets $B_1, \ldots, B_n$ of $\mathcal{P}_\sigma$ such that

$$t'(B) \subset \bigcup_{i=1}^{n} B_i, \quad B_i \times B_i \subset \frac{1}{3} U_A.$$  

Since $A$ is $\sigma(\mathcal{P}, \mathcal{P}')$-bounded, it is $u$-precompact with respect to the $\sigma(\mathcal{P}, \mathcal{P}')$, by Theorem 2.21. Thus there are subsets $A_1, \ldots, A_n$ of $\mathcal{P}$ such that

$$A \subset \bigcup_{j=1}^{m} A_j, \quad A_j \times A_j \subset \frac{1}{3} U_k \quad (j = 1, \ldots, m),$$

where $k = \{t'\mu_1, \ldots, t'\mu_n\}$, $\mu_i \in B$. Now let $x, y \in A \cap A_j$ and $\mu \in B$, then

$$\|\langle t(x), \mu \rangle - \langle t(y), \mu \rangle\| = \|\langle x, t'\mu \rangle - \langle y, t'\mu \rangle\|$$

$$\leq \|\langle x, t'\mu \rangle - \langle x, \mu_i \rangle\| + \|\langle \mu_i, y \rangle - \langle t'\mu, y \rangle\| + \|\langle \mu_i, x \rangle - \langle \mu_i, y \rangle\|$$

$$\leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$  

In a similar way (a) implies (b).  

**Corollary 3.12.** Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair, and $X, X'$ be collection of subsets of $\mathcal{Q}$ and $\mathcal{P}$, respectively, such that both have the properties $(P_0)$–$(P_2)$. Then each $A \in X$ is $u$-precompact in the $X'$-topology if and only if each $A' \in X'$ is $u$-precompact in the $X$-topology.

We prove that the dual of an inductive limit of locally convex cones is a projective limit. First we bring two theorems.

**Theorem 3.13.** For each $\gamma \in \Gamma$, let $\mathcal{P}_\gamma$ be a cone with convex quasi-uniform structure $U_\gamma$. Let $\mathcal{P}$ be a cone and for each $\gamma$, $\varphi_\gamma$ be a linear mapping of $\mathcal{P}$ into $\mathcal{P}_\gamma$. Then there is a convex quasi-uniform structure $U$ on $\mathcal{P}$ that is the coarsest one under which every $\varphi_\gamma$ is $u$-continuous.

**Proof.** See [3, 2.1].  

The locally convex cone $\mathcal{P}$ with the preorder and (abstract) 0-neighborhood system induced by the above convex quasi-uniform structure is called the projective limit of the locally convex cones $\mathcal{P}_\gamma$ by the mappings $\varphi_\gamma$.

**Theorem 3.14.** For each $\gamma \in \Gamma$, let $\mathcal{P}_\gamma$ be a locally convex cone with a convex quasi-uniform structure $U_\gamma$. Let $\mathcal{P}$ be a cone and for each $\gamma$, $f_\gamma : \mathcal{P}_\gamma \to \mathcal{P}$ be a linear mapping such that $\mathcal{P} = \text{span} \bigcup_{\gamma \in \Gamma} f_\gamma(\mathcal{P}_\gamma)$. Let $U$ be the set of all convex subsets of $\mathcal{P}$ such that:

(a) For each $U \in U$ and each $\gamma \in \Gamma$, we have $F^{-1}_\gamma(U) \in U_\gamma$.
(b) Each $U \in U$ satisfies (u$_3$).
(c) If $U_1, \ldots, U_n \in U$, then $U_1 \cap \cdots \cap U_n \in U$.

Then $U$ is the finest convex quasi-uniform structure on $\mathcal{P}$ which makes each $f_\gamma$ $u$-continuous.
The locally convex cone $P$ with the preorder and (abstract) 0-neighborhood system induced by the above convex quasi-uniform structure is called the inductive limit of the locally convex cones $P_\gamma$ by the mapping $f_\gamma$.

**Theorem 3.15.** For each $\gamma \in \Gamma$, let $P_\gamma$ be a locally convex cone and let its dual $P^*_\gamma$ have the $X_\gamma$-topology. Let $P$ be the inductive limit of the locally convex cones $P_\gamma$ by the mappings $\varphi_\gamma$. Then, if $X$ is the set of finite unions of the sets $\{\varphi_\gamma(A_\gamma): \gamma \in \Gamma\}$, the dual $P^*$ of $P$ under the $X$-topology is the projective limit of the $P^*_\gamma$’s by the mappings $\varphi'_\gamma$, where $\varphi'_\gamma$ is the adjoint of $\varphi_\gamma$.

**Proof.** Each $\varphi'_\gamma$ maps $P^*$ into $P^*_\gamma$, by Theorem 3.2(a). Let us denote the set of finite intersections of the sets $\varphi'^{-1}_{\gamma}(U_{A_\gamma}, \gamma \in \Gamma)$, by $U$, where $\varphi = \varphi \times \varphi$. First, we show that $U$ forms a convex quasi-uniform structure on $P^*$.

