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Coarser connected topologies

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Abstract

We investigate which spaces have coarser connected topologies. If in a collectionwise normal space X, the density equals the extent, which is attained and at least $\mathfrak{c}$, then X has a coarser connected collectionwise normal topology. In the previous sentence, the separation property collectionwise normal can be replaced by other separation properties—for example, Hausdorff, Urysohn, regular, metrizable. A zero-dimensional metrizable space X of density at least $\mathfrak{c}$ has a coarser connected metrizable topology. A non-H-closed Hausdorff space with a σ-locally finite base has a coarser connected Hausdorff topology. We give necessary conditions and sufficient conditions for an ordinal to have a coarser connected Urysohn topology. In particular, every indecomposible ordinal of countable cofinality has a coarser connected topology. We present a nowhere locally compact Hausdorff space X with no coarser connected Hausdorff topology, yet X is dense in a connected Hausdorff space Y.

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1. Introduction

This paper continues the quest for coarser connected topologies, begun in [13]. Much of this paper is motivated by the following remarkable theorem:

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Theorem 1.1 [7]. *A noncompact metrizable space has a coarser connected Hausdorff topology.*

It is natural to ask whether the conclusion Hausdorff can be strengthened. In Section 2, we show (Theorem 2.2) that if the extent is attained and equal to the density, at least $\mathfrak{c}$, a collectionwise normal space has a coarser connected collectionwise normal topology. In this theorem, "collectionwise normal" can be replaced by various other separation properties. For weak separation properties like Hausdorff and Urysohn, we can omit "at least $\mathfrak{c}$" from the hypothesis (Theorem 2.3). The technique works with metrics, too. If a metrizable space attains its extent, at least $\mathfrak{c}$, then it has a coarser connected metrizable topology (Theorem 2.5).

In Section 3, we describe the special form of metrizable spaces which do not attain their extent. We use this special form to show that zero-dimensional metrizable spaces with extent at least $\mathfrak{c}$ have a coarser connected metrizable topology, even if the extent is not attained (Theorem 3.4).

After proving Theorem 1.1, Gruenhage, Tkachuk, and Wilson asked two questions (3.9 and 3.10) of the form, Can the hypothesis metrizable be weakened?

Question 3.9 [7]. Let X be a non-H-closed Hausdorff space with a σ-locally finite base. Is it true that X has a weaker connected Hausdorff topology?

In Section 4, Theorem 4.12 gives the affirmative answer. Along the way, Theorem 4.11 shows that every space with a σ-locally finite base is the perfect, irreducible image of a metric space.

Question 3.10 [7]. Does every paracompact noncompact space X have a weaker Hausdorff connected topology? What happens if X is hereditarily paracompact or perfect?

In Example 3.4 of [6], we constructed a Hausdorff compactification Z of a discrete space D ($|D| = \mathfrak{c}$) and $Z\setminus D \approx \mathbf{I}^{2^{\mathfrak{c}}}$. Z has a π-base of clopen sets. An application of Lemma 2.3 [6] yields that $Z \oplus \omega$ does not have a coarser connected topology. Clearly, $Z \oplus \omega$ is paracompact and not compact. In Example 3.1 of this paper, we present a hereditarily paracompact space with no coarser connected Hausdorff topology.

We present other results on the topic of coarser connected topologies. In [6] we investigated when spaces had coarser connected Hausdorff topologies. It is natural to ask whether similar results hold for the Urysohn property. We give answers, especially for ordinals, in Section 5. For example, every indecomposible ordinal of countable cofinality has a coarser connected Urysohn topology (Theorem 5.14). If an ordinal α has a coarser connected Urysohn topology, it is necessary that α has cofinality ω (Corollary 5.4), and if $\alpha = \delta + \beta$, then necessarily $|\alpha| \leqslant |\beta|^{\aleph_0}$ (Theorem 5.8). If α has cofinality ω and cardinality at most $\mathfrak{c}$, then α has a coarser connected Urysohn topology (Theorem 5.18).

Section 6 explores the connections with connectifications. We present a space, Example 6.1, with a coarser connected Hausdorff topology which cannot be embedded densely in a connected Hausdorff space. In the other direction, we present a nowhere locally compact space, Example 6.6 with no coarser connected Hausdorff topology which can be embedded densely in a connected Hausdorff space. The nowhere local compactness is interesting because a major step of the proof of Theorem 1.1 in [7] is to show that a noncompact metrizable space X has a coarser nowhere locally compact topology.

A space X is called *Urysohn* if for every pair x, x' of distinct points, there is a discrete open family $\{V_x, V_{x'}\}$ with $x \in V_x$ and $x' \in V_{x'}$. We will need a countable connected Urysohn space. The Roy space is such a space.

Definition 1.2 [12]. Let $\{Q_n\colon n < \omega\}$ partition Q a countable dense subset of $\mathbb{R}$ into infinite dense sets. $R = Q \cup \{\infty\}$ is a Roy space with the following topology τ:

(1) If $x \in Q_{2n}$ then $(x - \varepsilon, x + \varepsilon) \cap Q_{2n} \in \tau$.
(2) If $x \in Q_{2n+1}$ then $(x - \varepsilon, x + \varepsilon) \cap (Q_{2n} \cup Q_{2n+1} \cup Q_{2n+2}) \in \tau$.
(3) For all n, $\{\infty\} \cup \bigcup_{k \geqslant 2n} Q_k \in \tau$.

A Roy space is Urysohn and connected. Note that if R is a Roy space and E is nowhere dense in $\mathbb{R}$ then the subspace topology on $R \setminus E$ is also Urysohn and connected. However, $R \setminus \{\infty\}$ is totally disconnected.

Choose a point $r_0 \in R$. We let R_σ denote the σ-product of countably many copies of R with base point r_0:

$$R_\sigma = \bigl\{x \in R^\omega\colon \{i \in \omega\colon x_i \neq r_0\} \text{ is finite}\bigr\}.$$

R_σ, like R, is a countable connected Urysohn space. Unlike R, it enjoys the property that $R_\sigma \setminus S$ is connected for every finite subset S of R_σ.

We will use the fact that a Roy space has a proper Urysohn extension, $Z = R \cup \{z\}$. (Z is an *extension* of R means that R is dense in Z. *Proper* means that $Z \neq R$.) To define Z, choose $z \in \mathbb{R} \setminus Q$, then repeat the construction of R with $Q_0 \cup \{z\}$ in place of Q_0.

We will use the following method of defining coarser topologies in Sections 2 and 3.

Lemma 1.3. *Let (X, τ) and (Y, ρ) be spaces. Let $\varphi : Y \to X$ be a set function. Define*

$$\sigma(\tau, \rho, \varphi) = \sigma = \bigl\{V \in \tau\colon \varphi^{\leftarrow}[V] \in \rho\bigr\}.$$

- *σ is a topology on X.*
- *σ is coarser than τ.*
- *$\varphi : (Y, \rho) \to (X, \sigma)$ is continuous.*
- *If Y is connected and $\varphi[Y]$ is dense in (X, τ), then (X, σ) is connected.*

Note that $\sigma = \sigma(\tau, \rho, \varphi)$ is the coarsest topology on X such that the functions $\mathrm{id}_X : (X, \sigma) \to (X, \tau)$ and $\varphi : (Y, \rho) \to (X, \sigma)$ are continuous.

In general, (X, σ) does not inherit separation properties from (X, τ) and (Y, ρ). For example, let (X, τ) and (Y, ρ) be $\mathbb{R}$ with the usual topology and set $\varphi(q) = q$ for $q \in \mathbb{Q}$ and $\varphi(p) = -p$ otherwise. Then there are no disjoint open sets separating 1 and -1. So our goal is to find conditions where the separation properties are inherited.

2. When the extent is attained

Definition 2.1. Our notation for cardinal functions follows [5]. For a metrizable space X, most of the global functions are equal [5, 4.1.15]. In this paper, we will use the *density*, $d(X)$, and the *extent*, $e(X)$.

$$d(X) = \inf\{|D|:\ D \text{ is dense in } X\},$$
$$e(X) = \sup\{|E|:\ E \text{ is closed discrete in } X\}.$$

If there is a closed discrete set E satisfying $|E| = e(X)$, then we say that the extent is *attained*. Otherwise, the extent is not attained. We say that a closed discrete set E is *strongly separated* if there is a discrete open family $\{W_e:\ e \in E\}$ satisfying $e \in W_e$ for all $e \in E$. If X is metrizable, then every closed discrete subset of X is strongly separated. (This conclusion defines *strongly collectionwise Hausdorff*.)

Digression: Instead of the extent, we really use the following cardinal function, which we suggest calling "discrete cellularity".

$$dc(X) = \sup\{|C|:\ C \text{ is a strongly separated closed discrete subset of } X\}$$
$$= \sup\{|\mathcal{W}|:\ \mathcal{W} \text{ is a discrete family of nonempty open subsets of } X\}.$$

However, we will state our results in terms of the existence of strongly separated closed discrete families.

The proof of Theorem 1.1 presented in [7] has two cases. The first, easier, case is when the extent is attained. In this case, the hypothesis metrizability can be weakened considerably, to $e(X) = d(X)$, as we show in Section 2. The second, harder, case is when the extent is not attained. In Section 3, we see that the hypothesis metrizable cannot be weakened so much.

In this section, we prove theorems of the following form: If (X, τ) is a space enjoying separation property P with $d(X) = e(X)$ and the extent is attained, then there is σ, a topology on X coarser than τ, such that (X, σ) is connected and enjoys property P. We will present three theorems—where P is collectionwise normal, Urysohn, and metrizable.

We will often use a metric hedgehog as the space (Y, ρ) of Lemma 1.3. For each cardinal $\kappa \geqslant \mathfrak{c}$, there is metric space $J = (J(\kappa), \nu)$, called the *hedgehog of spininess* κ. (See [5, 4.1.5].) The point set is $J = \{0\} \cup (0, 1] \times \kappa$. The metric, ν, is defined by cases: $\nu(0, (s, \alpha)) = s$; $\nu((s, \alpha), (t, \alpha)) = |s - t|$, and $\nu((s, \alpha), (t, \beta)) = s + t$ for $\alpha \neq \beta$. These are the pertinent cardinal functions of J: $|J| = \kappa = d(J) = e(J)$. Moreover, $e(J)$ is attained—there is a closed discrete set $T = \{1\} \times \kappa$ of cardinality κ.

The next theorem is valid when "collectionwise normal" is replaced by "regular" or by "normal".

Theorem 2.2. *Let (X, τ) be a collectionwise normal space with a strongly separated closed discrete subset C such that $|C| = d(X) \geqslant \mathfrak{c}$. Then there is σ, a topology on X coarser than τ, such that (X, σ) is connected and collectionwise normal.*

Proof. Set $\kappa = d(X) = |C|$. By replacing C with a subset, we may assume that $|X \setminus C| \geqslant \kappa$. Let (Y, ρ) be the hedgehog with spininess κ and closed discrete set T. (I.e., $Y = J(\kappa)$,

$T = \{1\} \times \kappa$, and $|T| = \kappa$.) Set $S = Y \setminus T$. Choose a one-to-one function $\varphi : Y \to X$ so that $\varphi[S] = C$ and $\varphi[Y]$ is dense in X. Set $D = \varphi[T] = \varphi[Y] \setminus C$. Set $\sigma = \sigma(\tau, \rho, \varphi)$.

Towards showing that (X, σ) is collectionwise normal, let $\mathcal{H}$ be a discrete family of closed sets. We will find a family $\{V_H\colon\ H \in \mathcal{H}\}$ satisfying $H \subset V_H \in \sigma$ for each $H \in \mathcal{H}$. Define $C^\# = C \setminus \bigcup \mathcal{H}$. In the space (X, τ), the family $\mathcal{H}^+ = \mathcal{H} \cup \{\{c\}\colon\ c \in C^\#\}$ is discrete, so there is an open discrete family $\mathcal{W} = \{W_H\colon\ H \in \mathcal{H}\} \cup \{W_c\colon\ c \in C^\#\}$ separating $\mathcal{H}^+$. (Digression: In (X, σ), the set C is dense.)

Because $\varphi : (Y, \rho) \to (X, \sigma)$ is continuous, the family $\varphi^{\leftarrow}\mathcal{H} = \{\varphi^{\leftarrow}[H]\colon\ H \in \mathcal{H}\}$ is closed discrete. Set $T^\# = T \setminus \bigcup \varphi^{\leftarrow}\mathcal{H}$. Let $\mathcal{U} = \{U_H\colon\ H \in \mathcal{H}\} \cup \{U_t\colon\ t \in T^\#\}$ be an open discrete family such that $\varphi^{\leftarrow}[H] \subset U_H$ for $H \in \mathcal{H}$ and $t \in U_t$ for $t \in T^\#$.

