Reduced order models for the nonlinear dynamic analysis of shells

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Abstract

The non-linear dynamic analysis of continuous systems, such as thin plates and shells, is a problem of relevance in many engineering fields. The finite element method is the most used approach for nonlinear dynamic analyses of these structures. However, the computational effort is very high. As an alternative to complex numerical approaches, analytical methods using simplified models can be successfully used to understand the main nonlinear features of the problem and may constitute efficient tools in the initial design stages. For plates and shells, the derivation of efficient reduced order models is in fact essential due to the complex nonlinear response of these structures. The usual procedure is to reduce the partial differential equations of motion of the continuous system to an approximate system of time-dependent ordinary differential equations of motion, which are in turn solved by numerical methods or, approximately, by perturbation procedures. However, the use of inappropriate modal expansions usually leads to misleading results or may require a rather large number of terms. The aim of the present work is to show how the application of a perturbation analyses together with the Galerkin method can be used to derive precise low order models for plates and shells, by capturing the influence of modal couplings and interactions.

Keywords: Shells; reduced order models; modal coupling and interaction; internal resonances; nonlinear shell vibrations.

1. Introduction

Shells are common structural elements in several engineering fields and architecture due to their structural advantages, in particular their capacity to withstand several types of loading through membrane action leading to light and economic structures, usually with rather pleasing aesthetic properties. In engineering industry thin metal
shells of revolution are frequently used, in particular shells of revolution such as spherical, conical and cylindrical shells. Their optimal dimensions however may lead to a nonlinear geometric behavior under static and dynamic conditions in the elastic range of material. This has led to several buckling and vibrations problems, which must be dealt with care at the stage of design. Due to the initial curvature of the shell reference surface, the shell equations written in terms of the displacements, exhibit not only cubic nonlinear terms but also quadratic nonlinearities, which can in several cases dominate the shell nonlinear behavior. Several examples where negative quadratic nonlinearities leading to softening vibrations and unstable post-buckling behavior are found in literature. Unstable post-buckling behavior may lead in pre-loaded shells under compressive stresses to multiple potential wells. Another important aspect is the fact that shells exhibit a dense frequency spectrum, especially in the lowest frequency range. In addition they usually display a finite or infinite number of symmetries which may lead to several vibrations modes with the same natural frequencies. These features may give rise to multiple internal resonances. Thus, due to these intrinsic shell characteristics, they display a complex non-linear behavior and their analyses have become one of the major research fields in applied mechanics.

The nonlinear dynamic analysis of discrete and continuous mechanical system is a flourishing research area and in recent decades several numerical and analytical tools have been developed to unveil their bifurcation scenario under different control parameters and the existence of competing coexisting solutions. These studies are usually conducted using discrete or discretized models with a small number of degrees of freedom.

To capture the nonlinear response of thin-walled shells, large numerical models usually using the finite element method, or similar discretization techniques, are usually employed. These analyses, involving a large number of degrees of freedom, are very expensive with respect to both storage and CPU time. As a result, it is rather cumbersome, if not impossible, to conducted an appropriate parametric nonlinear dynamic analysis using such models. So, the derivation of consistent reduced order models becomes an important step in the nonlinear dynamic analysis of thin shells. These simplified models can be successfully used to understand the main nonlinear features of the problem and may constitute efficient tools in the initial design stages. For continuous structural systems such as plates and shells, one must reduce the partial differential equations of motion to a set of ordinary equations of motion in time domain, which can be appropriately analyzed using analytical methods, such as perturbation methods or Galerkin balance, or numerical techniques. A traditional technique for space discretization is to use the Ritz or Galerkin procedures. For this an appropriate set of interpolation functions must be defined. A usual procedure is to use a series of linear normal modes since they satisfy all boundary conditions and constitute a set of orthogonal functions in the linear case. This may lead to erroneous results when applied to shell nonlinear analysis or convergence may only be achieved using a large number of modes. The quest for appropriate interpolation function in shell nonlinear dynamic analysis has been had several different contributions since the 1960s. The aim of the present work, based on previous works by the authors, is to show how the application of perturbation analyses together with the Galerkin (or Ritz) method can be used to derive precise low order models for plates and shells, by capturing the influence of modal couplings and interactions. It is shown that such reduced order models can capture the main characteristics of the shell behavior and constitute a satisfactory tool for computing state solutions, in particular bifurcation diagrams under dynamic loads.

