Journal of the Association of Arab Universities for Basic and Applied Sciences (2015) **18**, 46–59

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

Effects of tensors coupling to Dirac equation with shifted Hulthen potential via SUSYQM

Akpan N. Ikot [a,*], Bentol Hoda Yazarloo [b], Elham Maghsoodi [b], Saber Zarrinkamar [c], Hassan Hassanabadi [b]

[a] *Theoretical Physics Group, Department of Physics, University of Uyo, Nigeria*
[b] *Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran*
[c] *Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran*

Received 18 November 2013; revised 17 February 2014; accepted 31 March 2014
Available online 26 April 2014

KEYWORDS
Dirac equation;
Shifted Hulthen potential;
Supersymmetry method;
Spin and pseudospin symmetries

Abstract The Dirac equation with shifted Hulthen potential in the presence of the Yukawa-like tensor (YLT) and generalized tensor (GLT) interactions using supersymmetric quantum mechanics (SUSYQM) is presented. The bound state energy spectra and the radial wave functions have been approximately obtained in the case of spin and pseudospin symmetries. We have also reported some numerical results and figures to show the effect of the tensor interactions. Furthermore, scattering state solution under the generalized tensor (GLT) interaction is reported.

1. Introduction

It is known that the concept of spin and pseudospin symmetries of the Dirac equation in nuclear and hadronic spectroscopies have been great attention (Hassanabadi and Yazarloo, 2013) However, in order to investigate nuclear shell model, the study of spin and pseudospin symmetries of the Dirac equation has become an important area of research in nuclear physics (Ginocchio, 2005). These symmetries have been introduced many years ago in nuclear theory (Ginocchio, 2005, Ginocchio, 2004) and have been used successfully to explain the feature of deformed nuclei (Bohr et al., 1982)and superdeformation (Dudek et al., 1987) and establish an effective shell-model coupling scheme (Troltenier et al., 1995). Within the framework of the Dirac theory, spin symmetry arises if the magnitude of the spherical attractive scalar potential $S(r)$ and repulsive vector potential $V(r)$ is nearly equal, i.e., $S(r) \approx V(r)$ [2-3]. On the other hand, the jargon pseudospin symmetry refers to the case where the magnitude of the attractive Lorentz scalar potential $S(r)$ and the repulsive vector potential $V(r)$ is equal but opposite in sign, i.e., $S(r) = -V(r)$ (Ginocchio, 2005, 2004). In recent times many authors have investigated the Dirac equation with various potential models (Ikot, 2012; Hassanabadi et al., 2014; Wei and Dong, 2010a; Setare and Nazari, 2009; Ikot et al., 2013; Oyewumi and Akoshile, 2010). The spin symmetry in nuclear theory is usually referred to as a quasi-degeneracy of the single nucleon doublets and can be characterized with the non-relativistic quantum numbers $\left(n, l, j = l + \frac{1}{2}\right)$ and $\left(n, l, j = l - \frac{1}{2}\right)$, where n,l and j are the single –nucleon radial, orbital and total angular momentum quantum numbers for a single particle respectively. Also, the pseudospin symmetry implies that

* Corresponding author.
E-mail address: ndemikotphysics@gmail.com (A.N. Ikot).
Peer review under responsibility of University of Bahrain.

1815-3852 © 2014 Production and hosting by Elsevier B.V. on behalf of University of Bahrain.
http://dx.doi.org/10.1016/j.jaubas.2014.03.005

$\left(n, l, j = l + \frac{1}{2}\right)$ and $\left(n-1, l+2, j = l + \frac{3}{2}\right)$ states are degenerates. It had been shown that tensor interaction removes the degeneracy between two states in the pseudospin and spin doublets. However, due to the mathematical structure of the problem, different authors have devoted their investigation of the Dirac equation with the as Coulomb-like (Wei and Dong, 2010b,c; Aydogdu and Sever, 2010) or Cornell interaction as tensor interactions. Recently, Hassanabadi et al. (2012) first introduced the Yukawa tensor interaction and studied the Dirac equation for Yukawa potential within spin and pseudospin symmetries' limits. In the present study, our aim is to obtain the approximate analytical solutions of the Dirac equation for the scalar and vector shifted Hulthen potential together with the YLT and GTI (combined Coulomb-like and Yukawa-like) within the framework of spin and pseudospin symmetries' limits.

The Hulthén potential plays a significant role in atom and molecular physics (Bahar and Yasuk, 2013). It has also been used to explain the electronic properties of some alkali halides (Lu et al., 2012). Moreover, as it resembles the Coulomb interaction in structure, it has also been investigated within the framework of Dirac theory (Akcay, 2009). In this paper, we will study the Dirac equation with shifted Hulthén potential with spin-orbit coupling term quantum number κ for spin and pseudospin symmetries' limit using SUSYQM (Cooper et al., 1995). The paper is organized as follows. In Section 2, the Dirac theory within the framework of spin and pseudospin symmetries' limits is presented. Bound state solutions for YLT are presented in Section 3. Section 4 is devoted to the bound state solutions with GLT. Discussions of the Numerical results are given in Section 5. Finally, we give a brief conclusion in Section 6.

2. Theory of Dirac equation for spin and pseudospin symmetries

In the relativistic units ($\hbar = c = 1$), the Dirac equation both scalar $S(r)$ and vector $V(r)$ potentials read,

$$[\alpha \cdot p + \beta(M + S(r))]\psi(r) = [E - V(r)]\psi(r), \tag{1}$$

where E is the relativistic energy, M is the mass of a single particle, $\hat{p}$ momentum operator, α and β defined as,

$$\hat{p} = -i\nabla, \quad \alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2}$$

where I is the unit matrix and σ_i is the Pauli matrix. The total angular momentum J and $\widehat{K} = -\beta(\alpha.L + 1)$ of a particle commute with the Dirac Hamiltonian in a central field, where L is orbital angular momentum. The eigenvalues of the $\hat{K}$ are $\kappa = -\left(j + \frac{1}{2}\right)$ for aligned spin and $\kappa = +\left(j + \frac{1}{2}\right)$ for unaligned spin to a given total angular momentum j. The wave function can then be classified according to the angular momentum j and the spin-orbit quantum number κ as follows:

$$\psi_{nk}(r,\theta,\varphi) = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r) & Y^{l}_{jm}(\theta,\varphi) \\ iG_{n\kappa}(r) & Y^{\tilde{l}}_{jm}(\theta,\varphi) \end{pmatrix}, \tag{3}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the upper and the lower components of the wave function, $Y^{l}_{jm}(\theta,\varphi)$ and $Y^{\tilde{l}}_{jm}(\theta,\varphi)$ represent the spherical harmonics, n is the radial quantum number, m is the projection of the angular momentum on the z-axis and $l(l+1) = \kappa(\kappa+1), \tilde{l}(\tilde{l}+1) = \kappa(\kappa-1)$, where l is the orbital angular momentum spin quantum number and $\tilde{l}$ is the orbital angular momentum pseudospin quantum number respectively. The substitution of Eq. (3) into Eq. (1) yields two couple Dirac equations as follows:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n\kappa}(r) = [M + E_{n\kappa} - \Delta(r)]G_{n\kappa}(r), \tag{4}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n\kappa}(r) = [M - E_{n\kappa} + \Sigma(r)]F_{n\kappa}(r), \tag{5}$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$. By eliminating $G_{n\kappa}(r)$ in Eq. (4) and $F_{n\kappa}(r)$ in Eq. (5), we obtain two uncouple Schrödinger-like equations for the upper and lower components as follows:

$$\left\{\begin{array}{l} \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) \\ -(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) + \\ \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)}{M + E_{n\kappa} - \Delta(r)} \end{array}\right\} F_{n\kappa}(r) = 0, \tag{6}$$

$$\left\{\begin{array}{l} \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa U(r)}{r} + \frac{dU(r)}{dr} - U^2(r) \\ -(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) + \\ \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)}{M - E_{n\kappa} + \Sigma(r)} \end{array}\right\} G_{n\kappa}(r) = 0. \tag{7}$$

where $\kappa(\kappa - 1) = \tilde{l}(\tilde{l} + 1)$, $\kappa(\kappa + 1) = l(l + 1)$.

3. Bound state solutions

In this section, we intend to study the properties of the spin and pseudospin symmetries' limit using SUSYQM.

3.1. The spin symmetry limit with YLT

In the spin symmetry limit, $\frac{d\Delta(r)}{dr} = 0$ or $\Delta\ (r) = C_s =$ constant [2-3] and Eq. (6) reduces to

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) - \left(M + E^s_{n\kappa} - \Delta(r)\right)\left(M - E^s_{n\kappa} + \Sigma(r)\right)\right\}F^s_{n\kappa}(r) = 0, \tag{8}$$

where $\kappa = l$ for $\kappa < 0$ and $\kappa = -(l + 1)$ for $\kappa > 0$. We take the sum potential $\Sigma(r)$ as the shifted Hulthen potential, Ikot et al. (2013)

$$\Sigma(r) = \frac{V^s_0 + \frac{4}{b^2}}{\left(e^{\frac{2}{b}r} - 1\right)} + \frac{V^s_1}{\left(e^{\frac{2}{b}r} - 1\right)^2}, \tag{9}$$

where the parameters V^s_0, V^s_1 and b are real. In addition to the Yukawa tensor interaction, Yukawa (1935)

$$U(r) = \frac{-H_Y e^{-\frac{r}{b}}}{r} \tag{10}$$

where H_Y is the Yukawa parameter.

Substitution of the Eqs. (9) and (10) into Eq. (8) yields

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - \frac{2\kappa H_Y e^{-\frac{r}{b}}}{r^2} - \frac{H_Y^2 e^{-\frac{2r}{b}}}{r^2} - \frac{\frac{H_Y}{b}e^{-\frac{r}{b}}}{r} - \frac{H_Y e^{-\frac{r}{b}}}{r^2} - \left(M + E^s_{n\kappa} - C_s\right)\left(M - E^s_{n\kappa}\right)\right\}F^s_{n\kappa}(r) - \left(M + E^s_{n\kappa} - C_s\right)\left(\frac{V^s_0 + \frac{4}{b^2}}{\left(e^{\frac{2}{b}r} - 1\right)} + \frac{V^s_1}{\left(e^{\frac{2}{b}r} - 1\right)^2}\right)F^s_{n\kappa}(r) = 0. \tag{11}$$

Eq. (11) cannot be exactly solved due to the appearance of exponential and the centrifugal terms together. Therefore, we introduce the approximation, Hassanabadi et al. (2013)

$$\frac{1}{r^2} \approx \frac{\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}, \tag{12}$$

$$\frac{1}{r^2} \approx \frac{\left(\frac{2}{b}\right)^2 e^{-\frac{r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} \tag{13}$$

Inserting the latter into Eq. (11), we obtain

$$\begin{aligned}&\left\{\frac{d}{dr^2} - \frac{\left(\frac{2}{b}\right)^2 \kappa(\kappa+1)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{\left(\frac{2}{b}\right)^2 2\kappa H_Y e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}\right.\\ &\left. - \frac{\left(\frac{2}{b}\right)\frac{H_Y}{b} e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)} - \frac{\left(\frac{2}{b}\right)^2 H_Y e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{\left(\frac{2}{b}\right)^2 H_Y^2 e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}\right\} F^s_{n\kappa}(r)\\ &- \left(\frac{\beta_s\left(V^s_0 + \frac{4}{b^2}\right)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)} + \frac{\beta_s V^s_1 e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}\right) F^s_{n\kappa}(r)\\ &= \left(M + E^s_{n\kappa} - C_s\right)\left(M - E^s_{n\kappa}\right) F^s_{n\kappa}(r)\end{aligned} \tag{14}$$

where $\beta_s = \left(M + E^s_{n\kappa} - C_s\right)$.