The construction of V_H from H can be described in words: back, expand to ρ, forth, expand to τ; repeat ω times. When considering expansion to τ, it is helpful to observe that $C \subset \bigcup \mathcal{H}^+$; hence $(\bigcup \mathcal{W} \setminus \bigcup \mathcal{H}^+) \cap C = \emptyset$. Thus, for all $c \in C^\#$, $(W_c \setminus \{c\}) \cap \varphi[Y] \subset D$; and then $\varphi^{\leftarrow}[W_c \setminus \{c\}] \subset T^\#$. By the same argument, for all $H \in \mathcal{H}$, $\varphi^{\leftarrow}[W_H \setminus H] \subset T^\#$. Similarly, $T \subset \bigcup(\varphi^{\leftarrow}\mathcal{H}^+)$ leads to for all $t \in T^\#$, $\varphi[U_t \setminus \{t\}] \subset C^\#$ and for all $H \in \mathcal{H}$, $\varphi[U_H \setminus \varphi^{\leftarrow}[H]] \subset C^\#$.

Now the precise definition. By induction on $n \in \omega$, we define

$$G_H^0 = U_H,$$
$$V_H^0 = W_H \cup \bigcup \{W_c\colon\ c \in C^\# \cap \varphi[G_H^0]\},$$
$$G_H^{n+1} = G_H^n \cup \bigcup \{U_t\colon\ t \in T^\# \cap \varphi^{\leftarrow}[V_H^n]\},$$
$$V_H^{n+1} = V_H^n \cup \bigcup \{W_c\colon\ c \in C^\# \cap \varphi[G_H^{n+1}]\}.$$

Having completed the inductive definition, set $V_H = \bigcup \{V_H^n\colon\ n \in \omega\}$ and $G_H = \bigcup \{G_H^n\colon\ n \in \omega\}$. First, note that $V_H \in \tau$ and $G_H \in \rho$. Second, we will prove by induction that

$$\varphi^{\leftarrow}[V_H^n] \subset G_H^{n+1} \subset \varphi^{\leftarrow}[V_H^{n+1}].$$

Taking the union as n varies over ω we obtain $\varphi^{\leftarrow}[V_H] = G_H \in \rho$, so $V_H \in \sigma$ by the definition of σ. Here are some details of the inclusion above:

$$\varphi^{\leftarrow}[V_H^n] = G_H^n \cup (\varphi^{\leftarrow}[V_H^n] \cap T^\#) \subset G_H^n \cup \bigcup \{U_t\colon\ t \in T^\# \cap \varphi^{\leftarrow}[V_H^n]\} = G_H^{n+1},$$
$$G_H^{n+1} \subset G_H^{n+1} \cup (\varphi^{\leftarrow}[V_H^{n+1}] \cap T^\#) = \varphi^{\leftarrow}[V_H^{n+1}].$$

It is straightforward to prove by induction on n that $\{G_H^n\colon\ H \in \mathcal{H}\}$ is pairwise disjoint and $\{V_H^n\colon\ H \in \mathcal{H}\}$ is pairwise disjoint. Then $\mathcal{V} = \{V_H\colon\ H \in \mathcal{H}\}$ is pairwise disjoint, and we have separated $\mathcal{H}$ in (X, σ). □

The next theorem is valid when “Urysohn” is replaced by any of “Hausdorff”, “collectionwise Hausdorff”, and “strongly collectionwise Hausdorff”. We will use the Roy fan intead of the metric hedgehog as the space Y. Let I be the discrete space of cardinality κ. Choose a point $r^* \in R$, the Roy space (Definition 1.2) and let D be the discrete space of cardinality κ. Define an equivalence relation $\sim$ on the product $R \times I$: $(r, i) \sim (s, j)$ iff $((r, i) = (s, j)$ or $r = r^* = s)$. The *Roy fan* with κ spines, denoted F_κ, is the quotient space of $R \times I$ with the equivalence relation $\sim$. The density of F_κ is κ and F_κ has a closed discrete subset of cardinality κ.

Theorem 2.3. *Let (X, τ) be a Urysohn space with a strongly separated closed discrete subset C such that $|C| = d(X)$. Then there is σ, a topology on X coarser than τ, such that (X, σ) is connected and Urysohn.*

Proof. Let $\kappa = d(X)$. We use the Roy hedgehog as the space (Y, ρ) in place of the hedgehog of spininess κ. Define T, S, φ, and D in Theorem 2.2.

Let x, x' be distinct points of X. Let $C^\# = C \cup \{x, x'\}$. Observe that our hypotheses are strong enough to guarantee a discrete open family, $\mathcal{W} = \{W_c\colon c \in C^\#\}$ separating $C^\#$ in (X, τ). Similarly, let $T^\# = T \cup \{\varphi^{-1}(x), \varphi^{-1}(x')\}$, and find a discrete open family $\mathcal{U} = \{U_t\colon t \in T^\#\}$ separating $T^\#$ in (Y, ρ). The back and forth construction from Theorem 2.2 gives a disjoint open family $\{V_x, V_{x'}\}$ separating x and x' in (X, σ). To show that (X, σ) is Urysohn, we must show that this doubleton is in fact discrete. Let $z \in X$ be arbitrary; we will find $V_z \in \sigma$ containing z and disjoint from at least one of V_x and $V_{x'}$.

Because $\mathcal{W}$ is discrete, there is at most one c such that $z \in \mathrm{cl}_\tau\, W_c$. Because $\mathcal{U}$ is discrete, if $\varphi^{-1}(z)$ is defined, there is at most one t such that $\varphi^{-1}(z) \in \mathrm{cl}_\rho\, U_t$.

Claim. *The conjunction $z \in \mathrm{cl}_\tau\, W_x$, $z \in \varphi[Y]$, and $\varphi^{-1}(z) \in \mathrm{cl}_\rho\, U_{\varphi^{-1}(x')}$ does not occur.*

Case 1. If $z \in W_x$, then $\varphi^{-1}(z) \in G_x$, an open set disjoint from $U_{x'}$.

Case 2. If $\varphi^{-1}(z) \in U_{\varphi^{-1}(x')}$, then $z \in V_{x'}$, an open set disjoint from W_x.

Case 3. If $z \in \mathrm{cl}_\tau\, W_x \setminus W_x$, then $z \in D$ and $\varphi^{-1}(z) \in T$. However $\varphi^{-1}(z) \in \mathrm{cl}_\rho\, U_{\varphi^{-1}(x')} \setminus U_{\varphi^{-1}(x')}$ implies $\varphi^{-1}(z) \notin T$.

Via the claim (and possibly switching x and x') we may assume that $z \notin \mathrm{cl}_\tau\, W_x$ and, if $\varphi^{-1}(z)$ is defined, $\varphi^{-1}(z) \notin \mathrm{cl}_\rho\, U_{\varphi^{-1}(x)}$. We now repeat the inductive construction of V's and G's, replacing x, x' with x, z. Let $C^\flat = C \cup \{x, z\}$ and $T^\flat = T \cup \{\varphi^{-1}(x), \varphi^{-1}(z)\}$. (If $\varphi^{-1}(z)$ is not defined, let $T^\flat = T \cup \{\varphi^{-1}(x)\}$). We can choose the discrete open family $\mathcal{W}^\flat = \{W^\flat_c\colon c \in C^\flat\}$ to differ from $\mathcal{W}$ in at most two elements. First, $W^\flat_z$ replaces $W_{x'}$; second, we may have to modify the at most one W_c having z in its closure. Observe that if $W_c \subset V_x$, then $W^\flat_c = W_c$. Similarly define $\mathcal{U}^\flat = \{U^\flat_t\colon t \in T^\flat\}$ to differ from $\mathcal{U}$ in at most two elements.

By induction on n, we see that $V_x^{n,\flat} = V_x^n$. So V_z is disjoint from $V_x^\flat = V_x$, which demonstrates that $\{V_x, V_{x'}\}$ is discrete. We have shown that (X, σ) is Urysohn. □

In the last theorem of this section, the separation property is metrizability. It is perhaps surprising that the construction is the same as in the first two theorems, and that of the requirements on the new metric, the hardest to verify is (essentially) the Hausdorff property.

We start with a preparatory lemma.

Lemma 2.4. *If C is a closed discrete set in a metric space (X, μ), there is a compatible metric μ' such that $\mu(c, c') \geqslant 1$ for all distinct $c, c' \in C$.*

Proof. Choose a discrete open family $\{U_c\colon c \in C\}$ separating C. For each $c \in C$, choose a continuous, real-valued function f_c satisfying $f_c(c) = 1$ and $f_c[X \setminus U_c] = \{0\}$. Define

$$\mu'(x, x') = \mu(x, x') + \sum_{c \in C} \bigl| f_c(x) - f_c(x') \bigr|.$$

The sum is defined because at most two summands are nonzero. □

The following theorem was also proved by Irina Druzhinina, a doctorate student of Wilson, in [4, Theorem 3.3].

Theorem 2.5. *Let (X, τ) be a metrizable space with a closed discrete subset C such that $|C| = d(X) \geqslant \mathfrak{c}$. Then there is σ, a topology on X coarser than τ, such that (X, σ) is connected and metrizable.*

Proof. Let the metric μ generate the topology τ on X. By replacing C with a subset and using Lemma 2.4, we may assume that $|X \setminus C| \geqslant \kappa$ and that $\mu(c, c') \geqslant 1$ for all distinct $c, c' \in C$. As in the proof of Theorem 2.2, let (Y, ρ) be the hedgehog of spininess κ, with metric ν, and closed discrete set T. Define S, φ, and D as above.

For $x, x' \in \varphi[Y]$, let $\lambda_0(x, x')$ be the lesser of $\mu(x, x')$ and $\nu(\varphi^{-1}(x), \varphi^{-1}(x'))$. For other $x, x' \in X$, set $\lambda_0(x, x') = \mu(x, x')$. Because λ_0 does not satisfy the triangle inequality, we set

$$\lambda(x, x') = \inf\bigl\{\lambda_0(x_0, x_1) + \lambda_0(x_1, x_2) + \cdots + \lambda_0(x_{n-1}, x_n)\bigr\},$$

where $x_0, x_1, \ldots, x_n$ varies over all finite sequences with $x = x_0$ and $x' = x_n$.

Observe that in the definition of λ, it is sufficient to take the infimum of "alternating" sums

$$\mu(x_0, x_1) + \nu\bigl(\varphi^{-1}(x_1), \varphi^{-1}(x_2)\bigr) + \mu(x_2, x_3) + \cdots$$

(including those starting $\nu(\varphi^{-1}(x_0), \varphi^{-1}(x_1)) + \mu(x_1, x_2) + \cdots$) because μ and ν satisfy the triangle inequality. Also useful is this trivial observation: If the sum is less than ε, then each summand is less than ε.

Note that λ is symmetric and satisfies the triangle inequality. To complete the verification that λ is a metric, let x and x' be distinct points of X. We will show that $\lambda(x, x') > 0$. Choose $\varepsilon \in (0, 1)$ to be less than $\mu(x, x')$, $\mu(x, C \setminus \{x\})$, and $\mu(x', C \setminus \{x'\})$. Moreover, ε should be less than $\nu(\varphi^{-1}(x), \varphi^{-1}(x'))$, $\nu(\varphi^{-1}(x), T \setminus \{\varphi^{-1}(x)\})$, and $\nu(\varphi^{-1}(x'), T \setminus \{\varphi^{-1}(x')\})$, whenever these are defined.

We now follow the proof of Theorem 2.2. Set $C^{\#} = C \cup \{x, x'\}$. For $c \in C^{\#}$, let W_c be the ball of μ radius $\varepsilon/3$ around c. Similarly, set $T^{\#} = T \cup \{\varphi^{-1}(x), \varphi^{-1}(x')\}$ (if the latter are defined), and let U_t be the ball of ν radius $\varepsilon/3$ around t. Define V_x and $V_{x'}$ via the back and forth induction.

We chose $\varepsilon/3$ small enough that if $\mu(x_i, x_{i+1})$ is a term in an alternating sum, then $x_{i+1} \notin C$. Similarly, if $\nu(\varphi^{-1}(x_i), \varphi^{-1}(x_{i+1}))$ is a term in an alternating sum, then $\varphi^{-1}(x_{i+1}) \notin T$. Observe how the alternating sum parallels the back and forth induction. We see that the λ ball of radius $\varepsilon/3$ around x is a subset of V_x. Hence $\lambda(x, x') \geqslant \varepsilon/3 > 0$. □

When we used the metric hedgehog as the space Y, we additionally assumed that $d(X) \geqslant \mathfrak{c}$. When the new space (X, σ) is at least Tychonoff, some assumption of bigness is necessary because a connected Tychonoff space has cardinality at least $\mathfrak{c}$. (For another necessary condition, see [3].) However, to obtain a coarser topology which is Urysohn (or even less, Hausdorff), this necessary condition disappears. We removed the assumption $d(X) \geqslant \mathfrak{c}$ by using the Roy fan instead of the metric hedgehog.

3. When the extent is not attained

We begin with an example showing that "extent is attained" cannot be omitted from the hypotheses of the theorems of the previous section.

Example 3.1. A hereditarily paracompact space X with $d(X) = e(X)$ with no coarser connected Hausdorff topology.