2. Shell Formulation

The Donnell nonlinear kinematic relations for thin shells are:

\[
\begin{align*}
\varepsilon_x &= \frac{u_x}{A_1} + \frac{A_{1,y} v}{A_1 A_2} + \frac{w}{R_x} ; & \varepsilon_y &= \frac{v_y}{A_2} + \frac{A_{2,x} u}{A_1 A_2} + \frac{w}{R_y} ; & \gamma_{xy} &= \frac{v_x}{A_1} + \frac{u_x}{A_2} - \frac{A_{1,y} v + A_{1,x} u}{A_1 A_2} ; & \beta_x &= \frac{w_x}{A_1} ; & \beta_y &= -\frac{w_y}{A_2} \\
\chi_x &= \frac{\beta_{x,x}}{A_1} + \frac{A_{1,y} \beta_y}{A_1 A_2} ; & \chi_y &= \frac{\beta_{y,y}}{A_2} + \frac{A_{2,x} \beta_x}{A_1 A_2} ; & \chi_{xy} &= \frac{1}{2} \left( \frac{\beta_{y,x}}{A_1} + \frac{\beta_{x,y}}{A_2} - \frac{A_{1,y} \beta_x + A_{2,x} \beta_y}{A_1 A_2} \right) .
\end{align*}
\]

where \(x\) and \(y\) are the principal directions, \(\varepsilon_x, \varepsilon_y, \gamma_{xy}\) are the extensional and shearing strain components at a point on the shell middle surface, \(\beta_x, \beta_y\) are the rotations, \(\chi_x, \chi_y, \chi_{xy}\) are, respectively, the curvature changes.
and twist, $u$ and $v$ are the in-plane displacements in the $x$ and $y$ directions respectively, $w$ is the transversal displacement, $A_1$ and $A_2$ are the Lamé coefficients, $R_x$ and $R_y$ are the principal radii of curvature, which define the shell geometry.

By considering an isotropic elastic material and using Hooke’s law and Eq. (1), the normal and shearing stresses are obtained. By integrating the resulting stresses along the shell thickness, the in-plane normal and shearing forces intensities per unit length along the edge of a shell element, $N_x$, $N_y$, and $N_{xy}$, and the bending and twisting moment intensities, $M_x$, $M_y$ and $M_{xy}$, are obtained. Based on these quantities, the following nonlinear equations of motion are derived:

$$
\begin{align*}
(A_2 N_x)_{,x} + (A_1 N_{xy})_{,y} - A_{2,xx} N_y + A_{1,yy} N_{xy} &= -A_1 A_2 p_x, \\
(A_1 N_y)_{,x} + (A_2 N_{xy})_{,y} - A_{1,xx} N_x + A_{2,yy} N_{xy} &= -A_1 A_2 p_y, \\
\left[\frac{1}{A_1} (A_2 M_x)_{,x} \right]_{,x} - \left[\frac{A_{1,yy}}{A_2} M_x \right]_{,y} + \left[\frac{1}{A_2} (A_1 M_y)_{,y} \right]_{,y} - \left[\frac{A_{2,xx}}{A_1} M_y \right]_{,x} \\
+ 2 \left[ M_{xy,xy} + \left(\frac{A_{1,yy}}{A_1} M_{xy} \right)_{,x} + \left(\frac{A_{2,xx}}{A_2} M_{xy} \right)_{,y} \right] - A_1 A_2 \left(\frac{N_x}{R_x} + \frac{N_y}{R_y} \right) \\
\left[ (A_2 N_x \beta_x + A_2 N_{xy} \beta_y)_{,x} + (A_1 N_y \beta_y + A_1 N_{xy} \beta_x)_{,y} \right] &= -A_1 A_2 p_z.
\end{align*}
$$

(2)