Or more explicitly, we write Eq. (14) as,

$$-\frac{d^2 F^s_{n\kappa}}{dr^2} + V_{eff}(r) F^s_{n\kappa}(r) = \widetilde{E}^s_{n\kappa} F^s_{n\kappa}(r), \tag{15}$$

where,

$$V_{eff}(r) = \frac{A^s e^{-\frac{4r}{b}} + B^s e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}, \tag{16}$$

$$\begin{aligned}A^s = &\left(\frac{2}{b}\right)^2\left(H_Y - \frac{1}{2}\right)H_Y + MV^s_1 + E^s_{n\kappa}V^s_1 - C_s V^s_1\\ &- MV^s_0 - V^s_0 E^s_{n\kappa} + V^s_0 C_s - \frac{4M}{b^2} - \frac{4E^s_{n\kappa}}{b^2} + \frac{4C_s}{b^2},\end{aligned} \tag{17}$$

$$\begin{aligned}B^s = &\left(\frac{2}{b}\right)^2 \kappa(\kappa+1) + \left(\frac{2}{b}\right)^2\left(2\kappa + \frac{3}{2}\right)H_Y + MV^s_0\\ &+ E^s_{n\kappa}V^s_0 - C_s V^s_0 + \frac{4M}{b^2} + \frac{4E^s_{n\kappa}}{b^2} - \frac{4C_s}{b^2}\end{aligned} \tag{18}$$

$$\widetilde{E}^s_{n\kappa} = E^{s2}_{n\kappa} - M^2 + C_s\left(M - E^s_{n\kappa}\right) \tag{19}$$

In the SUSYQM formulation, the ground-state wave function $F^s_{0,\kappa}(r)$ is given by (Wei and Dong, 2009; Hassanabadi et al., 2013) (see Appendix A for more detail)

$$F^s_{0,\kappa}(r) = \exp\left(-\int W(r)dr\right), \tag{20}$$

in which the integrand is called the superpotential and the Hamiltonian is composed of the raising and lowering operators as(Wei and Dong, 2009; Hassanabadi et al., 2013)

$$H_- = \widehat{A}^+\widehat{A} = -\frac{d^2}{dr^2} + V_-(r), \tag{21}$$

$$H_+ = \widehat{A}\widehat{A}^+ = -\frac{d^2}{dr^2} + V_+(r), \tag{22}$$

with

$$\widehat{A} = \frac{d}{dr} - W(r), \tag{23}$$

$$\widehat{A}^+ = -\frac{d}{dr} - W(r), \tag{24}$$

$$V_\pm(r) = W^2(r) \mp W'(r) \tag{25}$$

Thus, we have to first solve the associated Riccati equation

$$W^2(r) \mp W'(r) = V_{eff}(r) - \widetilde{E}^s_{0,\kappa}, \tag{26}$$

for which we propose a solution of the form

$$W(r) = \frac{p^s e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)} + q^s. \tag{27}$$

Thus, we can obtain the exact parameter of our study as,

$$\begin{aligned}&\frac{(p^s)^2 e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} + (q^s)^2 + \frac{2p^s q^s e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)} + \frac{\left(\frac{2}{b}\right)p^s e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}\\ &= \frac{A^s e^{-\frac{4r}{b}} + B^s e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \widetilde{E}^s_{0\kappa}\end{aligned} \tag{28}$$

or more explicitly,

$$\widetilde{E}^s_{0,\kappa} = -(q^s)^2 \tag{29}$$

$$p^s = -\left(\frac{1}{b}\right) \pm \sqrt{\left(\frac{1}{b}\right)^2 + (A^s + B^s)}, \tag{30}$$

$$q^s = -\left(\frac{(p^s)^2 - A^s}{2p^s}\right) \tag{31}$$

Now based on Eq. (25), we can obtain the supersymmetric partner potentials as,

$$\begin{aligned}V_+(r) &= \frac{p^s\left(p^s - \frac{2}{b}\right)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{A^s e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)} + \left(\frac{(p^s)^2 - A^s}{2p^s}\right)^2\\ V_-(r) &= \frac{p^s\left(p^s + \frac{2}{b}\right)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{A^s e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)} + \left(\frac{(p^s)^2 - A^s}{2p^s}\right)^2\end{aligned} \tag{32}$$

Therefore, it is shown that $V_+(r)$ and $V_-(r)$ are shape invariant, satisfying the shape-invariant condition

$$V_+(r, a_0) = V_-(r, a_1) + R(a_1), \tag{33}$$

with $a_0 = p^s$ and a_i is a function of a_0, i.e., $a_1 = f(a_0) = a_0 - \frac{2}{b}$. Therefore, $a_n = f(a_0) = a_0 - \frac{2n}{b}$. Thus, we can see that the shape invariance holds via a mapping of the form $p^s \to p^s - \frac{2}{b}$. From Eq. (33), we have

$$\begin{aligned}R(a_1) &= \left(\frac{(a_0)^2 - A^s}{2a_0}\right)^2 - \left(\frac{(a_1)^2 - A^s}{2a_1}\right)^2,\\ R(a_2) &= \left(\frac{(a_1)^2 - A^s}{2a_1}\right)^2 - \left(\frac{(a_2)^2 - A^s}{2a_2}\right)^2,\\ &\vdots\\ R(a_n) &= \left(\frac{(a_{n-1})^2 - A^s}{2a_{n-1}}\right)^2 - \left(\frac{(a_n)^2 - A^s}{2a_n}\right)^2,\end{aligned} \tag{34}$$

The energy eigenvalues can be obtained as follows

$$\widetilde{E}^s_{n\kappa} = \widetilde{E}^{s-}_{n\kappa} + \widetilde{E}^s_{0,\kappa}, \tag{35}$$

where

$$\widetilde{E}^{s-}_{n\kappa} = \sum_{k=1}^{n} R(a_k) = \left(\frac{(a_0)^2 - A^s}{2a_0}\right)^2 - \left(\frac{(a_n)^2 - A^s}{2a_n}\right)^2, \quad (36)$$

By substituting Eqs. (31) and (36) into Eq. (35), we get

$$E^{s2}_{n\kappa} - M^2 + C_s\left(M - E^s_{n\kappa}\right) = -\left(\frac{(a_n)^2 - A^s}{2a_n}\right)^2 \quad (37)$$

Eq. (37) gives the energy equation for the shifted Hulthén potential. Introducing the new variable of the form $y = e^{-\frac{2r}{b}}$, we obtain the upper component of the wave function as,

$$F^s_{n\kappa}(r) = N^s_{n\kappa}\left(e^{-\frac{2r}{b}}\right)^{\sqrt{w_3^s}}\left(1 - e^{-\frac{2r}{b}}\right)^{1/2+\sqrt{w_1^s - w_2^s + w_3^s + 1/4}}\,{}_2F_1\left(-n, n+2\sqrt{w_3^s}+2\sqrt{\frac{1}{4}+w_1^s+w_3^s-w_2^s}+1; 2\sqrt{w_3^s}+1; e^{-\frac{2r}{b}}\right) \quad (38)$$

with

$$w_1^s = \frac{b^2}{4}\left(A^s - \widetilde{E}^s_{n\kappa}\right), \quad w_2^s = -\frac{b^2}{4}\left(B^s + 2\widetilde{E}^s_{n\kappa}\right), \quad w_3^s = -\frac{b^2\widetilde{E}^s_{n\kappa}}{4}. \quad (39)$$

where $N_{n\kappa}$ is the normalization constant. For the lower component, we can simply use

$$G^s_{n\kappa}(r) = \frac{1}{M + E^s_{n\kappa} - C_s}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F^s_{n\kappa}(r). \quad (40)$$

3.2. The pseudospin symmetry limit with YLT

In the pseudospin symmetry limit, $\frac{d\Sigma(r)}{dr} = 0$ or $\Sigma(r) = C_{ps} =$ constant (Ginocchio, 2005, 2004) and Eq. (7) reduces to

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{dU(r)}{dr} - U^2(r) - \left(M + E^{ps}_{n\kappa} - \Delta(r)\right)\left(M - E^{ps}_{n\kappa} + \Sigma(r)\right)\right\}G^{ps}_{n\kappa}(r) = 0, \quad (41)$$

where $\kappa = -\tilde{\ell}$ and $\kappa = \tilde{\ell} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Here, we take the difference of the potential as the shifted Hulthen potential,

$$\Delta(r) = \frac{V_0^{ps} + \frac{4}{b^2}}{\left(e^{\frac{2}{b}r} - 1\right)} + \frac{V_1^{ps}}{\left(e^{\frac{2}{b}r} - 1\right)^2}, \quad (42)$$

Substituting Eqs. (10) and (42) into Eq. (7), we arrive at

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa H_Y e^{-\frac{r}{b}}}{r^2} + \frac{\left(\frac{H_Y}{b}\right)e^{-\frac{r}{b}}}{r} + \frac{H_Y e^{-\frac{r}{b}}}{r^2} - \frac{H_Y^2 e^{-\frac{2r}{b}}}{r^2} - \varepsilon^2_{ps}\right\}G^{ps}_{n\kappa}(r) + \beta_{ps}\left(\frac{V_0^{ps} + \frac{4}{b^2}}{\left(e^{\frac{2}{b}r} - 1\right)} + \frac{V_1^{ps}}{\left(e^{\frac{2}{b}r} - 1\right)^2}\right)G^{ps}_{n\kappa}(r) = 0, \quad (43)$$

where $\varepsilon^2_{ps} = \left(M + E^{ps}_{n\kappa}\right)\left(M - E^{ps}_{n\kappa} + C_{ps}\right)$ and $\beta_{ps} = \left(M - E^{ps}_{n\kappa} + C_{ps}\right)$.

Substituting Eqs. (12) and (13) into Eq. (43), we obtain

$$\left\{\frac{d^2}{dr^2} - \frac{\left(\frac{2}{b}\right)^2\kappa(\kappa-1)e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} - \frac{2\left(\frac{2}{b}\right)^2\kappa H_Y e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} + \frac{\left(\frac{2}{b}\right)\left(\frac{H_Y}{b}\right)e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)} + \frac{\left(\frac{2}{b}\right)^2 H_Y e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} - \frac{\left(\frac{2}{b}\right)^2 H_Y^2 e^{-\frac{4r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2}\right\}G^{ps}_{n\kappa}(r) + \left\{-\varepsilon^2_{ps} + \beta_{ps}\left(\frac{V_0^{ps} + \frac{4}{b^2}}{\left(e^{\frac{2}{b}r} - 1\right)} + \frac{V_1^{ps}}{\left(e^{\frac{2}{b}r} - 1\right)^2}\right)\right\}G^{ps}_{n\kappa}(r) = 0, \quad (44)$$

Eq. (44) can be more neatly written as

$$-\frac{d^2G^{ps}_{n\kappa}}{dr^2} + V_{eff}(r)G^{ps}_{n\kappa}(r) = \widetilde{E}^{ps}_{n\kappa}G^{ps}_{n\kappa}(r), \quad (45)$$

with

$$V_{eff}(r) = \frac{A^{ps}e^{-\frac{4r}{b}} + B^{ps}e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2}, \quad (46)$$

$$A^{ps} = \left(\frac{2}{b}\right)^2\left(H_Y + \frac{1}{2}\right)H_Y + MV_0^{ps} - E^{ps}_{n\kappa}V_0^{ps} + C_{ps}V_0^{ps} + \frac{4M}{b^2} - \frac{4E^{ps}_{n\kappa}}{b^2} + \frac{4C_{ps}}{b^2} - MV_1^{ps} + E^{ps}_{n\kappa}V_1^{ps} - C_{ps}V_1^{ps} \quad (47)$$

$$B^{ps} = \left(\frac{2}{b}\right)^2\left(2\kappa - \frac{3}{2}\right)H_Y - MV_0^{ps} + E^{ps}_{n\kappa}V_0^{ps} - C_{ps}V_0^{ps} - \frac{4M}{b^2} + \frac{4E^{ps}_{n\kappa}}{b^2} - \frac{4C_{ps}}{b^2} + \left(\frac{2}{b}\right)^2\kappa(\kappa-1) \quad (48)$$