Let κ be a strong limit cardinal of cofinality ω. The example, (X, τ), is the free sum of a metrizable space S and compact space K. Let $S = \{\hat{s}\} \cup \{s_\alpha\colon \alpha < \kappa\}$. Let $K = \{\hat{k}\} \cup \{k_\alpha\colon \alpha < \kappa\}$. In (X, τ), all points s_α and k_α are isolated. A neighborhood of $\hat{s}$ contains $\{s_\alpha\colon \beta < \alpha < \kappa\}$ for some $\beta < \kappa$. A neighborhood of $\hat{k}$ contains all but a finite subset of $\{k_\alpha\colon \alpha < \kappa\}$.

It is easy to verify that the space (X, ρ) has many nice properties. For example, it is regular, hereditarily paracompact, first countable except at $\hat{k}$, locally compact except at $\hat{s}$, and $d(X) = e(X) = \kappa$. However the extent is not attained, and we now show that there is no coarser connected Hausdorff topology.

Let σ be a Hausdorff topology on X coarser than τ. We will show that (X, σ) has an isolated point, hence is not connected. First, note that K as a subspace of (X, σ) is homeomorphic to K as a subspace of (X, τ). (We can observe that K is compact and then quote a general theorem, e.g., [10, 7.5b] or [16, 17M]. However, it is straightforward to verify this special case.) In particular, $\{k_\alpha\} \cup S \in \sigma$ for every $\alpha < \kappa$. Because σ is Hausdorff, there are U and U' disjoint elements of σ satisfying $K \subset U$ and $\hat{s} \in U'$. Since $\hat{s} \in U' \in \sigma \subset \tau$, we observe that $|U \cap S| < \kappa$.

For each $\alpha < \kappa$, set $\mathcal{N}_\alpha = \{V \cap U \cap S\colon k_\alpha \in V \in \sigma\}$. Note that $\{k_\alpha\} \cup N \in \sigma$ for every $N \in \mathcal{N}_\alpha$ and that the intersection of two members of $\mathcal{N}_\alpha$ is again in $\mathcal{N}_\alpha$. Because κ is a strong limit cardinal, there are $\alpha < \alpha' < \kappa$ with $\mathcal{N}_\alpha = \mathcal{N}_{\alpha'}$. (Set $\delta = |U \cap S|$ and observe that $2^{2^\delta} < \kappa$.)

Let W, W' be disjoint elements of σ satisfying $k_\alpha \in W$ and $k_{\alpha'} \in W'$. Note that $W' \cap U \cap S \in \mathcal{N}_{\alpha'} = \mathcal{N}_\alpha$. Hence $(W' \cap U \cap S) \cap (W \cap U \cap S) = \emptyset \in \mathcal{N}_\alpha$. We conclude that $\{k_\alpha\} \in \sigma$, as desired.

Let (X, τ) be a metrizable space where the extent is not attained. Because of the above example, if we hope to find a coarser connected Hausdorff topology on X, we must use special properties of X. We now list some of these properties.

Theorem 3.2. *Let (X, τ) be a space with metric μ in which $e(X) = \kappa$ is not attained. Let K be set of points x of X such that every neighborhood of x has extent κ. Then*

(1) *κ is a singular cardinal of cofinality ω.*
(2) *K is a compact, nowhere dense subset of X.*
(3) *If U is an open subset of X such that $\mathrm{cl}_\tau U \cap K = \emptyset$, then $e(Y) < \kappa$.*
(4) *K is nonempty.*
(5) *For every open set U meeting K and every $\theta < \kappa$ there is an open subset U' of U such that $e(U') > \theta$ is attained and $\mathrm{cl}_\tau U' \cap K = \emptyset$. If X is zero-dimensional, we can require U' to be clopen.*

Proof. (1) is true because X has a σ-discrete base. Let $(\kappa_n\colon n \in \omega)$ be an increasing sequence of cardinals cofinal in κ.

Towards (2), assume that K is not compact. Then there is $\{x_n\colon n \in \omega\} \subset K$ which is closed discrete in X. Hence there is an open discrete family $\{U_n\colon n \in \omega\}$ with $x_n \in U_n$. For each n, choose E_n, a closed discrete subset of U_n with $|E_n| = \kappa_n$. Then $\bigcup\{E_n\colon n \in \omega\}$ is an closed discrete family of cardinality κ-contradicting the hypothesis that the extent is not attained. It follows that K is nowhere dense—if x were in the interior of K, then K would be a neighborhood of x with $e(K) \leqslant \omega < \kappa$, and $x \notin K$ (by the definition of K).

Towards (3), we assume that $e(U) = \kappa$ and again find E, a closed discrete subset of cardinality κ. We argue by cases. First assume that, for each n, there is a point $x_n \in U$ such that every neighborhood of x_n has extent at least κ_n. Then $\{x_n\colon n \in \omega\}$ is closed discrete, because a limit point would be in $\mathrm{cl}_\tau U \cap K = \emptyset$. Again there is an open discrete family $\{U_n\colon n \in \omega\}$ with $x_n \in U_n$. For each n, choose E_n, a closed discrete subset of U_n with $|E_n| = \kappa_n$. Then $\bigcup\{E_n\colon n \in \omega\}$ is an closed discrete family of cardinality κ.

The remaining case is that there is $n_0 \in \omega$ and an open cover $\mathcal{W}$ of U such that $e(W) \leqslant \kappa_{n_0}$ for every $W \in \mathcal{W}$. X is paracompact, so we may assume that $\mathcal{W}$ is locally finite. Because $e(U) = \kappa$, for each n there is a closed discrete subset E_n of U with $|E_n| = \kappa_n$. Set $S_m = \bigcup\{W \in \mathcal{W}\colon W \cap E_m \neq \emptyset\}$. Note that $e(S_m) \leqslant \kappa_{n_0} \cdot \kappa_m$. Set

$$E = \bigcup_{n\in\omega}\left\{E_n \Big\backslash \bigcup_{m<n} S_m\right\}$$

and observe that E is a closed discrete subset with $|E| = \kappa$.

Towards (4), assume that K is empty and apply (2) with $U = X$ to get a contradiction.

Towards (5), let U and θ be given. Let us call an open set V *e-homogeneous* if every nonempty open subset V' of V satisfies $e(V') = e(V)$. Observe that every nonempty open set has a nonempty open e-homogeneous subset. Moreover, the extent is attained in an e-homogeneous open subset (of a metrizable space).

Let $\mathcal{V}$ be a maximal pairwise disjoint family of sets V satisfying

- V is an open e-homogeneous subset of U;
- $\mathrm{cl}_\tau V \cap K = \emptyset$;
- V is clopen in X (if X is zero-dimensional).

If some $V \in \mathcal{V}$ has $e(V) > \theta$, let $V = U'$. Otherwise, $\theta \geqslant \sup\{e(V)\colon V \in \mathcal{V}\}$ and $|\mathcal{V}| = \kappa$. Choose $\varepsilon > 0$ so that $\mathcal{V}' = \{V \in \mathcal{V}\colon \mu(K, V) > \varepsilon\}$ has cardinality greater than θ. $\mathcal{V}'$ is disjoint, but not necessarily discrete. Let L be the set of limit points of $\mathcal{V}$. Choose $\varepsilon' \in (0, \varepsilon)$ so that $\mathcal{V}'' = \{V \in \mathcal{V}'\colon (\exists e \in V)\ \mu(e, L) > \varepsilon'\}$ has cardinality greater than θ. For each $V \in \mathcal{V}''$, choose a nonempty open (clopen if X is zero-dimensional) subset satisfying $\mu(V', L) > \varepsilon'$. Let $U' = \bigcup\{V':\ V \in \mathcal{V}''\}$. $\square$

We now show that K has a countable "base in X".

Lemma 3.3. *Let K be a Lindelöf subset of a zero-dimensional metric space (X, μ). There is a countable family $\mathcal{B}$ of clopen subsets of (X, μ) such that, for any $x \in K$ and any open $U \subset X$ with $x \in U$ there is $B \in \mathcal{B}$ for which $x \in B \subset U$.*

Proof. For each $n \in \omega$, let $\mathcal{G}_n$ be the family of clopen sets G of diameter less than $1/n$; that is, they satisfy $\sup\{\mu(x, x'):\ x, x' \in G\} < 1/n$. Because K is Lindelöf, there is a countable subfamily $\mathcal{G}'_n$ which covers K. Let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{G}'_n$. $\square$

We are ready to prove

Theorem 3.4. *If (X, τ) is zero-dimensional space with metric μ and $e(X) = \kappa > \mathfrak{c}$ is not attained, then there is σ, a topology on X coarser than τ, such that (X, σ) is connected and metrizable.*

Proof. There are four parts to the proof. First, we use the countable $\mathcal{B}$ from Lemma 3.3 to define a countable tree, P. Second, we partition $X\setminus K$ into a clopen family $\{W_p:\ p \in P\}$ indexed by P. Third, we use the method of Theorem 2.5 on each W_p, where Y is a connected tree of hedgehogs. The last step is to verify that λ is a metric.

Choose $\{B_i:\ i \in \omega\} \subset \tau$ as in the conclusion of Lemma 3.3. For $m \in \omega$ and $p \in {}^m2$, define

$$B^p = \bigcap \bigl\{B_i:\ p(i) = 1\bigr\} \cap \bigcap \bigl\{X\setminus B_i:\ p(i) = 0\bigr\}.$$

Let $P_m = \{p \in {}^m2:\ B^p \cap K \neq \emptyset\}$ and $P = \bigcup\{P_m:\ m \in \omega\}$. Say that p and q are *neighbors* if $p \in P_{m+1}$ and $q = p|m$.

Claim. *We can redefine $\{B_i:\ i \in \omega\}$ so that $B^p \cap K = \emptyset$ implies $B^p = \emptyset$.*

Let $\{B_i:\ i \in \omega\}$ be given. We will define $\widehat{B}_m$ by induction on m. We define $\widehat{B}_0$ by cases: $\widehat{B}_0 = X$ if $K \subset B_0$; $\widehat{B}_0 = \emptyset$ if $B_0 \cap K = \emptyset$; and $\widehat{B}_0 = B_0$ otherwise. If $\widehat{B}_i$ has been defined for $i < m$, let $\widehat{B}^p$ be defined analogously to B^p, using $\widehat{B}_i$ in place of B_i. Then we define

$$B_m^+ = \bigcup \bigl\{\widehat{B}^p:\ p \in P_m \text{ and } \widehat{B}^p \cap K \subset B_m\bigr\},$$

$$B_m^- = \bigcup \bigl\{\widehat{B}^p:\ p \in P_m \text{ and } \widehat{B}^p \cap K \cap B_m = \emptyset\bigr\},$$

$$\widehat{B}_m = B_m^+ \cup B_m \setminus B_M^-.$$

Observe that $\widehat{B}_m$ is clopen and that $\widehat{B}_m \cap K = B_m \cap K$. Moreover, $\widehat{B}^p \cap K = \emptyset$ implies $\widehat{B}^p = \emptyset$, as required.

By induction on m, using Theorem 3.2.5 repeatedly, we will define W_p, C_p, and κ_p for $p \in P_m$ to satisfy

- W_p is a clopen subset of B^p;
- $\{W_p\colon\ p \in P\}$ is pairwise disjoint;
- C_p is a closed discrete subset of W_p;
- $|C_p| = e(W_p) = d(W_p) = \kappa_p \geqslant \mathfrak{c}$;
- $\bigcup\{W_p\colon\ p \in P\} = X\backslash K$.

Let $\{O_m\colon\ m \in \omega\}$ be a nested, increasing sequence of clopen subsets of X satisfying $O_0 = \emptyset$, $\bigcup_{m\in\omega} O_m = X\backslash K$, and $\mu(K, O_m) < 2^{-m}$. A first approximation to W_p is $W'_p = B^p \cap (O_{m+1}\backslash O_m)$. However, W'_p may be empty, too small, or not attain its extent.

As the induction goes on, we define $W_m = \bigcup\{W_p\colon\ \operatorname{dom} p < m\}$, and verify the induction hypothesis $O_m \subset W_m$. When $m = 0$, we have $O_0 = \emptyset = W_0$.

For $p \in P_m$, set $\theta_p = \max\{\mathfrak{c}, d(B^p \cap (O_{m+1}\backslash W_m))\}$. Apply Theorem 3.2.5 with $U = B^p$ and $\theta = \theta_p$ to get a clopen set U'. Set $W_p = U' \cup (B^p \cap (O_{m+1}\backslash W_m))$. Set $\kappa_p = e(W_p)$. Let C_p be a closed discrete subset of U' with $|C_p| = e(W_p)$. (The existence of C_p is guaranteed by Theorem 3.2.5).

Claim. *We can redefine the metric μ on X so that it satisfies, for each $m \in \omega$ and each $p \in P_m$,*

- $\mu(c, c') \geqslant 2^{-m}$ *for distinct $c, c' \in C_p$; and*
- $\mu(C_p, X\backslash W_p) \geqslant 2^{-m}$.