3. Nonlinear vibration modes and modal coupling

Based on Eqs. (1-2), the nonlinear equations of motion for the undamped, unforced thin shell can be written in terms of its displacement vector $U = \{u, v, w\}^T$ as

$$
L(U) - U_{,tt} = \delta D_1(U) + \delta^2 D_2(U),
$$

(3)

where $L(U)$ is the matrix of linear differential operators, $\delta$ is an appropriate small perturbation parameter, $D_1(U)$ is a vector of quadratic terms and $D_2(U)$ is a vector of cubic terms. Other common shell theories, such as Sanders and Koiter shell theories, lead to similar results. One assumes that the components of the displacement vector $U$ can be expanded in terms of the small perturbation parameter $\delta$ as

$$
\begin{align*}
u = \sum_{i=0}^{\infty} \delta^i U(t_i) u_i(x, y) \\
w = \sum_{i=0}^{\infty} \delta^i V(t_i) v_i(x, y) \\
w = \sum_{i=0}^{\infty} \delta^i W(t_i) w_i(x, y).
\end{align*}
$$

(4)

Substituting Eq. (4) into Eq. (3), collecting terms of the same order in $\delta$ and equating them to zero, one obtains the following set of linear systems of partial differential equations of motion:

$$
\begin{align*}
L\left(U^0\right) - U^0_{,tt} &= 0, \\
L\left(U^1\right) - U^1_{,tt} &= D_1\left(U^0\right), \\
L\left(U^2\right) - U^2_{,tt} &= D_1\left(U^0, U^1\right) + D_2\left(U^0\right).
\end{align*}
$$

(5)

The solution of the first set of equations with the appropriate boundary conditions is simply the linear vibration modes, $U^0$. Substitution $U^0$ into the second equation set, leads to a system of non-homogeneous differential
equations, which is linear in $U^1$. The first order solution $U^1$ arises from the quadratic nonlinearity. Substituting $U^0$ and $U^1$ in the third equation set, one can identify the second order modes $U^2$, which captures the influence of the cubic nonlinearity. These equations have a well-defined pattern, in a way that one could continue developing higher order modes up to the desired order. Finally, by inspecting the solution for $U^1$, $U^2$, $U^3$, ..., $U^N$, a general modal expansion with time varying coefficients for the displacement field can be derived, which can be used in the Galerkin method to generate a set of ordinary differential equations of motion in time domain.

3.1. Application to a cylindrical shell

Consider a cylindrical shell of length $L$, radius $R$ and thickness $h$. The solution of the first set of linear equations in Eq. (5) with the appropriate boundary conditions is simply the linear vibration modes. For a simply supported cylindrical shell, this leads to

$$
\begin{align*}
    u_0 &= \bar{U}_0(t) \cos(n\theta) \cos\left(\frac{m\pi x}{L}\right), \\
    v_0 &= \bar{V}_0(t) \sin(n\theta) \sin\left(\frac{m\pi x}{L}\right), \\
    w_0 &= \bar{W}_0(t) \cos(n\theta) \sin\left(\frac{m\pi x}{L}\right),
\end{align*}
$$

(6)

where $\bar{U}_0$, $\bar{V}_0$, and $\bar{W}_0$ are the modal amplitudes, $n$ is the number of circumferential waves, $m$ is the number of axial half-waves in the axial direction and $\theta = y/R$.

Substitution of Eq. (6) into the second equation in Eq. (5), leads to a system of non-homogeneous differential equations, which is linear in $U^1$. The particular solution of this system gives the second order modes in the expansion Eq. (5), which are

$$
\begin{align*}
    u_1 &= \left[ \bar{U}_1(t) \sin\left(\frac{2m\pi x}{L}\right) + \bar{U}_3(t) \cos\left(\frac{2m\pi x}{L}\right) \cos(2n\theta) \right], \\
    v_1 &= \left[ \bar{V}_1(t) \sin(2n\theta) + \bar{V}_2(t) \cos\left(\frac{2m\pi x}{L}\right) \sin(2n\theta) \right], \\
    w_1 &= \left[ \bar{W}_1(t) \cos\left(\frac{2m\pi x}{L}\right) + \bar{W}_3(t) \cos(2n\theta) + \bar{W}_5(t) \cos(2n\theta) \cos\left(\frac{2m\pi x}{L}\right) + \bar{W}_4 \right].
\end{align*}
$$

(7)

These modes arise from the quadratic nonlinearity and are the main responsible for the in-out asymmetry of the shell nonlinear displacement field.