The corresponding effective energy in this case is

$$\widetilde{E}^{ps}_{n\kappa} = \left(E^{ps}_{n\kappa}\right)^2 - M^2 - C_{ps}\left(M + E^{ps}_{n\kappa}\right), \quad (49)$$

and the superpotential posses the form

$$W^{ps}(r) = \frac{f^{ps}e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)} + g^{ps}, \quad (50)$$

In this case, the coefficients are explicitly given as

$$\widetilde{E}^{ps}_{0\kappa} = -(g^{ps})^2 \quad (51)$$

$$f^{ps} = -\left(\frac{1}{b}\right) \pm \sqrt{\left(\frac{1}{b}\right)^2 + (A^{ps} + B^{ps})}, \quad (52)$$

$$g^{ps} = \frac{(f^{ps})^2 - A^{ps}}{2f^{ps}} \quad (53)$$

From Eq. (A.2), we can obtain the supersymmetric partner potentials as

$$V_+(r) = \frac{f^{ps}\left(f^{ps} - \frac{2}{b}\right)e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} - \frac{A^{ps}e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)} + \left(\frac{(f^{ps})^2 - A^{ps}}{2f^{ps}}\right)^2 \quad (54)$$

$$V_-(r) = \frac{f^{ps}\left(f^{ps} + \frac{2}{b}\right)e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} - \frac{A^{ps}e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)} + \left(\frac{(f^{ps})^2 - A^{ps}}{2f^{ps}}\right)^2 \quad (55)$$

With the mapping being

$$\rho_n = f(\rho_0) = \rho_0 - \left(\frac{2n}{b}\right) = f^{ps} - \left(\frac{2n}{b}\right). \quad (56)$$

Conversely, we have

$$R(\rho_1) = \left(\frac{(\rho_0)^2 - A^{ps}}{2\rho_0}\right)^2 - \left(\frac{(\rho_1)^2 - A^{ps}}{2\rho_1}\right)^2,$$
$$R(\rho_2) = \left(\frac{(\rho_1)^2 - A^{ps}}{2\rho_1}\right)^2 - \left(\frac{(\rho_2)^2 - A^{ps}}{2\rho_2}\right)^2, \quad (57)$$
$$\vdots$$
$$R(\rho_n) = \left(\frac{(\rho_{n-1})^2 - A^{ps}}{2\rho_{n-1}}\right)^2 - \left(\frac{(\rho_n)^2 - A^{ps}}{2\rho_n}\right)^2$$

The energy eigenvalues can be obtained as follows

$$\widetilde{E}^{ps}_{n\kappa} = \widetilde{E}^{ps-}_{n\kappa} + \widetilde{E}^{ps}_{0,\kappa}, \quad (58)$$

where,

$$\widetilde{E}^{ps-}_{n\kappa} = \sum_{k=1}^{n} R(\rho_k) = \left(\frac{(\rho_0)^2 - A^{ps}}{2\rho_0}\right)^2 - \left(\frac{(\rho_n)^2 - A^{ps}}{2\rho_n}\right)^2, \quad (59)$$

Finally, the energy relation can be written as

$$\left(E^{ps}_{n\kappa}\right)^2 - M^2 - C_{ps}\left(M + E^{ps}_{n\kappa}\right) + \left(\frac{(\rho_n)^2 - A^{ps}}{2\rho_n}\right)^2 = 0, \quad (60)$$

where

$$\rho_n = -\left(\frac{2}{b}\right)[n + \sigma^{ps}], \quad (61)$$

$$\sigma^{ps} = \frac{1}{2}\left(1 \pm \sqrt{1 + b^2(A^{ps} + B^{ps})}\right) \quad (62)$$

and the corresponding lower component of the wave function becomes

$$G^{ps}_{n\kappa}(r) = N^{ps}_{n\kappa}\left(e^{-\frac{2r}{b}}\right)^{\sqrt{w_3^{ps}}}\left(1 - e^{-\frac{2r}{b}}\right)^{1/2 + \sqrt{w_1^{ps} - w_2^{ps} + w_3^{ps} + 1/4}}$$
$$\times {}_2F_1\left(-n, n + 2\sqrt{w_3^{ps}} + 2\sqrt{\frac{1}{4} + w_1^{ps} + w_3^{ps} - w_2^{ps}} + 1; 2\sqrt{w_3^{ps}} + 1; e^{-\frac{2r}{b}}\right) \quad (63)$$

where,

$$w_1^{ps} = \frac{b^2}{4}\left(A^{ps} - \widetilde{E}^{ps}_{n\kappa}\right),$$
$$w_2^{ps} = -\frac{b^2}{4}\left(B^{ps} + 2\widetilde{E}^{ps}_{n\kappa}\right), \quad (64)$$
$$w_3^{ps} = -\frac{b^2\widetilde{E}^{ps}_{n\kappa}}{4}.$$

The upper component can be found by using the following relation

$$F^{ps}_{n\kappa}(r) = \frac{1}{\left(E^{ps}_{n\kappa} - M + C_{ps}\right)}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G^{ps}_{n\kappa}(r). \quad (65)$$

4. Pseudospin and spin symmetries' limits under GLT interaction

In the following section, we intend to investigate the Dirac equation with shifted Hulthen potential in the presence of GLT interactions. The presence of GLT potential as the tensor term in the Dirac equation also removes the degeneracies in addition to the Coulomb-like and Yukawa-like tensor interactions. Thus, it is pertinent to investigate this potential under consideration with GLT as interaction term.

4.1. Pseudospin symmetry in the Dirac equation with GLT

The pseudospin symmetry occurs in the Dirac theory as $\frac{d\Sigma(r)}{dr} = 0$ or equivalently $\Sigma(r) = C_{ps} = const$ (Ginocchio, 2005, 2004). In order to find the approximate analytical solution of the Dirac equation under the pseudospin symmetry limit, we take the difference of the scalar and vector potentials as the shifted Hulthen potential,

$$\Delta(r) = \frac{V_0^{ps} + \frac{4}{b^2}}{\left(e^{\frac{2}{b}r} - 1\right)} + \frac{V_1^{ps}}{\left(e^{\frac{2}{b}r} - 1\right)^2}, \quad (66)$$

In addition, we proposed a novel generalized tensor interaction of the form,

$$U(r) = -(U_C(r) + U_Y(r)), \quad (67)$$

where $U_C(r)$ and $U_Y(r)$ are the Coulomb–like and Yukawa-like potentials (Wei and Dong, 2010a;Yukawa, 1935) defined as,

$$U_C(r) = -\frac{H_c}{r},$$
$$U_Y(r) = -H_Y\frac{e^{-\frac{r}{b}}}{r} \quad (68)$$

If we identify $U_C(r)$ as the standard Coulomb potential, the potential parameter H_c is Coulomb parameter and H_Y is the Yukawa potential.

Substituting Eq. (68) into Eq. (67), we obtained our proposed GTI as,

$$U(r) = -\frac{1}{r}\left(H_c + H_Y e^{-\frac{r}{b}}\right) \quad (69)$$

Substituting the above equations into Eq. (7) yields

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa H_c}{r^2} + \frac{H_c}{r^2} - \frac{H_c^2}{r^2} - \left(M + E^{ps}_{n,\kappa}\right)\left(M - E^{ps}_{n,\kappa} + C_{ps}\right)\right\}G^{ps}_{n,\kappa}(r)$$
$$\left(-\frac{2H_Y\kappa e^{-\frac{r}{b}}}{r^2} + \frac{\left(\frac{H_Y}{b}\right)e^{-\frac{r}{b}}}{r} + \frac{H_Y e^{-\frac{r}{b}}}{r^2} - \frac{2H_c H_Y e^{-\frac{r}{b}}}{r^2} - \frac{H_Y^2 e^{-\frac{2r}{b}}}{r^2}\right)G^{ps}_{n\kappa}(r)$$
$$+\left(M - E^{ps}_{n,\kappa} + C_{ps}\right)\left(\frac{V_0^{ps} + \frac{4}{b^2}}{\left(e^{\frac{2}{b}r} - 1\right)} + \frac{V_1^{ps}}{\left(e^{\frac{2}{b}r} - 1\right)^2}\right)G^{ps}_{n,\kappa}(r) = 0. \quad (70)$$

It is well known that the above equation cannot be solved exactly due to the centrifugal term r^{-2}. In order to get rid of the centrifugal term, we make use of the following approximation (Hassanabadi et al., 2013)

$$\frac{1}{r^2} \approx \frac{\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2}, \quad (71)$$

$$\frac{1}{r^2} \approx \frac{\left(\frac{2}{b}\right)^2 e^{-\frac{r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} \quad (72)$$

By using the approximations (71) for ($\frac{\kappa(\kappa-1)}{r^2}$, $\frac{2\kappa H_c}{r^2}$, $\frac{H_c}{r^2}$, $\frac{H_c^2}{r^2}$, $\frac{\left(\frac{H_Y}{b}\right)e^{-\alpha r}}{r}$ and $\frac{H_Y^2 e^{-\frac{2r}{b}}}{r^2}$), and the approximation (72) for ($\frac{2H_Y\kappa e^{-\frac{r}{b}}}{r^2}$, $\frac{H_Y e^{-\frac{r}{b}}}{r^2}$ and $\frac{2H_c H_Y e^{-\frac{r}{b}}}{r^2}$), we obtain

$$\left\{\frac{d^2}{dr^2} - \frac{\left(\frac{2}{b}\right)^2\kappa(\kappa-1)e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} - \frac{2\kappa H_c\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} - \frac{2\kappa H_Y\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} + \frac{H_c\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2}\right\}G^{ps}_{n\kappa}$$
$$+\left\{\frac{\left(\frac{2}{b}\right)\left(\frac{H_Y}{b}\right)e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)} + \frac{H_Y\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} - \frac{H_c^2\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2}\right\}G^{ps}_{n\kappa}$$
$$+\left\{-\frac{2H_c H_Y\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} - \frac{H_Y^2\left(\frac{2}{b}\right)^2 e^{-\frac{4r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2} + \frac{\beta_{ps}\left(V_0^{ps} + \frac{4}{b^2}\right)e^{-\frac{2r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)} + \frac{\beta_{ps}V_1^{ps}e^{-\frac{4r}{b}}}{\left(1 - e^{-\frac{2r}{b}}\right)^2}\right\}G^{ps}_{n\kappa}(r)$$
$$= \left(M + E^{ps}_{n\kappa}\right)\left(M - E^{ps}_{n\kappa} + C_{ps}\right)G^{ps}_{n\kappa}, \quad (73)$$