We use the technique of Lemma 2.4. For each $p \in P$, choose a discrete open family $\{U_c\colon\ c \in C_p\}$ separating C_p, whose union is contained in W_p. For each $c \in C_p$, choose a continuous, real-valued function f_c satisfying $f_c(c) = 1$ and $f_c[X\backslash U_c] = \{0\}$. For $p \in P_m$, define $f^p(x) = 2^{-m} \cdot \sum_{c\in C_p} f_c(x)$. Next, define $f(x) = \sum_{p\in P} f^p(x)$. Finally, define $\mu'(x, x') = \mu(x, x') + |f(x) - f(x')|$.

Let J_p be the metric hedgehog of spininess κ_p with closed discrete set T_p, $|T_p| = \kappa_p$. If q is a neighbor of p, choose a point $t_{pq} \in T_p$, and set $N_p = \{t_{pq}\colon\ q \text{ is a neighbor of } p\}$. The metric space (Y, ρ) with metric ν will be the quotient of the free sum $\bigoplus\{J_p\colon\ p \in P\}$ created by identifying t_{pq} with t_{qp} for all neighbors p and q. (Note that Y is connected.) To avoid denoting equivalence classes, we will abuse notation and consider Y to be $\bigcup\{J_p\colon\ p \in P\}$. (In case $p \in P_{m+1}$ and $q = p|m$, there is an ambiguous point $\{t_{pq}, t_{qp}\}$. We will consider this point to be in N_q and not in J_p). Notice that if $y \in J_p$, $y' \in J_{p'}$, and $\nu(y, y') < 1$, then either $p = p'$ or p and p' are neighbors.

For each $p \in P$, define $T'_p = T_p\backslash N_p$. As in the proof of Theorem 2.5, choose a one-to-one function $\varphi_p : J_p \to W_p$ such that $\varphi_p[J_p\backslash T'_p] = C_p$ and $\varphi_p[J_p]$ is dense in W_p. Set $\varphi = \bigcup\{\varphi_p\colon\ p \in P\}$; then $\varphi : Y \to X$ is one-to-one and $\varphi[Y]$ is dense in X.

Define λ_0 and λ as in the proof of Theorem 2.5. Then λ is a pseudometric on X, and we must show that $\lambda(x, x') > 0$ for distinct $x, x' \in X$. We proceed by cases.

Case 1. $x \notin K$ and x is not of the form $\varphi(t_{pq})$. Let $x \in W_p$, $p \in P_m$. In case $x' \in W_p$, too, we argue as in Theorem 2.5. If $x' \notin W_p$, choose $\varepsilon \in (0, 2^{-m})$ so that $\mu(x, C_p\backslash\{x\} \cup (X\backslash W_p)) > \varepsilon$. Moreover, if $x \in \varphi[Y]$, then $\nu(\varphi^{-1}(x), T_p\backslash\{\varphi^{-1}(x)\} \cup (Y\backslash J_p)) > \varepsilon$. Now the λ ball of radius ε centered at x is contained in W_p.

Case 2. x has the form $\varphi(t_{pq})$. Proceed as in case 1, with $W_p \cup W_q$ replacing W_p, $J_p \cup J_q$ replacing J_p, $2^{-(m+1)}$ replacing 2^{-m}, etc. We conclude that the λ ball of radius ε centered at x is contained in $W_p \cup W_q$.

Case 3. $x \in K$ and $x' \in K$. Find i so that $x \in B_i$ and $x' \notin B_i$. Choose $\varepsilon \in (0, 2^{-(i+1)})$ so that $\varepsilon < \mu(B_i, X \setminus B_i)$.

Towards a contradiction, assume that $\lambda(x, x') = 0$. Fix a finite sequence $x, x_1, \dots, x'$ with $\lambda_0(x, x_1) + \lambda_0(x_1, x_2) + \cdots + \lambda_0(x_{n-1}, x') < \varepsilon$.

Because $x = x_0 \in B_i$ and $x' = x_n \in X \setminus B_i$, there is $j < n$ such that $x_j \in B_i$ and $x_{j+1} \in (X \setminus B_i)$.

Claim. *At least one of* x_j, x_{j+1} *is in* W_i.

Towards a contradiction, assume that $x_j \notin W_i$ and $x_{j+1} \notin W_i$. First, observe that $\mu(x_j, x_{j+1}) \geqslant \mu(B_i, X \setminus B_i) \geqslant \varepsilon$. Second, observe that if $\varphi^{-1}(x_j) \in J_p$ and $\varphi^{-1}(x_{j+1}) \in J_q$, then $p(i) = 1$ and $q(i) = 0$. Hence p and q are neither equal nor neighbors, and $\nu(\varphi^{-1}(x_j), \varphi^{-1}(x_{j+1})) \geqslant 1$. Combining these inequalities, we see that $\lambda_0(x_j, x_{j+1}) \geqslant \varepsilon$. However, this contradicts

$$\lambda_0(x_j, x_{j+1}) < \lambda_0(x, x_1) + \lambda_0(x_1, x_2) + \cdots + \lambda_0(x_{n-1}, x') < \varepsilon.$$

The claim establishes that there is $x^* \in W_i \cap \{x_j, x_{j+1}\}$. In more detail, there are $m < i$ and $p \in P_m$ with $x^* \in W_p$. If $\mu(x^*, \varphi[N_p]) \geqslant \varepsilon$, we argue as in case 1 and conclude that the λ ball of radius ε centered at x_j is contained in W_p. Otherwise, $\mu(x^*, \varphi(t_{pq})) < \varepsilon$ for some neighbor q of p. We argue as in case 2, and conclude that the λ ball of radius ε centered at x^* is contained in $W_p \cup W_q$. In either case, x is not in the λ ball of radius ε centered at x^*. However, $\lambda(x, x^*) \leqslant \lambda(x, x') < \varepsilon$. Contradiction. □

The following question is essentially the same as Question 3.1 of [4].

Question 3.5. Does every metrizable space of weight at least $\mathfrak{c}$ have a coarser connected metrizable topology?

In this context, notice that König's Theorem (see [8, Corollary 10.41]) implies that $\mathfrak{c}$ is not a singular cardinal of countable cofinality. If X is a metrizable space of weight $\mathfrak{c}$, then $d(X) = \mathfrak{c} = e(X)$, and the extent is attained. Hence the hypothesis of Theorem 2.5 is satisfied, and X has a coarser connected metrizable topology.

4. σ-locally finite bases

We show that every non-H-closed Hausdorff space with a σ-locally finite base has a weaker connected Hausdorff topology using a modification and extension of the technique provided in [9]. First, some additional background material is needed.

In this section, if we introduce a space X without specifying a topology, then implicitly that topology is called $\tau(X)$. We say that a Hausdorff space X is *H-closed* if whenever X is a subspace of a Hausdorff space Y, then X is closed in Y. For a Hausdorff space X, this is equivalent to every open ultrafilter on X converges and to the property that for every open cover $\mathcal{C}$ of X, there is a finite subset $\mathcal{D} \subset \mathcal{C}$ such that $X = \mathrm{cl}_X(\bigcup \mathcal{D})$. All of these results and more can be found in [10].

Let X and Y be two spaces. A function $f : Y \to X$ is *θ-continuous* if for each $p \in Y$ and open set $U \in \tau(X)$ such that $f(p) \in U$, there is an open set $V \in \tau(Y)$ such that $p \in V$ and $f[\mathrm{cl}_Y V] \subset \mathrm{cl}_X U$. A function $f : Y \to X$ is *perfect* if the image of every closed set is closed and the preimage of every point is compact. Note that perfect does not imply continuous.

Theorem 4.1 [9]. *Let X and Y be spaces and $f : Y \to X$ be a θ-continuous surjection. If Y is connected, then so is X.*

Let X and Y be sets and $f : Y \to X$ be a function. For $A \subset Y$, define $f^{\#}[A] = \{x \in X\colon f^{\leftarrow}(x) \subset A\}$. Note that for subsets $A, B \subset Y$, $f^{\#}[Y \backslash A] = X \backslash f[A]$ and $f^{\#}[A \cap B] = f^{\#}[A] \cap f^{\#}[B]$. The topology on Y generated by $\{f^{\#}[U]\colon U \in \tau(Y)\}$ is called the *θ-quotient topology* induced by f. The function f is called *irreducible* if for each nonempty open set $U \in \tau(Y)$, there is some $x \in X$ such that $f^{\leftarrow}(x) \subset U$.

Theorem 4.2 [9]. *Let $f : Y \to X$ be perfect, irreducible, and onto where X and Y are spaces. Let σ be the θ-quotient topology induced on X by f. Then (X, σ) is a Hausdorff space, $\sigma \subset \tau(X)$, and $f : Y \to (X, \sigma)$ is θ-continuous.*

Let X be a Hausdorff space and let $\Theta X = \{\mathcal{U}\colon \mathcal{U}$ is an open ultrafilter on $X\}$. For $U \in \tau(X)$, let $O(U) = \{\mathcal{U}\colon U \in \mathcal{U}\}$. For $U, V \in \tau(X)$, it is easy to verify (see [10]) that $O(\emptyset) = \emptyset$, $O(X) = \Theta X$, $O(U \cap V) = O(U) \cap O(V)$, $O(U \cup V) = O(U) \cup O(V)$, $\Theta X \backslash O(U) = O(X \backslash \mathrm{cl}_X U)$, and $O(U) = O(\mathrm{int}_X \mathrm{cl}_X U)$. ΘX with the topology generated by $\{O(U)\colon U \in \tau(X)\}$ is an extremally disconnected compact Hausdorff space. The subspace $EX = \{\mathcal{U} \in \Theta X\colon \mathcal{U}$ is fixed$\}$ is called the *absolute* of X. The function $k : EX \to X$ defined by $k(\mathcal{U})$ is the unique convergent point of $\mathcal{U}$ is called a covering function. The subspace EX is dense in ΘX (in particular, EX is an extremally disconnected Tychonoff space and $\Theta X = \beta EX$), and the covering function $k : EX \to X$ is irreducible, θ-continuous, perfect and onto and has the property that if T is a nonempty open subset of EX, then $k^{\#}[T]$ is a nonempty open subset of X.

The next technical result is pivotal in solving Question 3.9 of [7].

Theorem 4.3. *Let X be a space, Y a connected space, $f : EX \to Y$ a continuous surjection, and $g : Y \to X$ a function such that $g \circ f = k$. Then X has a coarser connected topology.*

Proof. First, we show that g is perfect, irreducible and onto. Clearly, g is onto. For $x \in X$, $f[k^{\leftarrow}(x)] = f[f^{\leftarrow} g^{\leftarrow}(x)] = g^{\leftarrow}(x)$ is compact. If A is closed in Y, $f^{\leftarrow}[A]$ is closed and $k[f^{\leftarrow}[A]] = g[A]$ is closed in X. If U is a nonempty open subset of Y, $f^{\leftarrow}[U]$ is a nonempty open subset of EX. For some $x \in X, k^{\leftarrow}(x) \subset f^{\leftarrow}[U]$. Thus, $g^{\leftarrow}(x) = f[k^{\leftarrow}(x)] \subset U$. Let σ be the θ-quotient topology on X induced by g. By

Theorem 4.2, (X,σ) is Hausdorff, $\sigma \subset \tau(X)$, and $g: Y \to (X,\sigma)$ is θ-continuous. By Theorem 4.1, (X,σ) is connected. □

Corollary 4.4. *Let X be a space. If EX has a coarser connected topology, then so does X.*

A space is called *feebly compact* if every locally finite family of open sets is finite.

Theorem 4.5 [10, 1.11(b)]. *The following are equivalent for a space X:*

(1) *X is feebly compact.*
(2) *Every locally finite family of pairwise disjoint open sets is finite.*
(3) *If $\{U_n\colon n \in \omega\}$ is a decreasing family of nonempty open sets of X, then $\bigcap\{\mathrm{cl}_X U_n\colon n \in \omega\} \neq \emptyset$.*
(4) *Every countable open cover of X has a finite subfamily whose union is dense in X.*

Theorem 4.6 [10, 1.11(c)].

(1) *Every feebly compact space is pseudocompact.*
(2) *A Tychonoff space is feebly compact iff it is pseudocompact.*

Theorem 4.7.