Substituting $U^0$ and $U^1$ in the third equation in Eq. (5), one can obtain the third order terms of the particular solution $U^2$. They are

$$
\begin{align*}
    u_2 &= \left[ \bar{U}_3(t) \cos(n\theta) \cos\left(\frac{m\pi x}{L}\right) + \bar{U}_4(t) \cos(n\theta) \cos\left(\frac{3m\pi x}{L}\right) + \right. \\
    \bar{U}_5(t) \cos(3n\theta) \cos\left(\frac{m\pi x}{L}\right) + \bar{U}_6(t) \cos(3n\theta) \cos\left(\frac{3m\pi x}{L}\right) \right], \\
\end{align*}
$$
Finally, by imposing the boundary and continuity conditions a general expression for the nonlinear vibration modes is obtained:

\[
\begin{align*}
    v_2 &= \left[ V_3(t) \sin(n \theta) \sin \left( \frac{m \pi x}{L} \right) + V_4(t) \sin(n \theta) \sin \left( \frac{3m \pi x}{L} \right) + \\
    &+ V_5(t) \sin(3n \theta) \sin \left( \frac{m \pi x}{L} \right) + V_6(t) \sin(3n \theta) \sin \left( \frac{3m \pi x}{L} \right) \right], \\
    w_2 &= \left[ \tilde{W}_5(t) \cos(n \theta) \sin \left( \frac{m \pi x}{L} \right) + \tilde{W}_6(t) \cos(n \theta) \sin \left( \frac{3m \pi x}{L} \right) + \\
    &+ \tilde{W}_7(t) \cos(3n \theta) \sin \left( \frac{m \pi x}{L} \right) + \tilde{W}_8(t) \cos(3n \theta) \sin \left( \frac{3m \pi x}{L} \right) \right].
\end{align*}
\] (8)

In a similar fashion the in-plane modes \( u \) and \( v \) are also derived. There is a basic difference between a truly multi-degree-of-freedom model with \( n \) independent co-ordinates and the nonlinear solution Eq. (9). Taking as a seed mode the buckling mode \((m, n)\), all higher order modes in Eq. (9) are due to modal coupling, as shown by the perturbation solution, leading to a modal representation of the nonlinear buckling mode.

Now it is possible to select the necessary number of terms to attain convergence up to large amplitude vibrations within the validity of a given shell theory. This is obtained by substituting the derived modal expansion into Eq. (2) and by applying the Galerkin method. Figure 1(a) shows the convergence of the frequency-amplitude relation of the cylindrical shell. It is observed that the inclusion of the first order \((02, 20, 22)\) and second order modes \((31, 13, 33)\) obtained by the perturbation procedure leads to convergence for vibration amplitudes up to three times the shell thickness (here \( \varepsilon_{ij}=W_{ij}/h \)). Figure 1(b) shows the influence of the nonlinear modal coupling between the seed mode and selected second and third order modes. It is observed that, while in most cases the coupling leads to a hardening behavior, the coupling between the seed mode and the axisymmetric mode with twice the number of waves in the axial direction as the seed mode leads to the expected softening behavior for this shell geometry and mode shape. This is the essential modal coupling due to quadratic nonlinearity for cylindrical shells, and is the main responsible for the in-out asymmetry of the shell nonlinear displacement field, as explained in detail by McRobie et al.\(^5\). The in-out asymmetry is due to the initial shell curvature. This cannot be observed in a linear analysis or in system with only cubic nonlinearity (e.g., perfect nonlinear plate). This axisymmetric mode has been sometimes approximated by a series of axisymmetric linear modes, increasing thus the model order. Gonçalves \( et \ al.\) have used a 2dof model based on this modal expansion to investigate the global dynamics and integrity of a parametrically excited cylindrical shell based on the evolution of the basing of attraction of the several coexisting solution. At present, such analysis could not be properly conducted using higher dimensional models.
4. Influence of symmetries