Or more explicitly, we write

$$-\frac{d^2G^{ps}_{n\kappa}}{dr^2}+V_{eff}(r)G^{ps}_{n\kappa}(r)=\widetilde{E}^{ps}_{n\kappa}G^{ps}_{n\kappa}, \tag{74}$$

where

$$V_{eff}(r)=\frac{D^{ps}e^{-\frac{4r}{b}}+H^{ps}e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} \tag{75}$$

$$D^{ps}=\left(\frac{2}{b}\right)^2\left(H_Y+\frac{1}{2}\right)H_Y+MV^{ps}_0 - E^{ps}_{n\kappa}V^{ps}_0+C_{ps}V^{ps}_0+\frac{4M}{b^2}-\frac{4E^{ps}_{n\kappa}}{b^2}+\frac{4C_{ps}}{b^2}-MV^{ps}_1 + E^{ps}_{n\kappa}V^{ps}_1-C_{ps}V^{ps}_1, \tag{76}$$

$$H^{ps}=\left(\frac{2}{b}\right)^2\gamma+\left(\frac{2}{b}\right)^2\left(2\kappa+2H_c-\frac{3}{2}\right)H_Y-MV^{ps}_0 + E^{ps}_{n\kappa}V^{ps}_0-C_{ps}V^{ps}_0-\frac{4M}{b^2}+\frac{4E^{ps}_{n\kappa}}{b^2}-\frac{4C_{ps}}{b^2}, \tag{77}$$

$$\gamma=2\kappa H_c+H_c^2-H_c+\kappa(\kappa-1)=(\kappa+H_c)(\kappa+H_c-1) =\eta_\kappa(\eta_\kappa-1)\rightarrow\eta_\kappa=(\kappa+H_c) \tag{78}$$

$$\widetilde{E}^{ps}_{n\kappa}=-M^2+ME^{ps}_{n\kappa}-MC_{ps}-ME^{ps}_{n\kappa}+\left(E^{ps}_{n\kappa}\right)^2-E^{ps}_{n\kappa}C_{ps} \tag{79}$$

In order to solve Eq. (74), we have to first solve the associated Riccati equation

$$W^2(r)\mp W'(r)=V_{eff}(r)-\widetilde{E}^s_{0,\kappa}, \tag{80}$$

for which we propose a solution of the form

$$W(r)=\frac{p^{ps}e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+q^{ps}. \tag{81}$$

Thus, we can obtain the exact parameter of our study as,

$$\frac{(p^{ps})^2e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}+(q^{ps})^2+\frac{2p^{ps}q^{ps}e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+\frac{\left(\frac{2}{b}\right)p^{ps}e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} =\frac{D^{ps}e^{-\frac{4r}{b}}+H^{ps}e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}-\widetilde{E}^s_{0\kappa} \tag{82}$$

or more explicitly,

$$\widetilde{E}^s_{0,\kappa}=-(q^{ps})^2, \tag{83}$$

$$p^{ps}=-\left(\frac{1}{b}\right)\pm\sqrt{\left(\frac{1}{b}\right)^2+(D^{ps}+H^{ps})}, \tag{84}$$

$$q^{ps}=-\left(\frac{(p^{ps})^2-D^{ps}}{2p^{ps}}\right), \tag{85}$$

Now based on Eq. (A.2), we can obtain the supersymmetric partner potentials as,

$$V_+(r)=\frac{p^{ps}\left(p^{ps}-\frac{2}{b}\right)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}-\frac{D^{ps}e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+\left(\frac{(p^{ps})^2-D^{ps}}{2p^{ps}}\right)^2$$
$$V_-(r)=\frac{p^{ps}\left(p^{ps}+\frac{2}{b}\right)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}-\frac{D^{ps}e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}+\left(\frac{(p^{ps})^2-D^{ps}}{2p^{ps}}\right)^2 \tag{86}$$

Therefore, it is shown that $V_+(r)$ and $V_-(r)$ are shape invariant, satisfying the shape-invariant condition

$$V_+(r,a_0)=V_-(r,a_1)+R(a_1), \tag{87}$$

with $a_0=f^s$ and a_i is a function of a_0, i.e., $a_1=f(a_0)=a_0-\left(\frac{2}{b}\right)$. Therefore, $a_n=f(a_0)=a_0-\left(\frac{2n}{b}\right)$. Thus, we can see that the shape invariance holds via a mapping of the form $f^{ps}\rightarrow f^{ps}-\left(\frac{2}{b}\right)$. From Eq. (A.5), we have

$$\begin{aligned} R(a_1)&=\left(\frac{(a_0)^2-D^{ps}}{2a_0}\right)^2-\left(\frac{(a_1)^2-D^{ps}}{2a_1}\right)^2,\\ R(a_2)&=\left(\frac{(a_1)^2-D^{ps}}{2a_1}\right)^2-\left(\frac{(a_2)^2-D^{ps}}{2a_2}\right)^2,\\ &\vdots\\ R(a_n)&=\left(\frac{(a_{n-1})^2-D^{ps}}{2a_{n-1}}\right)^2-\left(\frac{(a_n)^2-D^{ps}}{2a_n}\right)^2, \end{aligned} \tag{88}$$

The energy eigenvalues can be obtained as follows

$$\widetilde{E}^{ps}_{n\kappa}=\widetilde{E}^{ps-}_{n\kappa}+\widetilde{E}^{ps}_{0,\kappa}, \tag{89}$$

where

$$\widetilde{E}^{ps-}_{n\kappa}=\sum_{k=1}^{n}R(a_k)=\left(\frac{(a_0)^2-D^{ps}}{2a_0}\right)^2-\left(\frac{(a_n)^2-D^{ps}}{2a_n}\right)^2, \tag{90}$$

By substituting Eqs. (90) and (83) into Eq. (89), We get

$$\widetilde{E}^s_{n\kappa}=-\left(\frac{(a_n)^2-D^{ps}}{2a_n}\right)^2. \tag{91}$$

Substituting Eqs. (76), and (78) into Eq. (91), we obtain the energy equation for the shifted Hulthén potential as

$$\left(E^s_{n\kappa}\right)^2-M^2+C_s\left(M-E^s_{n\kappa}\right) +\frac{1}{b^2}\left[(n+\delta^{ps})-\left(\frac{b}{2}\right)^2\left\{\frac{D^{ps}}{n+\delta^{ps}}\right\}\right]^2=0, \tag{92}$$

where

$$a_n=-\left(\frac{2}{b}\right)[n+\delta^{ps}], \tag{93}$$

$$\delta^{ps}=\frac{1}{2}\left(1\pm\sqrt{1+b^2(D^{ps}+H^{ps})}\right) \tag{94}$$

The corresponding lower component of the wave function becomes,

$$G^{ps}_{n\kappa}(r)=N^{ps}_{n\kappa}\left(e^{-\frac{2r}{b}}\right)^{\sqrt{w^{ps}_3}}\left(1-e^{-\frac{2r}{b}}\right)^{1/2+\sqrt{w^{ps}_1-w^{ps}_2+w^{ps}_3+1/4}} {}_2F_1\left(-n,n+2\sqrt{w^{ps}_3}+2\sqrt{\frac{1}{4}+w^{ps}_1+w^{ps}_3-w^{ps}_2}+1;2\sqrt{w^{ps}_3}+1;e^{-\frac{2r}{b}}\right) \tag{95}$$

with

$$w^{ps}_1=\frac{b^2}{4}\left(D^{ps}-\widetilde{E}^{ps}_{n\kappa}\right), \tag{96}$$

$$w^{ps}_2=-\frac{b^2}{4}\left(H^{ps}+2\widetilde{E}^{ps}_{n\kappa}\right), \tag{97}$$

$$w^{ps}_3=-\frac{b^2\widetilde{E}^{ps}_{n\kappa}}{4}. \tag{98}$$

where $N_{n\kappa}$ is the normalization constant. For the lower component, we can simply use

$$F^{ps}_{n\kappa}(r)=\frac{1}{M-E^{ps}_{n\kappa}+C_{ps}}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G^{ps}_{n\kappa}(r). \tag{99}$$

4.2. Spin symmetry in the Dirac equation with GLT

In the spin symmetry limit condition, we take the sum potential $\Sigma(r)$ as the shifted Hulthen potential, the difference potential $\Delta(r)$ as constant and the tensor potential $U(r)$ as the GTI term. Thus, we have the following

$$\Sigma(r) = \frac{V_0^s + \frac{4}{b^2}}{\left(e^{\frac{2}{b}r} - 1\right)} + \frac{V_1^s}{\left(e^{\frac{2}{b}r} - 1\right)^2}, \Delta(r) = C_s, \quad U(r) = -\frac{1}{r}\left(H_c + H_Y e^{-\frac{r}{b}}\right). \tag{100}$$

Substituting Eq. (100) into Eq. (6) yields,

$$\left\{\begin{array}{l} \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - \frac{2\kappa H_c}{r^2} - \frac{H_c}{r^2} - \frac{H_c^2}{r^2} - (M+E_{n\kappa}-C_s)(M-E_{n\kappa}) \\ -(M+E_{n\kappa}-C_s)\left(\frac{V_0^s+\frac{4}{b^2}}{\left(e^{\frac{2}{b}r}-1\right)} + \frac{V_1^s}{\left(e^{\frac{2}{b}r}-1\right)^2}\right) \end{array}\right\} F^s_{n,\kappa}(r) - \left(\frac{2\kappa H_Y e^{-\frac{r}{b}}}{r^2} + \frac{\left(\frac{H_Y}{b}\right)e^{-\frac{r}{b}}}{r} + \frac{H_Y e^{-\frac{r}{b}}}{r^2} + \frac{2H_c H_Y e^{-\frac{r}{b}}}{r^2} + \frac{H_Y^2 e^{-\frac{2r}{b}}}{r^2}\right) F^s_{n\kappa}(r) = 0 \tag{101}$$

By using approximation (71) for ($\frac{\kappa(\kappa+1)}{r^2}$, $\frac{2\kappa H_c}{r^2}$, $\frac{H_c}{r^2}$, $\frac{H_c^2}{r^2}$, $\frac{\left(\frac{H_Y}{b}\right)e^{-\frac{r}{b}}}{r}$ and $\frac{H_Y^2 e^{-\frac{2r}{b}}}{r^2}$), and approximation (72) for ($\frac{2H_Y\kappa e^{-\frac{r}{b}}}{r^2}$, $\frac{H_Y e^{-\frac{r}{b}}}{r^2}$ and $\frac{2H_c H_Y e^{-\frac{r}{b}}}{r^2}$), the above second-order differential equation becomes,

$$\left\{\frac{d^2}{dr^2} - \frac{\left(\frac{2}{b}\right)^2\kappa(\kappa+1)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{2\kappa H_c\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{2\kappa H_Y\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{H_c\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{\left(\frac{H_Y}{b}\right)\left(\frac{2}{b}\right)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)}\right\} F^s_{n\kappa}(r)$$
$$-\left\{\frac{H_Y\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} + \frac{\left(\frac{2}{b}\right)^2 H_c^2 e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} + \frac{2H_cH_Y\left(\frac{2}{b}\right)^2 e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} + \frac{\left(\frac{2}{b}\right)^2 H_Y^2 e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} + \frac{\beta_s\left(V_0^s+\frac{4}{b^2}\right)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} + \frac{\beta_s V_1^s e^{-\frac{4r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}\right\} F^s_{n\kappa}(r)$$
$$= \left(M+E^s_{n\kappa}-C_s\right)\left(M-E^s_{n\kappa}\right)F^s_{n\kappa}(r) \tag{102}$$