(1) *If $\mathcal{U}$ is a locally finite family of open sets on X, then $\{OU \cap EX\colon U \in \mathcal{U}\}$ is a locally finite family of open sets on EX,*
(2) *A space X is feebly compact iff EX is feebly compact.*

Proof. To show (1), let $\mathcal{V} \in EX$ and $k(\mathcal{V}) = x$. There is an open set T in X such that $x \in T$ and $T \cap U = \emptyset$ except for a finite number of elements U of $\mathcal{U}$. Now, $\mathcal{V} \in OT \cap EX$ and $OT \cap OU = \emptyset$ except for a finite number of elements U of $\mathcal{U}$. To show (2), suppose EX is feebly compact and $\mathcal{U}$ is a locally finite family of pairwise disjoint open sets on X. By (1), $\{OU \cap EX\colon U \in \mathcal{U}\}$ is locally finite and hence finite by feebly compactness. Thus, $\mathcal{U}$ is finite. Conversely suppose that X is feebly compact. Let $\mathcal{U}$ be a locally finite family of pairwise disjoint open sets on EX. Then $\{k^{\#}[U]\colon U \in \mathcal{U}\}$ is a family of pairwise disjoint open sets on X. If $x \in X$, then as $k^{\leftarrow}(x)$ is compact, there is an open set T in EX such that $k^{\leftarrow}(x) \subset T$ and T meets only a finite number of elements of $\mathcal{U}$. Now, $x \in k^{\#}[T]$ and $k^{\#}[T]$ meets only a finite number of elements of $\{k^{\#}[U]\colon U \in \mathcal{U}\}$. Thus, $\{k^{\#}[U]\colon U \in \mathcal{U}\}$ is locally finite from which it follows that $\mathcal{U}$ is finite. □

Definition 4.8. Let $\mathcal{U}$ be a locally finite family of clopen sets on a space X. For $x, y \in X$, let $\rho_{\mathcal{U}}(x, y) = \sum\{|f_U(x) - f_U(y)|\colon U \in \mathcal{U}\}$ where $f_U : X \to \{0, 1\}$ is the characteristic function on U.

Theorem 4.9. *Let $\mathcal{U}$ be a locally finite family of clopen sets on a space X. Then $\rho_{\mathcal{U}}$ is a pseudometric on X taking only integer values. For $x \in X$ and $0 < r < 1$,*

$$B(x,r) = B(x,1) = \bigcap\{U \in \mathcal{U}\colon x \in U\} \cap \bigcap\{X \backslash U\colon x \notin U\},$$

which is a clopen set. Hence (X,ρ) *is a zero-dimensional pseudometric space* (*but not necessarily Hausdorff*).

Theorem 4.10. *If* X *is not H-closed and has a* σ*-locally finite base* $\mathcal{B}=\{\mathcal{B}_n\colon n\in\omega\}$, *then* X *and* EX *are not feebly compact and there is a continuous surjection* $u : EX\to\omega$.

Proof. If X is feebly compact, then $\mathcal{B}_n$ is finite for all $n\in\omega$. So, X is second countable. However, second countable plus feebly compact implies H-closed. So, X is not feebly compact. By Theorem 4.7, EX is not feebly compact. As EX is zero-dimensional, there is an infinite locally finite family of pairwise disjoint clopen sets $\{U_n\colon n\in\omega\}$. The function $u : EX\to\omega$ defined by $u[U_n]=n$ and $u[EX\setminus\bigcup\{U_n\colon n\in\omega\}]=0$ is continuous and onto ω. □

Let (Y,ρ) be a pseudometric space. For $y\in Y$, let $\bar{y}=\{x\in Y\colon \rho(x,y)=0\}$ and consider the partition $\overline{Y}=\{\bar{y}\colon y\in Y\}$ of Y. Define $\bar{\rho}$ on $\overline{Y}$ by $\bar{\rho}(\bar{x},\bar{y})=\rho(x,y)$ and $h : Y\to\overline{Y}$ by $h(x)=\bar{x}$. By Exercise 2C in [16], $(\overline{Y},\bar{\rho})$ is a metric space, and for $A\subset Y$, A is closed (open) in Y iff $h[A]$ is closed (open) in $\overline{Y}$.

Let X be a non-H-closed space with a σ-locally finite base $\mathcal{B}=\{\mathcal{B}_n\colon n\in\omega\}$. By Theorem 4.7, $\mathcal{U}_n=\{OU\cap EX\colon U\in\mathcal{B}_n\}$ is a locally finite family of open sets on EX for each $n\in\omega$. Let E_n denote EX with $\rho_{\mathcal{U}_n}$ the pseudometric defined in Theorem 4.8. Thus, E_n is a zero-dimensional pseudometric space but may not be Hausdorff. Define $f : EX\to\omega\times\prod\{E_n\colon n\in\omega\}$ by $f=u\times\Delta$ where u is as in Theorem 4.10 and where Δ is the infinite diagonal map, $\Delta(x)=(x,x,\ldots)$. Now, $\omega\times\prod\{E_n\colon n\in\omega\}$ is a zero-dimensional pseudometric space. Let

$$h:\omega\times\prod\{E_n\colon n\in\omega\}\to\overline{\omega\times\prod\{E_n\colon n\in\omega\}}$$

be defined as above.

Theorem 4.11. *Let* X *be a non-H-closed space with a* σ*-locally finite base* $\mathcal{B}=\{\mathcal{B}_n\colon n\in\omega\}$. *Using the notation defined in the preceding paragraph, the function*

$$h\circ f : EX\to\overline{\omega\times\prod\{E_n\colon n\in\omega\}}$$

is continuous and $(h\circ f)[EX]$ *is a noncompact metric space.*

Proof. $h\circ f$ is continuous because both f and h are continuous. As u is unbounded and $\overline{\omega\times\prod\{E_n\colon n\in\omega\}}$ is a metric space, $(h\circ f)[EX]$ is a noncompact metric space. □

The following theorem was also proved by Alas and Wilson in [1, Theorem 3.4]. The case when X has a countable network was proved by Tkachuk and Wilson in [14, Theorem 3.4], and by Porter in [9] (the main theorem).

Theorem 4.12. *A non-H-closed space with a* σ*-locally finite base* $\mathcal{B}=\{\mathcal{B}_n\colon n\in\omega\}$ *has a coarser connected topology.*

Proof. This follows from Theorems 4.3, 4.11, and 1.1 if we can find a function $g : h \circ f[EX] \to X$ such that $g \circ h \circ f = k$. If $x, y \in EX$ and $k(x) \neq k(y)$, it suffices to show that $h \circ f(x) \neq h \circ f(y)$. If $k(x) \neq k(y)$, there is some $n \in \omega$ and $B \in \mathcal{B}_n$ such that $k(x) \in B$ and $k(y) \in X \setminus \mathrm{cl}_X B$. Thus, $x \in OB \cap EX$ and $y \notin OB \cap EX$. Thus, $\rho_n(x, y) > 0$ as $f_{OB \cap EX}(x) = 1$ and $f_{OB \cap EX}(y) = 0$. Hence, $\rho(f(x), f(y)) > 0$. This shows that $h(f(x)) \neq h(f(y))$. □

Question 4.13. Improve Theorem 4.12 to: A non-H-closed space with a σ-locally finite point separating family $\mathcal{B} = \{\mathcal{B}_n\colon n \in \omega\}$ has a coarser connected topology.

Question 4.14. If X is non-H-closed space with a σ-locally finite base, does there exist a non-compact metric space M and a perfect, irreducible, θ-continuous surjection $f : M \to X$?

5. Coarser connected Urysohn topologies on ordinals

We begin with some general information about ordinals, ordinal notation, and ordinal arithmetic. In this section, we must be especially careful about notation. Here is a potentially confusing pair: $\kappa^{\aleph_0}$ is cardinal exponentiation, $\kappa^{\aleph_0} = |[\kappa]^{\aleph_0}|$; while κ^ω is ordinal exponentiation, $\kappa^\omega = \sup\{\kappa^n\colon n \in \omega\}$. Here is another potentially confusing pair: $\beta \cdot \omega$ is ordinal multiplication, while $\beta \times \omega$ is the product of topological spaces. If β is indecomposible, then $\beta \cdot \omega \cong (\beta + 1) \times \omega \ncong \beta \times \omega$.

Definition 5.1. An ordinal β is called *indecomposible* if $\delta + \beta = \beta$ for all $\delta < \beta$. An ordinal β is indecomposible iff there is ξ such that $\beta = \omega^\xi$ (ordinal exponentiation!) [8, p. 43(5)]. For an ordinal $\alpha > 0$, set $\beta_\alpha = \min\{\beta > 0\colon \exists\delta,\ \alpha = \delta + \beta\}$; set $\kappa(\alpha) = |\beta_\alpha|$. Observe that β_α is indecomposible.

We also recall the Cantor normal form theorem (see [8, p. 43(6)]): Every nonzero ordinal α may be represented

$$\alpha = \omega^{\xi_1} \cdot m_1 + \cdots + \omega^{\xi_n} \cdot m_n$$

where $1 \leqslant n < \omega$, $\alpha \geqslant \xi_1 > \cdots > \xi_n$, and $1 \leqslant m_i < \omega$ for $i = 1, \ldots, n$. Now we present a topological normal form.

Lemma 5.2. *Every nonzero ordinal α is homeomorphic to an ordinal of the form $\omega^\eta \cdot m$ or of the form $\omega^\xi \cdot m + \omega^\eta$, where $\xi > \eta$ and $m \in \omega$.*

Proof. Suppose that the Cantor normal form of α is $\omega^{\xi_1} \cdot m_1 + \cdots + \omega^{\xi_n} \cdot m_n$. Set $\zeta = \omega^{\xi_2} \cdot m_2 + \cdots + \omega^{\xi_n} \cdot (m_n - 1) + 1$. Then

$$\begin{aligned}\alpha &\cong \big(\omega^{\xi_1} \cdot m_1 + 1\big) \oplus \zeta \oplus \omega^{\xi_n} \cong \zeta \oplus \big(\omega^{\xi_1} \cdot m_1 + 1\big) \oplus \omega^{\xi_n} \\ &\cong \big(\omega^{\xi_1} \cdot m_1 + 1\big) \oplus \omega^{\xi_n} \cong \omega^{\xi_1} \cdot m_1 + \omega^{\xi_n}.\end{aligned}$$

If α is a successor ordinal (i.e., if $\xi_n = 0$), the details are slightly different. □

The basic concepts established, we can now introduce the results of this section. In [6] we asked the question, which ordinal spaces have a coarser connected Hausdorff topology? The answer is easy to state and depends only on cardinal arithmetic. An ordinal α has a coarser connected Hausdorff topology iff α is a limit ordinal and $|\alpha| \leqslant 2^{|\beta_\alpha|}$.

Here we ask, which ordinals have coarser connected Urysohn topologies? Note that it suffices to consider only ordinals in the form of Lemma 5.2. We have some answers analogous to the Hausdorff case. If an ordinal α has a coarser connected Urysohn topology, then α is a limit ordinal of countable cofinality (Corollary 5.4), and $|\alpha| \leqslant |\beta_\alpha|^{\aleph_0}$ (Theorem 5.8). For example, $\mathfrak{c}^+ + \omega_1 \cdot \omega$ has no coarser connected Urysohn topology. There are some easy to state sufficient conditions, too. A "monomial" ordinal α of the form $\omega^\eta \cdot m$ has a coarser connected Urysohn topology iff $\omega^\eta \cdot m$ has cofinality ω; equivalently if η is a successor or has cofinality ω (Corollary 5.14). Also, for an ordinal α of cofinality ω to have a coarser connected Urysohn topology, it suffices that α have cardinality at most $\mathfrak{c}$ (Theorem 5.18). For example, $\mathfrak{c} \cdot \mathfrak{c} + \omega_1 \cdot \omega$ has a coarser connected Urysohn topology.

To express our results when $|\beta_\alpha| = |\alpha|$, we must use ordinal arithmetic. For example, let κ be a cardinal and λ an ordinal, $\lambda \leqslant \kappa$. The ordinal $\alpha = \kappa \cdot \lambda + \kappa \cdot \omega$ has a coarser connected Urysohn topology if $\lambda < \mathfrak{c}^+$ (Theorem 5.21), or if κ is singular with cofinality ω (Theorem 5.17). However, this α has no coarser connected Urysohn topology when $\lambda = \kappa = (2^{\mathfrak{c}})^+$ (Theorem 5.9). The ordinal $\kappa^\omega \cdot \kappa^\omega + \kappa^\omega \cdot \omega$ has a coarser connected Urysohn topology for all cardinals κ (Theorem 5.17).

First, the necessity results.

Recall that a set is called *relatively compact* iff it is a subset of a compact set.

Lemma 5.3. *Let τ be a 0-dimensional topology on X and σ a coarser connected Hausdorff topology. Then no nonempty $u \in \sigma$ is relatively compact. Hence if an ordinal α has a coarser connected Urysohn topology σ, then every nonempty set in σ is unbounded in α.*

Proof. Let $u \in \sigma$. If u is relatively compact then, since compact topologies are minimal Hausdorff, u is open in some compact 0-dimensional subspace K. So there is some τ-clopen set $C \subset u$ which is σ-open (because $u \in \sigma$) and σ-closed (because K is compact), a contradiction. □

R. Wilson showed that ω_1 cannot be condensed onto a dense-in-itself (and hence onto a connected) Urysohn space [17, Example 2.12].

Corollary 5.4. *An ordinal with a coarser connected Urysohn topology has countable cofinality.*

Proof. Let u be a nonempty set in σ a coarser connected Urysohn topology on some ordinal α. Then u is unbounded, so $\mathrm{cl}_\sigma u$ is closed unbounded. Hence if $\mathrm{cf}\,\alpha > \omega$, then $\mathrm{cl}_\sigma u \cap \mathrm{cl}_\sigma v \neq \emptyset$ for any nonempty $u, v \in \sigma$. □

Every continuous function from an ordinal to the reals has countable range. Hence a continuous function from a coarser topology also has countable range. Therefore, no ordinal has a coarser connected Tychonoff topology. We use the preceding corollary to weaken the hypothesis Tychonoff to regular.