1. Let $U \in U$. We have $U = \bigcap_{i=1}^n \varphi'^{-1}_{\gamma_i}(U_{A_{\gamma_i}})$ and clearly $\Delta \subset U$ ($\Delta = \{(v, v): v \in P^*\}$).

2. Let $U, V \in U$. We have $U = \bigcap_{i=1}^n \varphi'^{-1}_{\gamma_i}(U_{A_{\gamma_i}})$ and $V = \bigcap_{j=1}^m \varphi'^{-1}_{\gamma_j}(U_{A_{\gamma_j}})$. Put $W = U \cap V \in U$.

3. Let $U = \bigcap_{i=1}^n \varphi'^{-1}_{\gamma_i}(U_{A_{\gamma_i}}) \in U$ and $\lambda, \mu > 0$. If $(v, \eta) \in \lambda U \cap \mu U$, then there is some $\tau \in P^*$ such that $(v, \tau) \in \lambda U$ and $(\tau, \eta) \in \mu U$, or equivalently

$$(\varphi'^{\prime}_{\gamma_1}(v), \varphi'^{\prime}_{\gamma_1}(\tau)) \in \lambda U_{A_{\gamma_1}}, \quad (\varphi'^{\prime}_{\gamma_i}(\tau), \varphi'^{\prime}_{\gamma_i}(\eta)) \in \mu U_{A_{\gamma_i}} \quad (i = 1, \ldots, n).$$

Hence

$$\varphi'^{\prime}_{\gamma_1}(v, \eta) = (\varphi'^{\prime}_{\gamma_1}(v), \varphi'^{\prime}_{\gamma_1}(\eta)) = (\varphi'^{\prime}_{\gamma_1}(v), \varphi'^{\prime}_{\gamma_1}(\tau)) \circ (\varphi'^{\prime}_{\gamma_1}(\tau), \varphi'^{\prime}_{\gamma_1}(\eta)) \in \lambda U_{A_{\gamma_1}} \circ \mu U_{A_{\gamma_1}} \subset (\lambda + \mu)U_{A_{\gamma_1}} \quad (i = 1, \ldots, n),$$

that is

$$(v, \eta) \in (\lambda + \mu) \bigcap_{i=1}^n \varphi'^{-1}_{\gamma_1}(U_{A_{\gamma_i}}) = (\lambda + \mu)U \quad (i = 1, \ldots, n).$$

4. This property is trivial.

5. Let $v \in P^*$, $U = \bigcap_{i=1}^n \varphi'^{-1}_{\gamma_i}(U_{A_{\gamma_i}}) \in U$ and choose the strictly positive scalars $\lambda_1, \ldots, \lambda_n$ such that $(0, \varphi'^{\prime}_{\gamma_i}(v)) \in \lambda_i U_{A_{\gamma_i}}$. Put $\lambda = \max_{i=1}^n \lambda_i$, then we have

$$(0, v) \in \bigcap_{i=1}^n \lambda_i \varphi'^{-1}_{\gamma_i}(U_{A_{\gamma_i}}) \in \lambda U.$$

Now the projective limit on $P^*$ by the mappings $\varphi'_\gamma$ is the coarsest convex quasi-uniform structure such that the finite intersections of the sets $\varphi'^{-1}_{\gamma_i}(U_{A_{\gamma_i}}), A_\gamma \in X_\gamma$, are its members. Since

$$U_{\varphi_\gamma(A_{\gamma})} = \varphi'^{-1}_{\gamma}(U_{A_{\gamma}}), \quad A_\gamma \in X_\gamma, \quad \gamma \in \Gamma,$$

by Theorem 3.8, or

$$U_{\bigcup_{i=1}^n \varphi_\gamma(A_{\gamma_i})} = \bigcap_{i=1}^n \varphi'^{-1}_{\gamma_i}(U_{A_{\gamma_i}}), \quad A_\gamma \in U_{\gamma_i} \quad (i = 1, \ldots, n),$$

this convex quasi-uniform structure is that of $X$-topology. □
Corollary 3.16. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, $X$ be a collection of subsets of $\mathcal{P}$ satisfying the properties $(P_0)$–$(P_2)$. Then the $X$-topology on $\mathcal{P}^*$ induces on the dual $\mathcal{N}$ of $\mathcal{P}/\mathcal{M}$ the $k(X)$-topology.

Proof. The adjoint mapping $k'$ is the identity mapping of $\mathcal{N}$ into $\mathcal{P}^*$ and so the induced locally convex topology on $\mathcal{N}$ is the projective limit of $\mathcal{P}^*$ by $k'$. By the theorem, this is the $k(X)$-topology, since $\mathcal{P}/\mathcal{M}$ is the inductive limit of $\mathcal{P}$ by the mapping $k$. □

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