Or more neatly, we write Eq. (102) as,

$$-\frac{d^2F^s_{n\kappa}(r)}{dr^2} + V_{eff}(r)F^s_{n\kappa}(r) = \widetilde{E}^s_{n\kappa}F^s_{n\kappa}(r), \tag{103}$$

where

$$V_{eff}(r) = \frac{D^s e^{-\frac{4r}{b}} + H^s e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2}, \tag{104}$$

$$D^s = \left(\frac{2}{b}\right)^2\left(H_Y - \frac{1}{2}\right)H_Y - MV_0^s - E^s_{n\kappa}V_0^s + C_sV_0^s - \frac{4M}{b^2} - \frac{4E^s_{n\kappa}}{b^2} + \frac{4C_s}{b^2} + MV_1^s + E^s_{n\kappa}V_1^s - C_sV_1^s \tag{105}$$

$$H^s = \left(\frac{2}{b}\right)^2\gamma^s + \left(\frac{2}{b}\right)^2\left(2\kappa + 2H_c + \frac{3}{2}\right)H_Y + MV_0^s + E^s_{n\kappa}V_0^s - C_sV_0^s + \frac{4M}{b^2} + \frac{4E^s_{n\kappa}}{b^2} - \frac{4C_s}{b^2} \tag{106}$$

$$\widetilde{E}^s_{n\kappa} = -\left(M+E^s_{n\kappa}-C_s\right)\left(M-E^s_{n\kappa}\right) \tag{107}$$

$$\gamma^s = \kappa(\kappa+1) + 2\kappa H_c + H_c + H_c^2 = (\kappa+H_c)(\kappa+H_c+1) = \Lambda_\kappa(\Lambda_\kappa - 1) \rightarrow \Lambda_\kappa = (\kappa + H_c + 1) \tag{108}$$

Using the same procedure we obtain the energy equation in the Dirac theory for the spin symmetry limits written as

$$\left(E^s_{n\kappa}\right)^2 - M^2 + C_s\left(M - E^s_{n\kappa}\right) + \left(\frac{(\rho_n)^2 - D^s}{2\rho_n}\right)^2 = 0, \tag{109}$$

where

$$\rho_n = -\left(\frac{2}{b}\right)[n + \sigma^s], \tag{110}$$

$$\sigma^s = \frac{1}{2}\left(1 \pm \sqrt{1 + b^2(D^s + H^s)}\right). \tag{111}$$

The wave function can be obtained as follows:

$$F^s_{n\kappa}(r) = N^s_{n\kappa}\left(e^{-\frac{2r}{b}}\right)^{\sqrt{w_3^s}}\left(1-e^{-\frac{2r}{b}}\right)^{1/2+\sqrt{w_1^s - w_2^s + w_3^s + 1/4}} \times {}_2F_1\left(-n, n + 2\sqrt{w_3^s} + 2\sqrt{\frac{1}{4} + w_1^s + w_3^s - w_2^s} + 1; 2\sqrt{w_3^s} + 1; e^{-\frac{2r}{b}}\right) \tag{112}$$

with

$$w_1^s = \frac{b^2}{4}\left(D^s - \widetilde{E}^s_{n\kappa}\right), \quad w_2^s = -\frac{b^2}{4}\left(H^s + 2\widetilde{E}^s_{n\kappa}\right), \quad w_3^s = -\frac{b^2\widetilde{E}^s_{n\kappa}}{4}. \tag{113}$$

where $N_{n\kappa}$ is the normalization constant. For the lower component, we can simply use

$$G^s_{n\kappa}(r) = \frac{1}{M + E^s_{n\kappa} - C_s}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F^s_{n\kappa}(r). \tag{114}$$

5. Scattering state solution

In this section we are going to obtain the scattering properties of Dirac equation for the shifted Hulthen potential under the pseudospin and spin symmetries. First we consider the pseudospin symmetry.

5.1. Scattering state solution for the pseudospin symmetry

Introducing a new variable of the form $z = 1 - e^{-2r/b}$, changes Eq. (73) as

$$\left\{z(1-z)\frac{d^2}{dz^2} - z\frac{d}{dz} + \frac{\gamma_1}{z} + \frac{\gamma_2}{1-z} + \gamma_3\right\}G^{ps}_{n\kappa}(z) = 0, \tag{115}$$

where

$$\gamma_1 = A + \left(\frac{b}{2}\right)^2 C, \quad \gamma_2 = -\left(\frac{b}{2}\right)^2\varepsilon_{ps}, \quad \gamma_3 = \left(\frac{b}{2}\right)^2(B - C + \varepsilon_{ps}), \tag{116}$$

and

$$A = -\eta_\kappa(\eta_\kappa - 1) - 2\kappa H_Y + H_Y - 2H_C H_Y, \quad B = \frac{2H_Y}{b^2} + \left(M - E^{ps}_{n\kappa} + C_{ps}\right)\left(V_0^{ps} + \frac{4}{b^2}\right), \quad C = -\frac{4H_Y^2}{b^2} + V_1^{ps}\left(M - E^{ps}_{n\kappa} + C_{ps}\right). \tag{117}$$

Table 1 The energy of the spin symmetry in the presence of YTI.

l	n, k	State	$E^{s}_{n,k(H_Y=0)}$	$E^{s}_{n,k(H_Y=0.5)}$	n, k	State	$E^{s}_{n,k(H_Y=0)}$	$E^{s}_{n,k(H_Y=0.5)}$
1	0, −2	$0p_{\frac{3}{2}}$	4.941009893	4.938976489	0, 1	$0p_{\frac{1}{2}}$	4.941009893	4.943778986
2	0, −3	$0d_{\frac{5}{2}}$	4.947209317	4.943778986	0, 2	$0d_{\frac{3}{2}}$	4.947209317	4.951209494
3	0, −4	$0f_{\frac{7}{2}}$	4.955674003	4.951209494	0, 3	$0f_{\frac{5}{2}}$	4.955674003	4.960486231
4	0, −5	$0g_{\frac{9}{2}}$	4.965521537	4.960486231	0, 4	$0g_{\frac{7}{2}}$	4.965521537	4.970650204

Table 2 The energy of the pseudospin symmetry in the presence of YTI.

$\tilde{l}$	n, k	State	$E^{ps}_{n,k(H_Y=0)}$	$E^{ps}_{n,k(H_Y=0.5)}$	$n-1, k$	State	$E^{ps}_{n,k(H_Y=0)}$	$E^{ps}_{n,k(H_Y=0.5)}$
1	1, −1	$1s_{\frac{1}{2}}$	−4.971883316	−4.970112404	0, 2	$0d_{\frac{3}{2}}$	−4.971883316	−4.973260354
2	1, −2	$1p_{\frac{3}{2}}$	−4.975879919	−4.973260354	0, 3	$0f_{\frac{5}{2}}$	−4.975879919	−4.978040648
3	1, −3	$1d_{\frac{5}{2}}$	−4.981196728	−4.978040648	0, 4	$0g_{\frac{7}{2}}$	−4.981196728	−4.983811863
4	1, −4	$1f_{\frac{7}{2}}$	−4.987107693	−4.983811863	0, 5	$0h_{\frac{9}{2}}$	−4.987107693	−4.989777087

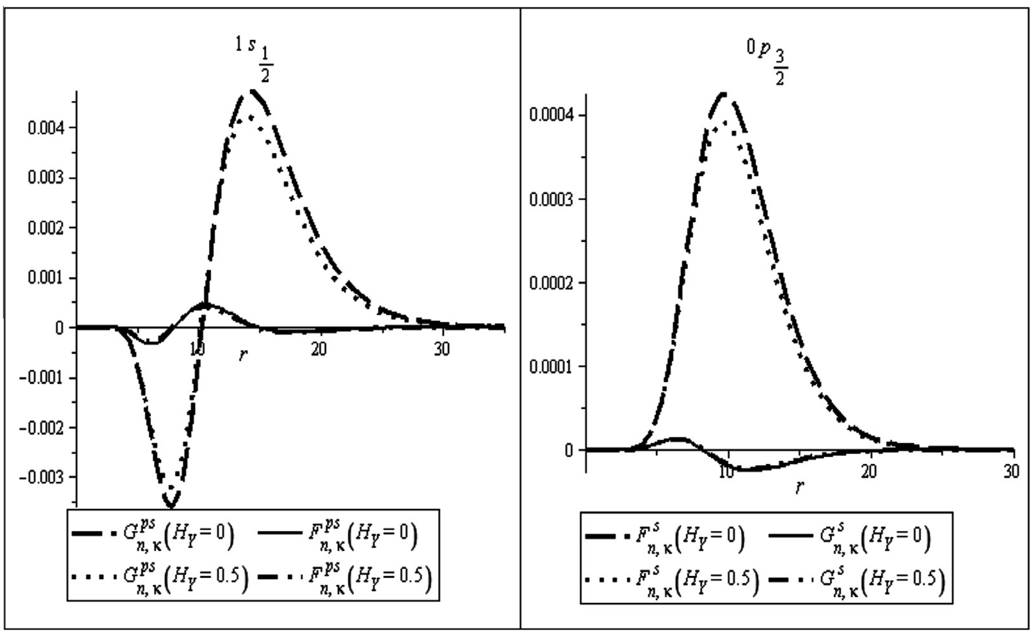

Fig. 1 Wave functions in the pseudospin and spin symmetries' limit in the presence and absence of YTI.

By taking the wave function of the system of the following form

$$G^{ps}_{n\kappa}(z) = z^{\beta_1}(1-z)^{\beta_2} g^{ps}_{n\kappa}(z), \tag{118}$$

and substituting into Eq. (115), we obtain a hypergeometric-type equation

$$\left\{ z(1-z)\frac{d^2}{dz^2} + (\eta_3 - [1+\eta_2+\eta_1]z)\frac{d}{dz} - \eta_1\eta_2 \right\} g^{ps}_{n\kappa}(z) = 0, \tag{119}$$

The solution of Eq. (119), is a hypergeometric function

$$g^{ps}_{n\kappa}(z) = {}_2F_1(\eta_1, \eta_2, \eta_3; z), \tag{120}$$

where

$$\eta_1 = \beta_1 + \beta_2 + \sqrt{\gamma_3},$$
$$\eta_2 = \beta_1 + \beta_2 - \sqrt{\gamma_3}, \tag{121}$$
$$\eta_3 = 2\beta_1,$$

with

$$\beta_1 = \frac{1}{2}\left(1 + \sqrt{1 - 4\gamma_1}\right), \tag{122}$$

$$\beta_2 = -i\frac{b}{2}k, \quad k = \sqrt{-\varepsilon_{ps}}. \tag{123}$$

From Eqs. (118) and (120), we write the wave function of the scattering states as

$$G^{ps}_{n\kappa}(z) = N^{ps}_{n\kappa} e^{ikr}\left(1 - e^{-\frac{2r}{b}}\right)^{\beta_1} {}_2F_1\left(\eta_1, \eta_2, \eta_3; 1 - e^{-\frac{2r}{b}}\right), \tag{124}$$

Here, to obtain the normalized constant and phase shifts we apply the following properties of heypergeometric function

$${}_2F_1(\eta_1, \eta_2, \eta_3; 0) = 1, \tag{125}$$

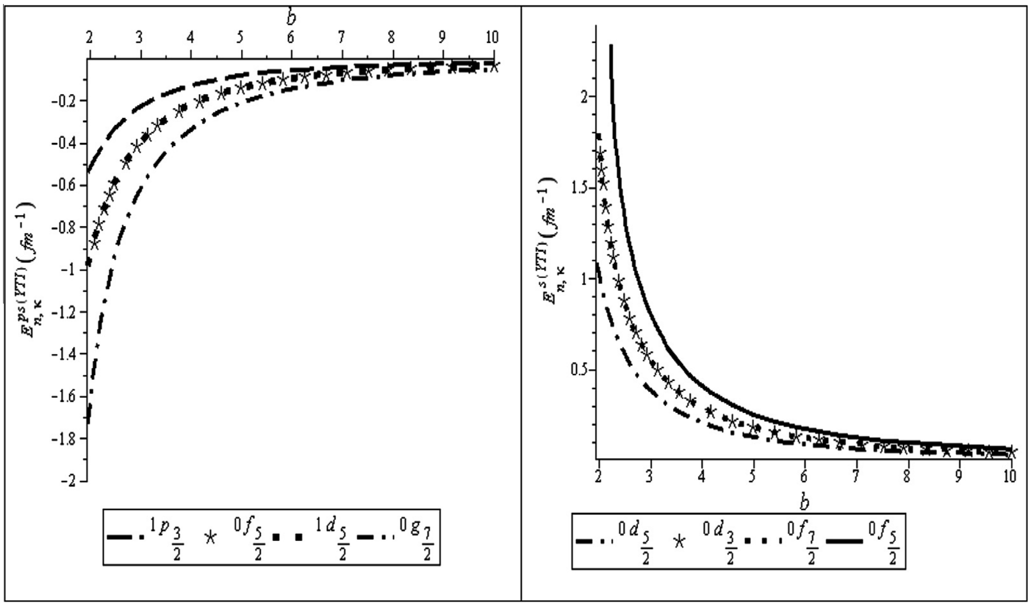

Fig. 2 Energy spectra in the pseudospin and spin symmetries versus b for YTI interaction.