Proposition 5.5. *No ordinal has a coarser connected regular Hausdorff topology.*

Proof. This follows from Corollary 5.4 and [13, 2.11] (citing dimension theory). Here is a direct proof: by Corollary 5.4 an ordinal with a coarser connected regular Hausdorff topology must have cofinality ω, and hence is the union of countably many compact subsets. Thus it is Lindelöf and hence Tychonoff. □

The next string of necessity results starts with a combinatorial condition and then gives cardinality results.

Definition 5.6.

(a) A sequence of sets $\{C_\eta\colon \eta < \nu\}$ is *right-independent* iff for all finite $H, G \subset \nu$ with $\sup H < \inf G$ $\bigcap_{\eta\in G} C_\eta \setminus \bigcup_{\eta\in H} C_\eta$ is infinite.
(b) If $\{C_\eta\colon \eta < \nu\}$ is right-independent we define $\mathcal{C} = \{\bigcap_{\eta\in G} C_\eta \setminus (c \cup \bigcup_{\eta\in H} C_\gamma)\colon \sup H < \inf G < \nu$ and H, G, c are finite$\}$.
(c) For $C = \bigcap_{\eta\in G} C_\eta \setminus (c \cup \bigcup_{\eta\in H} C_\gamma) \in \mathcal{C}$ we define $\mathrm{m}(C) = \sup H$, $\mathrm{M}(C) = \inf G$.

Theorem 5.7. *Let $\alpha = \delta + \beta$. If α has a coarser connected Urysohn topology σ, then β has a right-independent family of subsets $\{C_\eta\colon \eta < \delta\}$.*

Proof. It suffices to find such a family on the interval $(\delta, \delta + \beta)$. Let $\alpha = \delta + \beta$. Let σ be a coarser connected Urysohn topology on α. Since any two disjoint compact subsets of a Urysohn space can be separated by open sets with disjoint closures, for each $\eta \in [0, \delta)$ there are $u_\eta, v_\eta \in \sigma$ separating $[0, \eta]$ from $(\eta, \delta]$ with $\mathrm{cl}_\sigma u_\eta \cap \mathrm{cl}_\sigma v_\eta = \emptyset$. Define $C_\eta = (\delta, \delta + \beta) \cap u_\eta$. Note that $C_\eta \neq \emptyset$ by Lemma 5.3.

Suppose that $\sup H = \mu < \inf G = \gamma < \delta$, where H, G are finite. Then $(\mu, \delta] \cap \mathrm{cl}_\sigma \bigcup_{\eta\in H} u_\eta = \emptyset$ and $[0, \gamma] \subset \bigcap_{\eta\in G} u_\eta$, so $(\mu, \gamma] \subset \bigcap_{\eta\in G} u_\eta \setminus \mathrm{cl}_\sigma \bigcup_{\eta\in H} u_\eta$ which is open, hence has infinite intersection with $(\delta, \delta + \beta)$. □

Letting δ vary, we see that for all $\delta < \alpha$, there is a right-independent family on β_α indexed by δ.

Theorem 5.8. *Let δ and β be nonzero ordinals satisfying $|\delta| > |\beta|^{\aleph_0}$. If α' is an ordinal with a coarser connected Urysohn topology, σ, then no $U \in \sigma$ has order type $\delta + \beta$. In particular, $\alpha = \delta + \beta$ has no coarser connected Urysohn topology.*

Proof. Suppose, by way of contradiction, α' has a coarser connected Urysohn topology, σ, and that $U \in \sigma$ has order type $\delta + \beta$. Let $f : \delta + \beta \to U$ be the order preserving bijection. (N.b. f is not necessarily continuous). For each $\eta < \delta$, $[f(0), f(\eta)]$ and $[f(\eta + 1), f(\delta)]$ are disjoint compact sets in a Urysohn space. So there are open sets u'_η, v'_η with disjoint closures satisfying $[f(0), f(\eta)] \subset u'_\eta$ and $[f(\eta+1), f(\delta)] \subset v'_\eta$. Set $u_\eta = U \cap u'_\eta$ and $v_\eta = U \cap v'_\eta$.

Fix $\eta < \delta$. We define an increasing sequence $\rho^\eta = \{\rho^\eta_n\colon\ n \in \omega\}$, which alternates between v_η and u_η. Start with $\rho^\eta_0 = f(\delta) \in v_\eta$, then let ρ^η_1 be the least element of u_η greater than ρ^η_0, let ρ^η_2 be the least element of v_η greater than ρ^η_1, etc. Formally, set $\rho^\eta_0 = f(\delta)$; set $\rho^\eta_{2k+1} = \min(u_\eta \backslash (\rho^\eta_{2k}+1))$; and set $\rho^\eta_{2k+2} = \min(v_\eta \backslash (\rho^\eta_{2k+1}+1))$. This definition is valid because u_η and v_η are unbounded in α' (Lemma 5.3). Moreover, ρ^η is unbounded in α'; if $\sup\{\rho^\eta_n\colon\ n \in \omega\} = \xi < \alpha'$, then $\xi \in \operatorname{cl} u_\eta \cap \operatorname{cl} v_\eta = \emptyset$.

Because $|\delta| > |\beta|^{\aleph_0}$, there are $\eta < \eta' < \delta$ with $\rho^\eta = \rho^{\eta'}$. Observe that $u_{\eta'} \cap (\bigcup_{n\in\omega}[\rho^\eta_{2k}, \rho^\eta_{2k+1})) = \emptyset$ and that $v_\eta \cap (\bigcup_{n\in\omega}[\rho^\eta_{2k+1}, \rho^\eta_{2k+2})) = \emptyset$. Hence $u_{\eta'} \cap v_\eta \cap [f(\delta), \alpha') = \emptyset$.

On the other hand, $f(\eta') \in u_{\eta'} \cap v_\eta$, a nonempty element of σ, so by Lemma 5.3, $u_{\eta'} \cap v_\eta \cap [f(\delta), \alpha') \neq \emptyset$. Contradiction. □

Now we use the ideas of the previous two theorems to show that α can fail to have a coarser, connected Urysohn topology even when $|\beta_\alpha| = |\alpha|$ (and $\operatorname{cf}\alpha = \omega$, of course).

Theorem 5.9. *Suppose κ is a cardinal and γ and δ are ordinals which satisfy*

(1) $\operatorname{cf}\kappa > \omega$
(2) *if* $\lambda < \kappa$, *then* $|\lambda|^{\aleph_0} < \kappa$
(3) $|\delta| > 2^{2^{|\gamma|}}$.

If α' is an ordinal with a coarser connected Urysohn topology, σ, then no $U \in \sigma$ has order type $\kappa\cdot\delta + \kappa\cdot\gamma$. In particular, $\alpha = \kappa\cdot\delta+\kappa\cdot\gamma$ fails to have a coarser connected Urysohn topology.

Proof. Suppose, by way of contradiction, α' has a coarser connected Urysohn topology, σ, and that $U \in \sigma$ has order type $\kappa\cdot\delta+\kappa\cdot\gamma$. Let $f : \kappa\cdot\delta+\kappa\cdot\gamma \to U$ be the order preserving bijection. (N.b. f is not necessarily continuous).

Let Y be the final segment $[f(\kappa\cdot\delta), \alpha')$. Since σ is connected, Y is dense in the space (α', σ) (Lemma 5.3). For each $\xi < \delta$, let $\mu_\xi = \sup\{f(\iota)\colon\ \iota < \kappa\cdot(\xi+1)\}$. For each $\beta < \gamma$, let $\zeta_\beta = \sup\{f(\iota)\colon\ \iota < \kappa\cdot(\delta+\beta)\}$, and set $Z = \{\zeta_\beta\colon\ \beta < \gamma\}$.

Claim. *We cannot separate any point μ_ξ from the closed set Z. In symbols, if $\mu_\xi \in W \in \sigma$, then $\operatorname{cl}_\sigma W \cap Z \neq \emptyset$.*

Suppose that μ_ξ and W refute the claim. Choose $\nu < \kappa$ so that $f[(\kappa\cdot\xi+\nu, \kappa\cdot(\xi+1))] \subset W$. We may assume that W is disjoint from the compact set $[0, f(\kappa\cdot\xi+\nu)] \cup [\mu_\xi+1, f(\kappa\cdot\delta)]$. Assuming that $\mu_\xi \notin U$, the order type of $U \cap W\backslash Y$ is the order type of $(\kappa\cdot\xi+\nu, \kappa\cdot(\xi+1))$ which is κ. Assuming that $\mu_\xi \in U$, the order type of $U\cap W\backslash Y$ is the order type of $(\kappa\cdot\xi+\nu, \kappa\cdot(\xi+1)]$ which is $\kappa+1$.

Now we show that the order type of $U \cap W \cap Y$, call it λ, is less than κ. Because $\mathrm{cl}_\sigma W \cap Z = \emptyset$, there are $\{\beta_i\colon i \in \omega\}$, a sequence cofinal in γ, and $\{\lambda_i\colon i \in \omega\}$, a sequence of ordinals less than κ, satisfying

$$W \cap Y \subset \bigcup_{i \in \omega} \bigl(\zeta_{\beta_i}, f\bigl(\kappa \cdot (\delta + \beta_i) + \lambda_i\bigr)\bigr).$$

Hence $\lambda \leqslant \sum_{i<\omega} \lambda_i < \kappa$, by (1); and then $\lambda^{\aleph_0} < \kappa$, by (2). So the open set, $U \cap W$, with order type $\kappa + \lambda$, contradicts Theorem 5.8 and establishes the claim.

For each $\xi \in \delta$, define $\mathcal{U}_\xi = \{\mathrm{cl}_\sigma U \cap Z\colon \mu_\xi \in U \in \sigma\}$. By the claim $\mathcal{U}_\xi$ is a filterbase. Because $|\delta| > 2^{2^{|\gamma|}}$, by (3), there are $\xi < \xi'$ in M with $\mathcal{U}_\xi = \mathcal{U}_{\xi'}$. Let $V, V' \in \sigma$ separate $\mu_\xi, \mu_{\xi'}$. Then $(\mathrm{cl}_\sigma V \cap Z) \cap (\mathrm{cl}_\sigma V' \cap Z)$ is not empty. We have shown that σ is not Urysohn. Contradiction. □

Theorem 5.10. *Let κ be a cardinal of uncountable cofinality such that $2^{|\lambda|} < \kappa$ for all $\lambda < \kappa$. Then for all $n \in \omega$ and for all $\lambda < \kappa$, if α' is an ordinal with a coarser connected Urysohn topology σ, there is no open set $U \in \sigma$ having order type $\kappa^{n+1} + \kappa^n \cdot \lambda$. In particular, $\alpha = \kappa^{n+1} + \kappa^n \cdot \lambda$ fails to have a coarser connected Urysohn topology.*

Proof. By induction on n. Theorem 5.8 is (stronger than) the base step, $n = 0$. The general induction step follows closely the proof of Theorem 5.9, which is the induction step 0 to 1. □

This ends the necessity results. Now the sufficiency results.

Lemma 5.11. *Let β be an indecomposible ordinal of countable cofinality. Then $\beta \cong \bigoplus_{i<\omega} \delta_i$ whenever $\{\delta_i\colon i \in \omega\}$ is a nondecreasing sequence of successor ordinals cofinal in β. Moreover, $\beta \cong \beta \times \omega$.*

Proof. Let $\{\delta_i\colon i \in \omega\}$ be a nondecreasing sequence of successor ordinals cofinal in β. Set $Z_0 = \delta_0$, and for $0 < n < \omega$, set $Z_n = \sum_{i \leqslant n} \delta_i \setminus \sum_{i<n} \delta_i$. Then $Z_n \cong \delta_n$ and $\beta \cong \bigoplus_{i<\omega} Z_i$.

Let $\{P_i\colon i \in \omega\}$ partition ω into infinite pieces. Set $X_i = \bigoplus_{n \in P_i} Z_n$. Then $X_i \cong \beta$, and $\bigoplus_{i<\omega} X_i \cong \bigoplus_{i<\omega} Z_i \cong \beta$. □

Definition 5.12. Let X be a space and Y a space with a proper extension, $Z = Y \cup \{p\}$. For A a subspace of the product $X \times Y$, let $A_x = \{y \in Y\colon (x, y) \in A\}$ and let X' be the set of $x \in X$ such that A_x is connected and $p \in \mathrm{cl}_Z A_x$. We say that A is *vertically connected* if $A \cap (X' \times Y)$ is dense in A. The simplest instance is where Y is connected and $A = X \times Y$. Another instance is when for all $x \in X$, A_x is a dense connected subset of Y.