Symmetries are found in most engineering systems. In particular, shells used in engineering and architecture usually display several symmetry planes due to aesthetic and equilibrium reasons. This may lead to multiple linear vibration modes with the same natural frequency and, consequently, to internal resonances. Also due to symmetries discrete mechanical models may display a number of nonlinear normal modes greater than the number of degrees of freedom. These characteristics may lead to a complex nonlinear dynamic behavior. In particular, shells of revolution (including circular plates and rings), due to the circumferential symmetry, have at least two linear modes with the same natural frequency leading to 1:1 internal resonance. In shells of revolution, if \( \cos n\theta \) is a solution of the partial differential equations of motion (\( \theta \) being the circumferential coordinate), \( \sin n\theta \) is also a solution of these equations. Consequently, if \( \cos n\theta \) is directly excited (driven mode), the mode \( \sin n\theta \) (companion mode) can be indirectly excited due to the nonlinear modal coupling. The energy transfer may also be activated by a detuning parameter such as load or geometric imperfections. It is reported that this symmetry may lead in certain cases to travelling waves in the circumferential direction. So, in order to obtain a consistent reduced order model, the seed solution in the perturbation analysis is adopted as the sum of the two interacting mode. For the simply supported cylindrical shell, the solution

\[
W_0 = A l_0(t) \cos(n\theta) \sin\left(\frac{m\pi x}{L}\right) + B l_0(t) \sin(n\theta) \sin\left(\frac{m\pi x}{L}\right)
\]  

(10)

must be considered. Substituting Eq. (10) into equations (5), the higher order terms arising from this modal coupling can be easily deduced together with the terms already presented in Eq. (9), which are associated with each individual mode.

Figure 2 shows the variation of the modal amplitudes of the driven and companion modes for a pressure loaded cylindrical shell. The companion mode is activated when secondary bifurcation points occur along the resonant branch of the driven mode and displays large amplitude oscillations. If the companion mode with the associated coupling modes were not considered, the classical nonlinear resonance curve of the driven mode would be obtained with the two saddle-node bifurcations at the turning points delimiting the stable and unstable branches. Thus, the consideration of the companion mode and the associated coupling terms has a significant influence on the shell response and stability.
5. Multiple resonances

Another topic that must be considered with care in the derivation of reduced order models for shell structures is the fact that most shells exhibit a dense frequency spectrum with equal or nearly equal natural frequencies. Figure 3 illustrates the variation of the natural frequencies of a steel cylindrical shell with the shell length $L$, for selected numbers of circumferential waves $n$, considering one axial half waves ($m=1$) and the geometric parameter radius-to-thickness $R/h=100$. The gray dots represent the geometries for which two modes with different numbers of circumferential waves exhibit the same natural frequencies. A similar parametric analysis could be conducted considering any $m:n$ internal resonance and similar results obtained. Thus, one can conclude that, $m:n$ internal resonances may be the norm instead of an exception in shell structures.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Variation of natural frequency for different lengths and number of circumferential waves ($m=1$). $R = 0.2$ m; $h = 0.002$ m; $E = 210$ GPa; $\nu = 0.3$; $\rho = 7850$ kg/m$^3$.}
\end{figure}
\[ W = A_{11}(\tau) \cos(n \theta) \sin\left(\frac{m \pi x}{L}\right) + B_{11}(\tau) \sin(n \theta) \sin\left(\frac{m \pi x}{L}\right) \\
+ A_{21}(\tau) \cos(N \theta) \sin\left(\frac{m \pi x}{L}\right) + B_{21}(\tau) \sin(N \theta) \sin\left(\frac{m \pi x}{L}\right) \] (11)

Fig. 4 shows two coexisting solutions due to the interaction between modes (1,5) and (1,6), which have the same natural frequency (see Fig. 3), and their respective companion modes (Eq. (11)) in an axially loaded cylindrical shell\(^{11}\).

6. Conclusions

The results show that a perturbation technique used together with the Galerkin method can be effectively used to derive consistent reduced order models for any number of interacting modes, enabling the analysis of several types of internal resonances that may occur in shell structures. The present reduced order models enable the efficient use of several tools of nonlinear dynamics such as the calculation of bifurcation diagrams, coexisting solutions, basins of attraction, integrity factors, etc. As a rule, one must first identify all the spatial modes that contribute to a possible internal resonance and consider the sum of such modes as the initial solution in the perturbation procedure. The necessary coupling modes are then derived and the used in the Galerkin method. Usually the consideration of the first order modes is enough to obtain convergence of the vibration amplitudes up to the order of the shell thickness. The effectiveness of the present procedure has been confirmed using other techniques such as proper orthogonal decomposition and by comparisons with finite element results\(^{11}\).

References