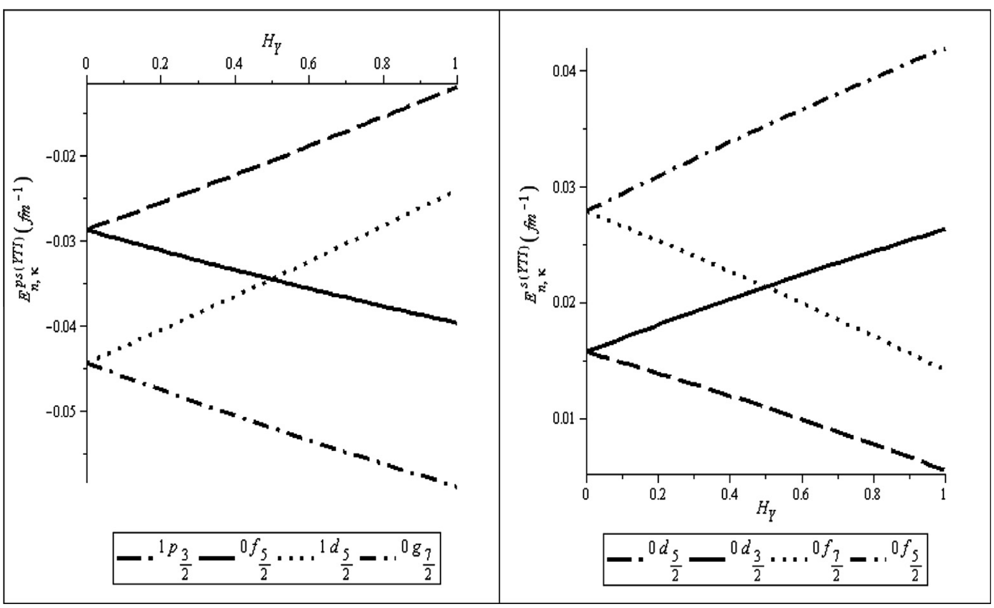

Fig. 3 Energy spectra in the pseudospin and spin symmetries versus H_Y for YTI.

$$
\begin{aligned}
{}_2F_1(\eta_1,\eta_2,\eta_3;z) &= \frac{\Gamma(\eta_3)\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)} \\
&\times {}_2F_1(\eta_1,\eta_2;\eta_1+\eta_2-\eta_3+1;1-z) \\
&+ (1-z)^{\eta_3-\eta_1-\eta_2}\frac{\Gamma(\eta_3)\Gamma(\eta_1+\eta_2-\eta_3)}{\Gamma(\eta_1)\Gamma(\eta_2)} \\
&\times {}_2F_1(\eta_3-\eta_1,\eta_3-\eta_2;\eta_3-\eta_1-\eta_2+1;1-z).
\end{aligned}
\tag{126}
$$

From the properties of the hypergeometric functions for $r\to\infty$ $(z\to 1)$ we have

$$
{}_2F_1(\eta_1,\eta_2,\eta_3;1-e^{-\frac{2r}{b}}) = \Gamma(\eta_3)\left\{\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)} + e^{-2ikr}\left(\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right)^*\right\}, \tag{127}
$$

where we have used the following relation

$$
\begin{aligned}
&\eta_3-\eta_1-\eta_2 = ibk = (\eta_1+\eta_2-\eta_3)^*, \\
&\eta_3-\eta_1 = \beta_1+\frac{ib}{2}k-\sqrt{\gamma_3} = \eta_2^*, \\
&\eta_3-\eta_2 = \beta_1+\frac{ib}{2}k+\sqrt{\gamma_3} = \eta_1^*.
\end{aligned}
\tag{128}
$$

By taking $\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)} = \left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right| e^{i\delta_{ps}}$ and inserting in Eq. (127) we can write the wave function Eq. (124) as

$$
G^{ps}_{n\kappa}(r) = 2N^{ps}_{n\kappa}\Gamma(\eta_3)\left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right| \sin\left(kr+\frac{\pi}{2}+\delta_{ps}\right). \tag{129}
$$

By comparing Eq. (129) with the boundary condition (Landau and Lifshitz, 1977) $r\to\infty \Rightarrow G^{ps}_{n\kappa}(\infty)\to 2\sin\left(kr-\frac{\kappa\pi}{2}+\delta^{ps}_{\kappa}\right)$ phase shifts and the normalized constant can be given by

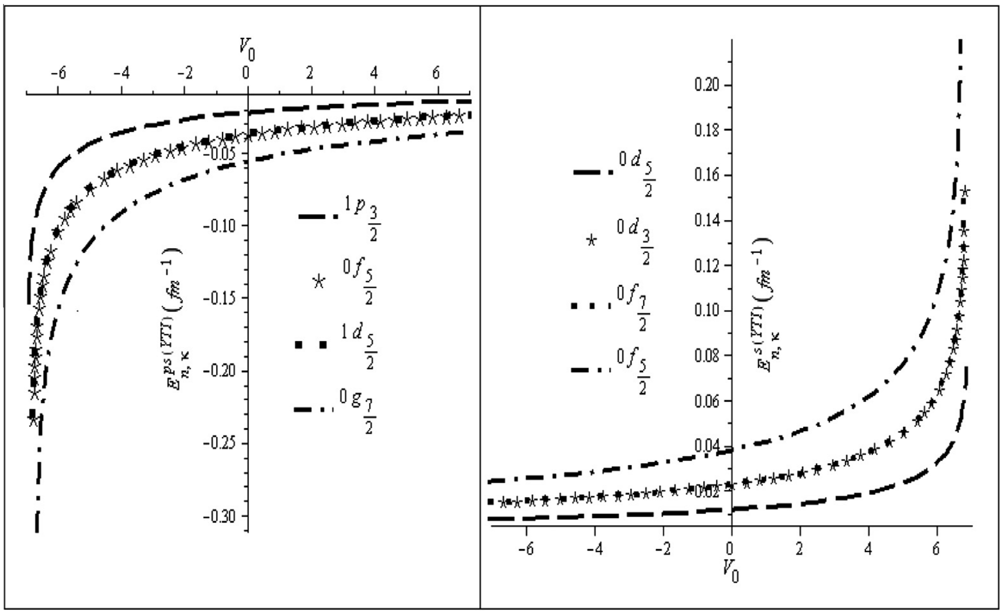

Fig. 4 Energy spectra in the pseudospin and spin symmetries versus V_0 for YTI.

Table 3 The energy of the spin symmetry in the presence of GTI.

l	n, k	State	$E^s_{n,k(H_c=H_Y=0)}$	$E^s_{n,k(H_c=H_Y=0.5)}$	n, k	State	$E^s_{n,k(H_c=H_Y=0)}$	$E^s_{n,k(H_c=H_Y=0.5)}$
1	0, −2	$0p_{\frac{3}{2}}$	4.941009893	4.937733858	0, 1	$0p_{\frac{1}{2}}$	4.941009893	4.947209317
2	0, −3	$0d_{\frac{5}{2}}$	4.947209317	4.941009893	0, 2	$0d_{\frac{3}{2}}$	4.947209317	4.955674003
3	0, −4	$0f_{\frac{7}{2}}$	4.955674003	4.947209317	0, 3	$0f_{\frac{5}{2}}$	4.955674003	4.965521537
4	0, −5	$0g_{\frac{9}{2}}$	4.965521537	4.955674003	0, 4	$0g_{\frac{7}{2}}$	4.965521537	4.975740155

Table 4 The energy of the pseudospin symmetry in the presence of GTI.

$\tilde{l}$	n, k	State	$E^s_{n,k(H_c=H_Y=0)}$	$E^s_{n,k(H_c=H_Y=0.5)}$	$n-1, k$	State	$E^s_{n,k(H_c=H_Y=0)}$	$E^s_{n,k(H_c=H_Y=0.5)}$
1	1, −1	$1s_{\frac{1}{2}}$	−4.971883316	−4.969291407	0, 2	$0d_{\frac{3}{2}}$	−4.971883316	−4.975482048
2	1, −2	$1p_{\frac{3}{2}}$	−4.975879919	−4.971450277	0, 3	$0f_{\frac{5}{2}}$	−4.975879919	−4.980849415
3	1, −3	$1d_{\frac{5}{2}}$	−4.981196728	−4.975482048	0, 4	$0g_{\frac{7}{2}}$	−4.981196728	−4.986824094
4	1, −4	$1f_{\frac{7}{2}}$	−4.987107693	−4.980849415	0, 5	$0h_{\frac{9}{2}}$	−4.987107693	−4.992558832

$$\delta^{ps}_{\kappa} = (\kappa+1)\frac{\pi}{2} + \delta_{ps} = (\kappa+1)\frac{\pi}{2} + \arg\left(\frac{\Gamma(ibk)}{\Gamma\left(\beta_1+\frac{ib}{2}k-\sqrt{\gamma_3}\right)\Gamma\left(\beta_1+\frac{ib}{2}k+\sqrt{\gamma_3}\right)}\right), \tag{130}$$

$$N^{ps}_{n\kappa} = \frac{1}{\Gamma\left(1+\sqrt{1-4\gamma_1}\right)}\left|\frac{\Gamma\left(\beta_1+\frac{ib}{2}k-\sqrt{\gamma_3}\right)\Gamma\left(\beta_1+\frac{ib}{2}k+\sqrt{\gamma_3}\right)}{\Gamma(ibk)}\right|. \tag{131}$$