Theorem 5.13. *Let X be a Urysohn space and Y a connected space with a proper Urysohn extension $Z = Y \cup \{p\}$. If A is a vertically connected subspace of $X \times Y$, then A has a coarser connected Urysohn topology.*

Proof. Let τ be the product topology on $X \times Y$. Choose $(q, r) \in A$. We define a coarser topology on $X \times Y$. Let σ be the set of $T \in \tau$ such that if $(q, r) \in T$, then there is V, open in Z and containing p, such that $X \times V \subset T$.

We show that $(X \times Y, \sigma)$ is Urysohn by cases. First, we separate (q, r) and (x, y) when $r \neq y$. In Z, there are open sets V_p, V_r, and V_y with disjoint closures containing p, r, and y, respectively. Then $X \times (V_p \cup V_r)$ and $X \times V_y$ are elements of σ with disjoint closures which separate (q, r) and (x, y).

Second, we separate (q, r) and (x, y) when $q \neq x$. In X, there are open sets U_q and U_x with disjoint closures containing q and x, respectively. In Z, there are open sets V_p and V_y with disjoint closures containing p and y, respectively. Then $(U_q \times Y) \cup (X \times V_p)$ and $(U_x \times Y) \cap (X \times V_y)$ are elements of σ with disjoint closures which separate (q, r) and (x, y).

The other cases are similar. Now the subspace topology, $\sigma \restriction A$, is a coarser Urysohn topology on A.

For each $x \in X'$, A_x is connected and $(q, r) \in \mathrm{cl}_{\sigma \restriction A} A_x$. Then $A \cap (X' \times Y)$ being the union of connected subspaces with a point in common, is connected. So A is connected because it has a dense connected subset. $\square$

Corollary 5.14. *If a Urysohn space X is homeomorphic to $X \times \omega$, then X has a coarser connected Urysohn topology. If $\alpha = \beta \cdot m$, where β is an indecomposible ordinal of countable cofinality and $m \in \omega$, then α has a coarser connected Urysohn topology.*

Proof. The Roy space R is a coarser connected Urysohn topology on ω, and it has a proper Urysohn extension. Hence X maps on-to-one continuously onto $X \times R$, which is vertically connected.

Similarly, the space $\beta \cdot m$ maps on-to-one continuously onto to the vertically connected subspace

$$A = (\beta \times R) \cup \big(\{\beta\} \times S\big)$$

of the product $(\beta + 1) \times R$, where S is an $m - 1$ element subspace of R. $\square$

Corollary 5.15. *If X is a Urysohn space and $\{X_n\colon n \in \omega\}$ is a family of subspases of X such that $X_n \subset X_{n+1}$ for all $n \in \omega$, then $\bigoplus_{n\in\omega} X_n$ has a coarser connected Urysohn topology.*

Proof. Let $\{r_n\colon n \in \omega\}$ enumerate R_σ, the σ-product of countably many copies of the Roy space, see Definition 1.2. Then

$$A = \bigcup_{n\in\omega} \big(X_n \times \{r_n\}\big)$$

is a vertically connected subset of $X \times R_\sigma$, and hence, by Lemma 5.13, has a coarser connected Urysohn topology. Then $\bigoplus_{n\in\omega} X_n$ has a coarser connected Urysohn topology because it maps one-to-one continuously onto A. $\square$

We remark that Theorem 5.13 is valid with "Hausdorff" or "regular" replacing "Urysohn". Corollary 5.15 is valid with "Hausdorff" replacing "Urysohn".

The following lemma is a variation on the Rado–Milner paradox.

Lemma 5.16. *Let δ be an ordinal of cardinality κ The space δ has the form $\bigcup_{n\in\omega} X_n$, where $\{X_n\colon n\in\omega\}$ is an increasing sequence of regular closed sets and each X_n has order type less than κ^ω. If κ is singular of cofinality ω, we can require that each X_n have order type less than κ.*

Proof. The general case is proved by induction on α [8, p. 45(20)]. The singular case can be done directly. (Let X_n be the closure of a small set of isolated points.) □

Theorem 5.17. *Let α be an ordinal of cofinality ω. If $\kappa = |\alpha|$ and $\kappa^\omega \leqslant \beta_\alpha$, then α has a coarser connected Urysohn topology. If $\kappa = |\alpha|$ is singular of cofinality ω and $\kappa \leqslant \beta_\alpha$, then α has a coarser connected Urysohn topology.*

Proof. If α is indecomposible, then we are done by Corollary 5.14. So let $\alpha = \delta + \beta$, where δ is a successor ordinal greater than $\beta = \beta_\alpha$. Note that $\alpha \cong \delta \oplus \beta$. Let $\delta = \bigcup_{n\in\omega} X_n$, where $\{X_n\colon n\in\omega\}$ is as in Lemma 5.16. Let ξ_n be the order type of X_n. We may assume that $\{\xi_n\colon n\in\omega\}$ is cofinal in β. (Let $\{\beta_n\colon n\in\omega\}$ be cofinal in β, and make β_n a subset of X_n.) Observe that $X_0 \cong [0,\xi_0)$, $X_1 \cong [\xi_0, \xi_0+\xi_1)$, etc. Hence $\beta \cong \bigoplus_{n\in\omega} X_n$.

Let R be the Roy space and enumerate $R\setminus\{\infty\}$ as $\{r_n\colon n\in\omega\}$. Then

$$A = \bigl(\delta\times\{\infty\}\bigr) \cup \bigcup_{n\in\omega}\bigl(X_n\times\{r_n\}\bigr)$$

is a vertically connected subset of $\delta\times R$, and hence, by Lemma 5.13, has a coarser connected Urysohn topology. Because A is a one-to-one continuous image of α, α has a coarser connected Urysohn topology, too. □

We start another series of sufficiency theorems.

Theorem 5.18. *If* $\operatorname{cf}\alpha = \omega$ *and* $|\alpha| \leqslant \mathfrak{c}$ *then α has a coarser connected Urysohn topology.*

Proof. Let D be an ω-sequence cofinal in α. Then D is closed discrete. Let τ be a connected Urysohn topology on D with a countable closed discrete set A covered by an open discrete family $\{v_a\colon a\in A\}$ where each $a\in v_a$, e.g., the countable fan on the Roy space.

Let $\mathcal{A}$ be an independent family on A, enumerated as $\{A_\gamma\colon \gamma<\alpha\}$, and let $\mathcal{C} = \{\bigcap_{\gamma\in G} A_\gamma \setminus (c \cup \bigcup_{\eta\in H} A_\eta)\colon\ G, H, c \text{ finite},\ \sup H < \inf G\}$. For $C = \bigcap_{\gamma\in G} A_\gamma\setminus(c\cup \bigcup_{\eta\in H} A_\eta) \in \mathcal{C}$ we write $\mathrm{m}(C) = \sup H$, $\mathrm{M}(C) = \inf G$. We define the topology σ on α as follows:

(1) $\tau \subset \sigma$;
(2) If $\gamma\in u\in\sigma$ and $\gamma\notin D$ then $\exists\eta<\gamma\ \exists C\in\mathcal{C}$ with $(\eta,\gamma]\subset u$, $\mathrm{m}(C)\leqslant\eta$, $\mathrm{M}(C)\geqslant\gamma$ and $\bigcup_{a\in C} u_a\subset u$.

σ is connected because it has a dense connected subspace. We show that σ is Urysohn.

Distinct points in D are easily seen to be separated by open sets whose closures are disjoint.

Suppose $\xi < \gamma$, $\xi, \gamma \notin D$. Then $[0, \xi] \cup \{v_a\colon a \in A_\xi\}$ and $(\xi, \gamma] \cup \{v_a\colon a \in A_\gamma \setminus A_\xi\}$ are open sets separating ξ, γ whose closures are disjoint.

Suppose $\xi \in A \in \mathcal{A}$ and $\gamma \notin D$. Let δ so $\xi \notin (\delta, \gamma]$ and let $C = A_\gamma \setminus \{\xi\}$. Then v_ξ, $(\delta, \gamma] \cup \{v_a\colon a \in C\}$ are open sets separating ξ, γ whose closures are disjoint.

Suppose $\xi \in D \setminus A$ and $\gamma \notin D$. Let $v \in \tau$ with $\mathrm{cl}_\tau\, v \cap v_a = \emptyset$ for all but at most one $a \in A$ (we will call this point, if it exists, a^*). Let $\delta < \gamma$ so $\xi, a^* \notin (\delta, \gamma]$. Then $(\delta, \gamma] \cup \{v_a\colon a \neq a^*\}$, v are open sets separating ξ, γ whose closures are disjoint. □

Here is a proof of Theorem 5.18 in the style of [6]. Let $Z = \{\zeta_n\colon n \in \omega\}$ be a set of isolated points cofinal in α. Then $\alpha \cong (\alpha \setminus Z) \oplus Z$. Choose a one-to-one function ψ from α to the product space $2^{\mathfrak{c}}$ so that $\psi \restriction (\alpha \setminus Z)$ is an embedding and $\psi[Z]$ is dense in $2^{\mathfrak{c}}$. Observe that ψ is continuous and that $\psi[\alpha]$ is separable, Tychonoff, and has a countable closed discrete subset. Theorem 2.3 gives a coarser connected Urysohn topology σ on $\psi[\alpha]$. Then $\{\psi^{-1}[U]\colon U \in \sigma\}$ is the desired connected Urysohn topology on α.

Definition 5.19. Let X and Y be spaces and let S be an open subset of X. We define $W = W(X, Y, S)$ to be the space with point set $(S \times Y) \cup (X \setminus S)$ and two types of basic open sets: rectangles $U \times V$ where U is open in S and V is open in Y; and $U^{\uparrow} = ((S \cap U) \times Y) \cup (U \setminus S)$, where U is open in X.

Here are a few easy observations about $W(X, Y, S)$.

Lemma 5.20. *If X and Y have the separation property Hausdorff, Urysohn, regular, or Tychonoff, then so does $W(X, Y, S)$. If X is connected, so is $W(X, Y, S)$. If X' has the same point set as X with a coarser topology, then $W(X', Y, S)$ has the same point set as $W(X, Y, S)$ with a coarser topology.*

Theorem 5.21. *If $\alpha = \beta \cdot \gamma$ and γ has a coarser connected Urysohn topology, then α does, too. In particular, if $\mathrm{cf}\, \gamma = \omega$ and $\gamma < \mathfrak{c}^+$, then α has a coarser connected Urysohn topology.*

Proof. Let S be the set of successor ordinals of γ; that is $S = \{\xi + 1\colon \xi \in \gamma\}$. Set $Y = (\beta + 1) \setminus \{0\}$. We define a homeomorphism from $W(\gamma, Y, S)$ onto α. For $(\xi + 1, \zeta) \in S \times Y$ set $h(\xi + 1, \zeta) = \beta \cdot \xi + \zeta$. For $\xi \in \gamma \setminus S$, set $h(\xi) = \beta \cdot \xi$. Let X be the space (γ, σ), where σ is the hypothesized coarser connected topology on γ. The space $W(X, Y, S)$ is connected and Urysohn by Lemma 5.20. Then $\{h^{-1}[O]\colon O \text{ open in } W(X, Y, S)\}$ is the desired coarser connected Urysohn topology on α. □

Question 5.22. Which ordinals have coarser connected Urysohn topologies? The least ordinal to which the results of this section do not apply is $\mathfrak{c}^+ \cdot \mathfrak{c}^+ + \mathfrak{c}^+ \cdot \omega$. Another interesting open case is $\kappa^+ + \kappa$, where κ is a singular cardinal of cofinality ω.

6. Connections with connectifications

In [15] Watson and Wilson asked which spaces X have Hausdorff connectifications. We say that Y is a *connectification* of X if X is dense in Y and Y is connected. It is natural to ask whether this property is related to having a coarser Hausdorff topology. In this section, we present examples showing that there is no direct implication. Afterwards, we show that for every $p \in \omega^*$, the space $\omega^* \backslash \{p\}$ has a coarser connected Hausdorff topology.

It is easy to see that a space with an isolated point, or, more generally, a nontrivial H-closed open set, has no Hausdorff connectification. It is harder to find nice nowhere locally compact spaces with no Hausdorff connectification. Example 4.1 of [15] (our 6.1 below) is a regular, Lindelöf, nowhere locally compact space with no Hausdorff connectification. We show that it does have a coarser connected Hausdorff topology.

Example 6.1 [15, Example 4.1]. Let $\kappa = 2^{\mathfrak{c}}$ and $\lambda = \kappa^+$. Let S be the set of successor ordinals of λ; that is $S = \{\xi + 1\colon\ \xi \in \lambda\}$. For each $\alpha \in S$, let be Z_α be the irrationals; for $\alpha \in (\lambda + 1)\backslash S$, let Z_α be a singleton, $\{p_\alpha\}$. The point set of X is the free sum $\bigoplus_{\alpha \leqslant \lambda} Z_\alpha$. If $\alpha \in S$, then Z_α is open and homeomorphic to the irrationals in the natural way. If $\alpha \notin S$, then a neighborhood of p_α contains $\bigcup\{Z_\gamma\colon\ \beta < \gamma \leqslant \alpha\}$ for some $\beta < \alpha$. It is easy to verify that X is a regular, Lindelöf, nowhere locally compact space.