5.2. Scattering state solution for the spin symmetry

Now, we want to investigate the phase shifts and normalized wave function of spin symmetry. If we consider two transformations, i.e., $z = 1 - e^{-\frac{2}{b}r}$ and $F^s_{n\kappa}(z) = z^{\beta'_1}(1-z)^{\beta'_2} f^s_{n\kappa}(z)$, Eq. (102) modified as

$$\left\{z(1-z)\frac{d^2}{dz^2} + \left(\eta'_3 - [1+\eta'_2+\eta'_1]z\right)\frac{d}{dz} - \eta'_1\eta'_2\right\} f^s_{n\kappa}(z) = 0, \tag{132}$$

where

$$\begin{aligned}
\beta'_1 &= \frac{1}{2}\left(1+\sqrt{1-4\gamma'_1}\right),\\
\beta'_2 &= -i\frac{b}{2}k',\ \ k' = \sqrt{-\varepsilon_s},\\
\eta'_1 &= \beta'_1+\beta'_2+\sqrt{\gamma'_3},\\
\eta'_2 &= \beta'_1+\beta'_2-\sqrt{\gamma'_3},\\
\eta'_3 &= 2\beta'_1,
\end{aligned} \tag{133}$$

with

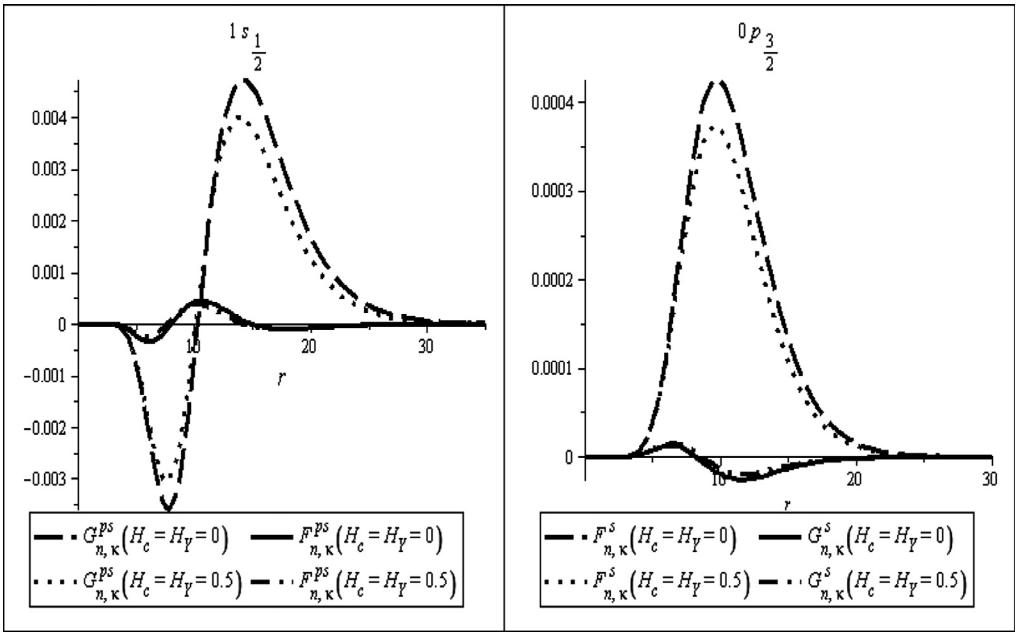

Fig. 5 Wave functions in the pseudospin and spin symmetries' limit in the presence and absence of GTI.

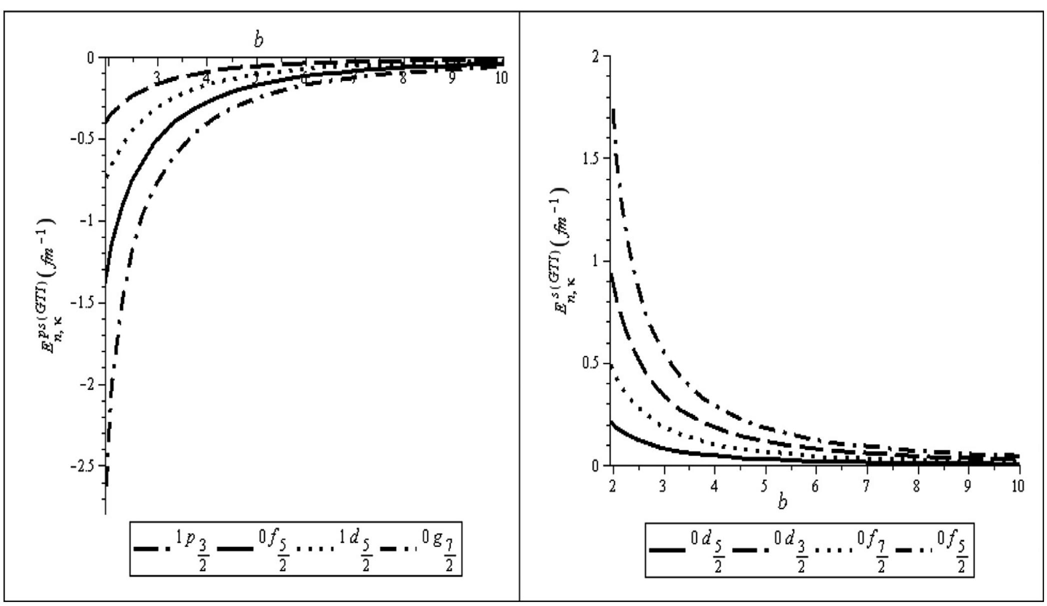

Fig. 6 Energy spectra in the pseudospin and spin symmetries versus b for GTI interaction.

$$
\begin{aligned}
\gamma_1' &= A' + \left(\frac{b}{2}\right)^2 C', \\
\gamma_2' &= -\left(\frac{b}{2}\right)^2 \varepsilon_s, \\
\gamma_3' &= \left(\frac{b}{2}\right)^2 (B' - C' + \varepsilon_s), \\
A' &= -\Lambda_\kappa(\Lambda_\kappa - 1) - 2\kappa H_Y - H_Y - 2H_C H_Y, \\
B' &= -\frac{2H_Y}{b^2} - \left(M + E^s_{n\kappa} - C_s\right)\left(V_0^s + \frac{4}{b^2}\right), \\
C' &= -\frac{4H_Y^2}{b^2} - V_1^s\left(M + E^s_{n\kappa} - C_s\right).
\end{aligned} \tag{134}
$$

By doing the same approach as the previous subsection, the phase shifts and the normalized constant of spin symmetry limit can be given by

$$
\delta^s_\kappa = (\kappa + 1)\frac{\pi}{2} + \delta_s = (\kappa + 1)\frac{\pi}{2} + \arg\left(\frac{\Gamma(ibk')}{\Gamma\left(\beta_1' + \frac{ib}{2}k' - \sqrt{\gamma_3'}\right)\Gamma\left(\beta_1' + \frac{ib}{2}k' + \sqrt{\gamma_3'}\right)}\right), \tag{135}
$$

$$
N^s_{n\kappa} = \frac{1}{\Gamma\left(1 + \sqrt{1 - 4\gamma_1'}\right)} \left|\frac{\Gamma\left(\beta_1' + \frac{ib}{2}k' - \sqrt{\gamma_3'}\right)\Gamma\left(\beta_1' + \frac{ib}{2}k' + \sqrt{\gamma_3'}\right)}{\Gamma(ibk')}\right|. \tag{136}
$$

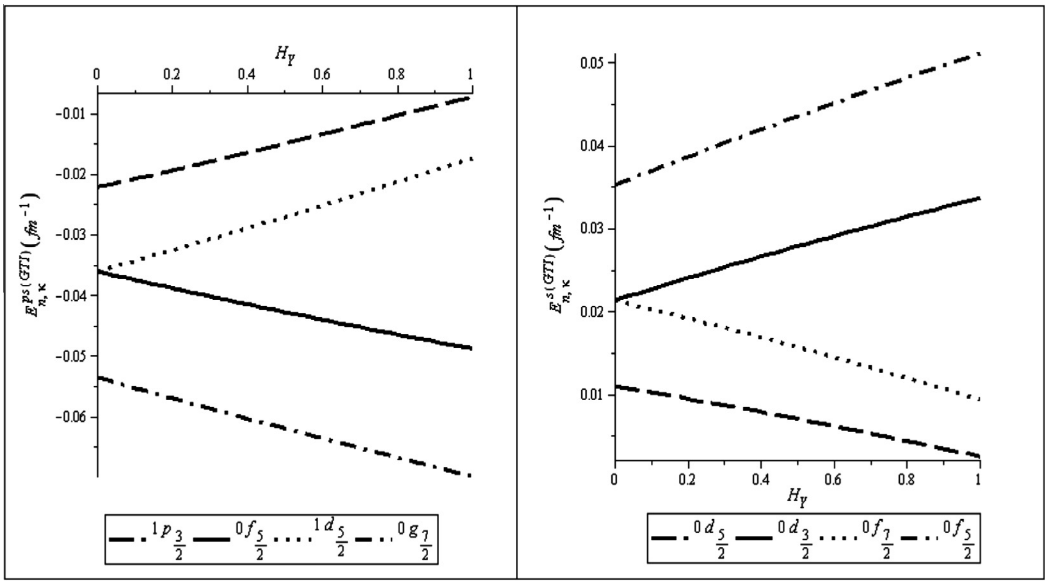

Fig. 7 Energy spectra in the pseudospin and spin symmetries versus H_Y for GTI interaction.

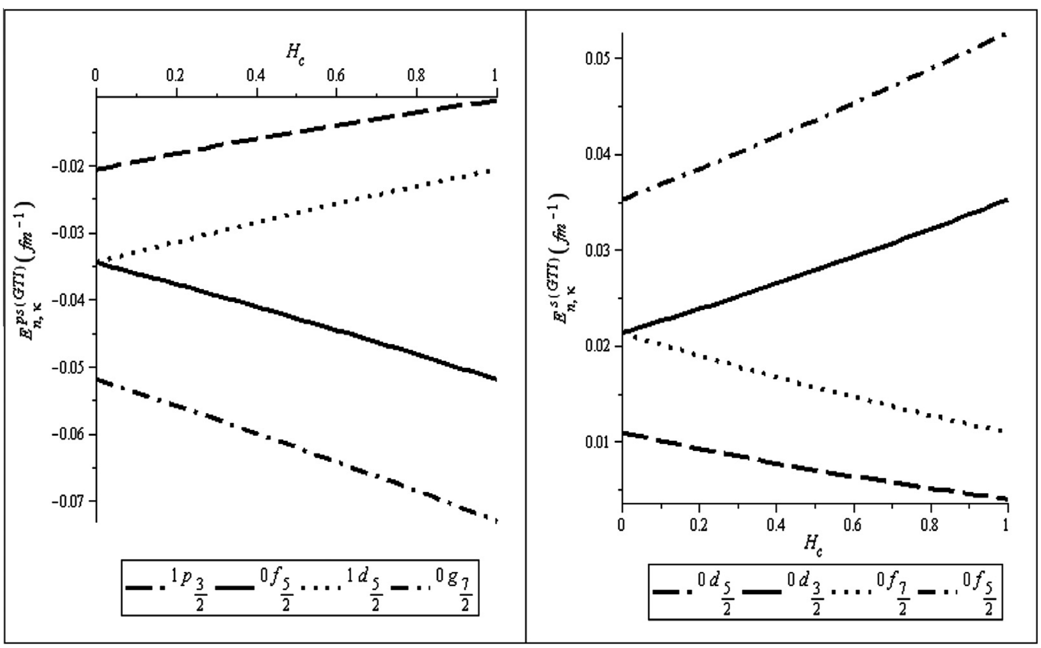

Fig. 8 Energy spectra in the pseudospin and spin symmetries versus H_c for GTI interaction.

6. Numerical results

In this section we discuss about the effect of the tensor interactions on the wave functions and energy of the Dirac equation. In our calculations we have taken $M = 5$, $b = 10$, $V_0 = 0.9$, $V_1 = -2$, $C_{ps} = -5$ for the pseudospin symmetry limit and $M = 5$, $b = 10$, $V_0 = -0.9$, $V_1 = 2$, $C_s = 5$ for the spin symmetry limit.