Suppose that Y is a Hausdorff connectification of X. We will find a nontrivial clopen set. For each $y \in Y \backslash X$, let U_y be an open set containing y whose closure misses the compact set $\bigcup\{S_\alpha\colon\ \alpha \in \lambda + 1 \backslash S\}$ (it is homeomorphic to $\lambda + 1$). Observe that $\{\alpha \in S\colon\ U_y \cap Z_\alpha \neq \emptyset\}$ is finite, and conclude that $F_y = \{\alpha \in S\colon\ y \in \mathrm{cl}_X\, Z_\alpha\}$ is finite. Because X is Hausdorff, $Y_\alpha = \{y \in Y \backslash X\colon\ y \in \mathrm{cl}_X\, Z_\alpha\}$ has cardinality at most κ, the number of open filters on Z_α.

Next a counting and closure argument gives a limit ordinal $\rho < \lambda$ such that if $\alpha < \rho$ and $y \in Y_\alpha$, then $F_y \subset \rho$. Then $\bigcup\{Z_\alpha\colon\ \alpha \leqslant \rho\} \cup \{y\colon\ F_y \subset \rho\}$ is a nontrivial clopen subset of Y. Hence X has no Hausdorff connectification.

Towards showing that X has a coarser connected Hausdorff topology, for each $\alpha \in S$ let Z'_α be a coarser connected Hausdorff topology on the irrationals with a proper Hausdorff extension, $Z'_\alpha \cup \{q_\alpha\}$. Designate a special point $p_\alpha \in Z'_\alpha$. We create the coarser topology on X in two steps. First, we repeat the construction of X using Z'_α in place of Z_α. Second, we require that open sets U satisfy, for all $\alpha < \lambda$, if $p_\alpha \in U$, then $\{q_{\alpha+1}\} \cup (U \cap Z'_{\alpha+1})$ is open in $Z'_\alpha \cup \{q_\alpha\}$.

We remark that the space X of the previous example can be expressed as $W(\lambda + 1$, irrationals, $S)$, using the notation of Definition 5.19. The method of Lemma 5.20 does not apply because $\lambda + 1$ has no (strictly) coarser Hausdorff topology. Further, we note that our methods give a coarser connected normal topology on X.

We denote the Stone–Čech remainder $\beta\omega\backslash\omega$ as ω^*. Recall that ω^* has a clopen base $\{A^*\colon\ A \in [\omega]^\omega\}$, where $A^* = \{p \in \omega^*\colon\ A \in p\}$. Note that $A^* \cap B^* = \emptyset$ iff A and B are almost disjoint (it means that $A \cap B$ is finite). A Tychonoff space X is called *extremally disconnected* if every pair of disjoint open subsets of X have disjoint closures. A Tychonoff space X is called an *F-space* if every pair of disjoint open F_σ subsets of X have disjoint closures. Note that $\beta\omega$ is extremally disconnected, and that ω^* is an F-space, but not extremally disconnected.

The following machinery was introduced in [6] specifically for the space ω^*, but it generalizes to F-spaces.

Definition 6.2. Let Y be a compact, zero-dimensional F-space without isolated points (for example, $\beta\omega\backslash\omega$). A sequence $s:\omega\to Y$ is *faithful* if s is one-to-one. A subset X of Y is called *pervasive* if for every pair of faithful sequences $\langle s_n\rangle$, $\langle t_m\rangle$ contained in Y, there is $p\in\beta\omega$ such that $p\lim s_n\in X$ and $p\lim t_n\in X$. Note that if X is pervasive in Y, then X is dense in Y.

The next lemma is 7.1 of [6].

Lemma 6.3. *Let $\langle s_n\rangle$ and $\langle t_n\rangle$ be disjoint pair of sequences in an F-space Y. There is $M\in[\omega]^\omega$ such that*

$$\mathrm{cl}_Y\{s_n\colon n\in M\}\cap\mathrm{cl}_Y\{t_n\colon n\in M\}=\emptyset.$$

For the rest of this section, let X be pervasive in Y. Let τ denote the subspace topology on X and let σ be a Hausdorff topology on X coarser than τ.

For $x\in X$, define $K(x)=\bigcap\{\mathrm{cl}_Y U\colon x\in U\in\sigma\}$.

Lemma 6.4. *With the notation and assumptions established above we have*

(1) $\{x\}=K(x)\cap X$;
(2) *For $x\in X$, $K(x)\backslash X$ is finite*;
(3) $\bigcup\{K(x)\colon x\in X\}\backslash X$ *is finite*;
(4) *Additionally assume that (X,σ) is also connected, $A\subset Y$ is clopen, and $A\cap X\neq\emptyset\neq X\backslash A$. Then there is some $x\in X$ such that $K(x)\cap A\neq\emptyset\neq K(x)\backslash A$. Hence, $\{x\in X\colon K(x)\neq\{x\}\}$ is dense in (X,τ).*

Proof. (1) follows because (X,σ) is Hausdorff and $\mathrm{cl}_Y U\cap X=\mathrm{cl}_{\tau(X)}U\subset\mathrm{cl}_{\sigma(X)}U$.

Towards (2) assume that $K(x)\backslash X$ infinite. Then there is a faithful sequence $\langle s_n\rangle$ contained in $K(x)\backslash X$. By Lemma 6.3, we can assume that $x\notin\mathrm{cl}_Y\{s_n\colon n\in\omega\}$. Since $K(x)\cap X=\{x\}$, we see that $\mathrm{cl}_Y\{s_n\colon n\in\omega\}\cap X=\emptyset$. This is a contradiction as X is pervasive.

Towards (3) assume that $\bigcup\{K(x)\colon x\in X\}\backslash X$ is infinite. Since $K(x)\backslash X$ is finite for each $x\in X$, we can find a faithful pair $\langle s_n\rangle$ and $\langle t_n\rangle$ of sequences satisfying $\{s_n\colon n\in\omega\}\subset X$, $\{t_n\colon n\in\omega\}\subset\mathrm{cl}_Y X\backslash X$, and $t_n\in K(s_n)$ for each $n\in\omega$. By Lemma 6.3, we may assume that $\mathrm{cl}_Y\{s_n\colon n\in\omega\}\cap\mathrm{cl}_Y\{t_n\colon n\in\omega\}=\emptyset$. There is $p\in\omega^*$ such that $\bar{s}=p\lim s_n\in X$ and $\bar{t}=p\lim t_n\in X$. Since (X,σ) is Hausdorff, there are disjoint open sets U and V with $\bar{s}\in U$ and $\bar{t}\in V$. There is some $A\in p$ such that $\{s_n\colon n\in A\}\subset U$. Thus, $\{t_n\colon n\in A\}\subset\mathrm{cl}_Y U$. Then $\widehat{V}=Y\backslash\mathrm{cl}_Y U$ is open in Y and contains V. As $\bar{t}\in V$, there is some $B\in p$ such that $\{t_n\colon n\in B\}\subset\widehat{V}$. Thus, $\{t_n\colon n\in A\cap B\}\subset\mathrm{cl}_Y U\cap\widehat{V}=\emptyset$, a contradiction as $A\cap B\neq\emptyset$.

Towards (4), assume that for all $x\in X$, $K(x)\subset A$ or $K(x)\subset X\backslash A$. For each $x\in X\backslash A$, $A\cap K(x)=A\cap\bigcap\{\mathrm{cl}_Y U\colon x\in U\in\sigma\}=\emptyset$. So, there is some $U\in\sigma$ such that $x\in U$ and $U\subset X\backslash A$. It follows that $X\backslash A\in\sigma$. By symmetry, it follows that $X\cap A\in\sigma$. This contradicts that (X,σ) is connected. □

Lemma 6.5. *If Y is an extremally disconnected space without isolated points and X is pervasive in Y, then X has no coarser connected Hausdorff topology.*

Proof. Let σ be a coarser Hausdorff topology on X. If $x, y \in X$ are distinct, there are disjoint $U, V \in \sigma$ such that $x \in U$, $y \in V$. There are $\widehat{U}, \widehat{V}$ open in Y such that $\widehat{U} \cap X = U$ and $\widehat{V} \cap X = V$. As X is dense in Y, $\widehat{U} \cap \widehat{V} = \emptyset$. But Y is extremally disconnected. So, $\mathrm{cl}_Y \widehat{U} \cap \mathrm{cl}_Y \widehat{V} = \emptyset$. As $\mathrm{cl}_Y \widehat{U} = \mathrm{cl}_Y(\widehat{U} \cap X) = \mathrm{cl}_Y U$, $K(x) \cap K(y) = \emptyset$. By Lemma 6.4, it follows that X has no coarser connected Hausdorff topology. □

Example 6.6. A separable, nowhere locally compact, extremally disconnected Tychonoff space X with no coarser connected Hausdorff topology but with a connectification Y such that $Y \setminus X$ is countable.

Recall that the absolute $E\mathbb{I}$ of the unit interval $\mathbb{I}$ is separable, compact, crowded, extremally disconnected and has a countable clopen π-base $\mathcal{B}$. By 6F in [10], $E\mathbb{I}$ can be embedded in $\beta\omega \setminus \omega$ in such a way that $|B| = 2^{\mathfrak{c}}$ for each $B \in \mathcal{B}$. Choose a countable subset $C_B \subset B$ such that $C_B \cap C_D = \emptyset$ for $B, D \in \mathcal{B}$. The space $X = E\mathbb{I} \setminus \bigcup\{C_B\colon B \in \mathcal{B}\}$ is dense in $E\mathbb{I}$, has a countable clopen π-base, is nowhere locally compact since a countable set has been removed from each element of the π-base, and for each $B \in \mathcal{B}$ and $p \in C_B$, there is an free open ultrafilter $\mathcal{U}(p, B)$ on X converging to p in Y. Using $\{\{\mathcal{U}(p, B)\colon p \in C_B\}\colon B \in \mathcal{B}\}$ and a slight modification of 2.7(a) in [11], we conclude that X has a connectification with a countable remainder. As $E\mathbb{I} \setminus X$ is countable, X is pervasive in $E\mathbb{I}$. By Lemma 6.5, X has no coarser connected Hausdorff topology.

The first step of the proof of Theorem 1.1 in [7] is to show that a noncompact metrizable space X has a coarser nowhere locally compact topology. In this context, we observe that Example 6.6 is the first known example of a nowhere locally compact space with no coarser connected Hausdorff topology.

Observe that in Lemma 6.4 we were careful not to assert the tempting but false: if $U, V \in \sigma$ are disjoint, then $\mathrm{cl}_Y U$ and $\mathrm{cl}_Y V$ are disjoint. Example 6.8 refutes Lemma 7.2 of [6], where we were not so careful. In Lemma 6.5 above, we repaired our error by strengthening the hypothesis Y is an F-space to Y is extremally disconnected.

Lemma 6.4 suggests what a coarser connected Hausdorff topology on a pervasive subspace X of $\beta\omega$ must look like. Because of clause (3), $\beta\omega \setminus X$ may as well be finite, so we let $X = \omega^* \setminus \{p\}$. Because of clause (4), we choose a dense set of points $D = \{x_E\colon E \in [\omega]^\omega\}$. For $x \in D$, we will have $K(x) = \{x, p\}$; for $x \in X \setminus D$, we will have $K(x) = \{x\}$. To actually construct such a space, we need the powerful Corollary 1.7 from [2].

Theorem 6.7. *Let $p \in \omega^*$ be arbitrary.*

(1) *There is an almost disjoint family $\{A_P\colon P \in p\} \subset [\omega]^\omega$ such that $A_P \subset P$ for all $p \in P$.*
(2) *There is a pairwise disjoint open family $\mathcal{U}$ of cardinality $\mathfrak{c}$ such that $p \in \mathrm{cl}\, U$ for all $U \in \mathcal{U}$.*

Example 6.8. For every $p \in \omega^*$, the space $\omega^* \setminus \{p\}$ has a coarser connected Hausdorff topology.

Let $p \in \omega^*$ be arbitrary. Let $\{A_P\colon\ P \in p\}$ be as in Theorem 6.7 (1). For each $P \in p$, choose a point q_P in A_P^* and apply Theorem 6.7(2) to get a disjoint open family $\{U_E^P\colon\ E \in [\omega]^\omega\}$ with $q_P \in \operatorname{cl} U_E^P$ for all $E \in [\omega]^\omega$. We may assume that $U_E^P \subset A_P^*$ for all P and E.

For all $E \in [\omega]^\omega$, choose $x_E \in E^* \setminus \{p\}$, and define $U_E = \bigcup\{U_E^P\colon\ P \in p\}$. Let τ be the topology on X as a subspace of ω^*. Let the new topology σ be the set of all $T \in \tau$ such that if $x_E \in T$, then for some $P \in p$ we have $U_E \cap P^* \subset T$.

It is straightforward to verify that (X, σ) is Hausdorff and nowhere Urysohn (it means that there is no pair of nonempty open sets with disjoint closures)—hence (X, σ) is connected.

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