6.1. The effect of Yukawa-like tensor interaction

To show the effect of YTI on the energy eigenvalues and the wave functions of the system we have calculated numerical results for different states both in pseudospin symmetry and spin symmetries' limits as shown in Tables 1 and 2 respectively. We can see that there are degeneracies among energy levels in the absence of GTI such as ($1p_{3/2}$, $0f_{5/2}$), ($1s_{1/2}$,$0d_{3/2}$),... in pseudospin symmetry and ($0p_{3/2}$,$0p_{1/2}$), ($0d_{5/2}$,$0d_{3/2}$), ... in the spin symmetry limits and when the GTI appears these degeneracies remove. Fig. 1 shows the effect of GTI on the components of Dirac spinors. The effect of the parameters b,H_Y and V_0 on the energy of the pseudospin symmetry limit for $1p_{3/2}$,$0f_{5/2}$,$1d_{5/2}$,$0g_{7/2}$ and spin symmetry limits for $0d_{5/2}$, $0d_{3/2}$,$0f_{7/2}$,$0f_{5/2}$ is plotted in Figs. 2–4. We can see that when the parameter b increases the energy of the pseudospin symmetry (spin symmetry) becomes less (more) bounded, and when V_0 increases the energy of spin symmetry and pseudospin

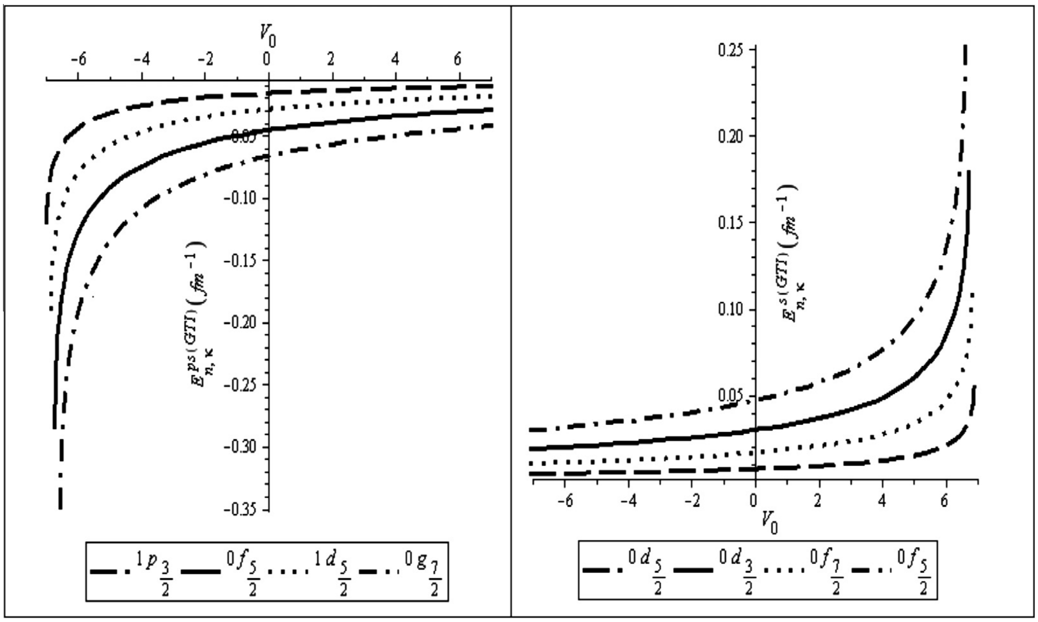

Fig. 9 Energy spectra in the pseudospin and spin symmetries versus V_0 for GTI interaction.

symmetry has an increasing behavior. It is clear that when $H_Y = 0$, ($1p_{3/2}$,$0f_{5/2}$) and ($1d_{5/2}$,$0g_{7/2}$) [($0d_{5/2}$,$0d_{3/2}$) and ($0f_{7/2}$, $0f_{5/2}$)] are degenerated in the pseudospin symmetry (spin symmetry).

6.2. The effect of generalized tensor interaction

In Tables 3 and 4 the energy of the Dirac equation in the absence and presence of YTI is reported. Fig. 5 shows the lower and upper components of the Dirac spinors under the pseudospin and spin symmetries' limits. The energy spectrum in the pseudospin symmetry and spin symmetry versus b is represented in Fig. 6. By taking $H_c = 0.5$ we have plotted the behavior of the energy of the system versus V_Y in Fig. 7 for both pseudospin symmetry and spin symmetry. When $V_Y = 0$, ($1d_{5/2}$,$0f_{5/2}$) in the pseudospin symmetry and ($0d_{3/2}$, $0f_{7/2}$) in the spin symmetry are degenerated. In Fig. 8, by choosing $V_Y = 0.5$ we have presented the effect of H_c on the energy of the pseudospin symmetry and spin symmetry. We can see that in the case of $H_c = 0$ we have degeneracies between ($1d_{5/2}$, $0f_{5/2}$) in the pseudospin symmetry and ($0d_{3/2}$, $0f_{7/2}$) in the spin symmetry. And finally the effect of parameter V_0 on the energy of the system is shown in Fig. 9. It is seen that as V_0 increases the energy of the pseudospin symmetry and spin symmetry increases.

7. Conclusions

In this work, we have studied bound and scattering state solutions of Dirac equation for the shifted Hulthen potential under the spin and pseudospin symmetries and YTI and GTI. We have obtained the energy eigenvalues, normalized wave function and scattering phase shifts approximately. We have presented our numerical results in Tables 1–4 for YTI and GTI, and showed that the degeneracy between two states in spin and pseudospin symmetries has been removed.

Acknowledgements

We wish to give our sincere gratitude to the referees for their instructive comments and careful reading of the manuscript.

Appendix A. We include this short introduction to SUSYQM to proceed on a more continuous manner. In SUSUQM we normally deal with the partner Hamiltonians, Cooper et al. (1995) and Hassanabadi et al. (2013)

$$H_{\pm} = \frac{p^2}{2m} + V_{\pm}(x), \tag{A.1}$$

where

$$V_{\pm}(x) = \Phi^2(x) \pm \Phi'(x). \tag{A.2}$$

In the case of good SUSY, i.e., $E_0 = 0$, the ground state of the system is obtained via

$$\phi_0^-(x) = Ce^{-U}, \tag{A.3}$$

where C is a normalization constant and

$$U(x) = \int_{x_0}^{x} dz\Phi(z). \tag{A.4}$$

Next, if the shape invariant condition

$$V_+(a_0, x) = V_-(a_1, x) + R(a_1), \tag{A.5}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping F:$a_0 \mapsto a_1 = F(a_0)$ and $R(a_1)$ does not include x, the higher state solutions are obtained via

$$E_n = \sum_{s=1}^{n} R(a_s), \tag{A.6}$$

$$\phi_n^-(a_0, x) = \Pi_{s=0}^{n-1}\left(\frac{A^\dagger(a_s)}{[E_n - E_s]^{1/2}}\right)\phi_0^-(a_n, x), \tag{A.7}$$

$$\phi_0^-(a_n, x) = C\exp\left\{-\int_0^x dz\Phi(a_n, z)\right\}, \tag{A.8}$$

where

$$A_s^\dagger = -\frac{\partial}{\partial x} + \Phi(a_s, x). \tag{A.9}$$

Therefore, this condition determines the spectrum of the bound states of the Hamiltonian

$$H_s = -\frac{\partial^2}{\partial x^2} + V_-(a_s, x) + E_s. \tag{A.10}$$

and the energy eigenfunctions of

$$H_s\phi_{n-s}^-(a_s, x) = E_n\phi_{n-s}^-(a_s, x), \quad n \geqslant s \tag{A.11}$$

are related via [17,22,26–27]

$$\phi_{n-s}^-(a_s, x) = \frac{A^\dagger}{[E_n - E_s]^{1/2}}\phi_{n-(s+1)}^-(a_{s+1}, x). \tag{A.12}$$

References

Akcay, H., 2009. Dirac equation with scalar and vector quadratic potentials and Coulomb-like tensor potential. Phys. Lett. A 373, 616.

Aydogdu, O., Sever, R., 2010. Exact pseudospin symmetric solution of the Dirac equation for pseudoharmonic potential in the presence of tensor potential. Few-Body Syst. 47, 193.

Bahar, M.K., Yasuk, F., 2013. Bound states of the Dirac equation with position-dependent mass for the Eckart potential. Chin. Phys. B 22, 010301.

Bohr, A., Hamarrnoto, I., Motelson, B.R., 1982. Pseudospin in rotating nuclear potentials. Phys. Scr. 26, 267.

Cooper, F., Khare, A., Sukhatme, U., 1995. Supersymmetry and quantum mechanics. Phys. Rep. 251, 267.

Dudek, J., Nazarewicz, W., Szymanski, Z., Lender, G.A., 1987. Abundance and systematics of nuclear superdeformed states; relation to the pseudospin and pseudo-SU (3) symmetries. Phys. Rev. Lett. 59, 1405.

Ginocchio, J.N., 2004. Relativistic harmonic oscillator with spin symmetry. Phys. Rev. C 69, 034318.

Ginocchio, J.N., 2005. Relativistic symmetries in nuclei and hadrons. Phys. Rep. 414, 165–261.

Hassanabadi, H., Yazarloo, B.H., 2013. Exact solutions of the Spinless–Salpeter equation under Kink-Like potential. Chin. Phys. C 37, 123101.

Hassanabadi, H., Maghsoodi, E., Zarrinkamar, S., 2012. Spin and pseudospin symmetries of Dirac equation and the Yukawa potential as the tensor interaction. Commun. Theor. Phys. 58, 807.

Hassanabadi, H., Yazarloo, B.H., Mahmoudieh, M., Zarrinkamar, S., 2013. Dirac equation under the Deng–Fan potential and the Hulth'en potential as a tensor interaction via SUSYQM. Eur. Phys. J. Plus 128, 111.

Hassanabadi, H., Yazarloo, B.H., Salehi, N., 2014. Pseudospin and spin symmetry of Dirac equation under Deng–Fan potential and Yukawa potential as a tensor interaction. Indian J. Phys. DOI http://dx.doi.org/10.1007/s12648-013-0426-x.

Ikot, A.N., 2012. Solutions of Dirac equation for generalized hyperbolical potential including Coulomb-like tensor potential with spin symmetry. Few-Body Syst. 53, 549.

Ikot, A.N., Hassanabadi, H., Yazarloo, B.H., Zarrinkamar, S., 2013. Approximate relativistic k-state solutions to the Dirac–hyperbolic problem with generalized tensor interaction. J. Mod. Phys. E 22, 1350048.

Ikot, A.N., Maghsoodi, E., Ibanga, E.J., Zarrinkamar, S., Hassanabadi, H., 2013. Spin and pseudospin symmetries of the Dirac equation with shifted Hulthén potential using supersymmetric quantum mechanics. Chin. Phys. B 22, 120302.

Landau, L.D., Lifshitz, E.M., 1977. Quantum Mechanics, Non-Relativistic Theory, 3rd ed. Pergamon, New York.

Lu, B.N., Zhao, E.G., Zhou, S.G., 2012. Pseudospin symmetry in single particle resonant states. Phys. Rev. Lett. 109, 072501.

Oyewumi, K.J., Akoshile, C.O., 2010. Bound state solutions of the Dirac–Rosen–Morse potential with spin and pseudospin symmetry. arXiv:1008.2358v1.

Setare, M.R., Nazari, Z., 2009. Solution of Dirac equations with five-parameter exponent-type potential. Acta Phys. Pol. B 40, 2809.

Troltenier, D., Bahri, C., Draayer, J.P., 1995. Generalized pseudo-SU (3) model and pairing. Nucl. Phys. A 586, 53.

Wei, G.F., Dong, S.H., 2009. Algebraic approach to pseudospin symmetry for the Dirac equation with scalar and vector modified Pöschl–Teller potentials. Europhys. Lett. 87, 40004.

Wei, G.F., Dong, S.H., 2010a. Pseudospin symmetry in the relativistic Manning–Rosen potential including a Pekeris-type approximation to the pseudo-centrifugal term. Phys. Lett. B 686, 288.

Wei, G.F., Dong, S.H., 2010b. A novel algebraic approach to spin symmetry for Dirac equation with scalar and vector second Pöschl–Teller potentials. Eur. Phys. J. A 43, 185.

Wei, G.F., Dong, S.H., 2010c. Pseudospin symmetry for modified Rosen–Morse potential including a Pekeris-type approximation to the pseudo-centrifugal term. Eur. Phys. J. A 46, 207.

Yukawa, H., 1935. On the interaction of elementary particles. Proc. Phys. Math. Soc. Jpn. 17, 48.