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Generalised Einstein mass-variation formulae: I Subluminal relative frame velocities

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ABSTRACT

Much of the formalism in special relativity is intimately bound up with Einstein's formula for the variation of mass m with its velocity v, namely $m(v) = m_0^*[1-(v/c)^2]^{-1/2}$, where m is the mass, v the velocity, c denotes the speed of light and m_0^* denotes the rest mass, noting that in these papers, we employ an asterisk to designate the rest mass. Einstein's formula together with the Lorentz transformations and their consequences are fundamental to the development of special relativity. Here we introduce the notion of the residual mass $m_0(v)$ which for $v < c$ is defined by the equation $m(v) = m_0(v)[1-(v/c)^2]^{-1/2}$ for the actual mass $m(v)$; namely the residual mass is the actual mass with the Einstein factor removed. We emphasise that we make no restrictions on $m_0(v)$, and that this formal device merely facilitates the analysis. Using this formal device we deduce corresponding new mass variation formulae, assuming only the Lorentz transformations and two invariants known to apply in special relativity. One is force invariance in the direction of relative motion applying to two non-accelerating frames, while the other is not so well known, but applies in special relativity. Together the two assumed invariances imply that the energy–mass transfer rates are frame invariant but not necessarily constant as in special relativity. The new formulae involving two arbitrary constants may be exploited so that the mass remains finite at the speed of light, and an illustrative example is provided for which this is the case, and from which a new comparison formula is derived that is singular at the speed of light. This new expression may be contrasted with the Einstein expression, and roughly speaking, the new formula predicts more mass than that given by the Einstein formula, since the singularity at the speed of light is steeper.

Introduction

While Einstein's formula $m(v) = m_0^*[1-(v/c)^2]^{-1/2}$, for the variation of mass m with its velocity v, where m_0^* denotes the rest mass, has been overwhelmingly verified in our own local environment, it is clear that on a cosmological scale our understanding of matter and mass are not so successful, and issues such as dark energy and dark matter remain improperly understood. In our local environment the rest mass m_0^* is deemed to be the sole critical parameter, and yet the mysteries associated with dark energy and dark matter indicate that matter itself may adopt other forms or possess other defining characteristics, see for example [8]. Einstein's formula is a necessary condition for force invariance in two non-accelerating frames, but it is by no means sufficient, and in this paper we produce new special relativistic mass variation formulae, which we exploit to produce explicit new formulae exhibiting finite mass at the speed of light.

The underlying philosophy of the formulae presented here, is first the recognition of the importance of the Einstein theory of special relativity, and second to seek to develop this theory in a manner that embraces the essential features of the existing theory. Now given the veracity of the special theory of relativity, it may not be too unreasonable to expect that somewhere embodied within the theory are clues as to the notions which have been termed dark matter and dark energy. However, since the special theory deals only with non-accelerating frames, we certainly would not expect any such extension to tell the complete story, but we might expect some definite pointers as to how a more complete picture may be subsequently developed. The present papers deal with a possible formal extension of the special theory of relativity that produces

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explicit formulae that might be tested with any experimental data arising is the search for new elementary particles associated with dark matter and dark energy.

The formula $m(v) = m_0^*[1-(v/c)^2]^{-1/2}$ is fundamental to the development of special relativity, and this expression together with the Lorentz transformations and the law for the addition of velocities are fundamental to the derivation of numerous other results in special relativity, including Lorentz invariant mass-momentum relations and the so-called Lorentz invariant energy–momentum relations through the formula $\mathscr{E} = mc^2$. The questions arise as to how much of special relativity can be generalised without assuming $m(v) = m_0^*[1-(v/c)^2]^{-1/2}$, and are there other expressions for the variation of mass with velocity? In this paper, assuming only the Lorentz transformations and their consequences, we seek to develop the formalism of special relativity without making any assumptions on the variation of mass with velocity. To facilitate the analysis, we introduce the concept of the residual mass $m_0(v)$ as being defined by the equation $m(v) = m_0(v)[1-(v/c)^2]^{-1/2}$ for the actual mass $m(v)$, namely the residual mass is the actual mass with the Einstein factor removed (see Eq. (2.10)).

Based on two invariances which are known to apply in special relativity, we then deduce new expressions for the variation of mass with velocity, and for which $m(v) = m_0^*[1-(v/c)^2]^{-1/2}$ arises as a special case. The two assumed invariances, are force invariance in the direction of relative motion and another invariance involving mass which is not so well known, but nevertheless applies in special relativity. In this sense the new results have a corresponding same status to $m(v) = m_0^*[1-(v/c)^2]^{-1/2}$ but involve an additional arbitrary constant. As an example, the additional degree of freedom can be exploited to ensure that the actual mass remains finite at $v = c$, namely the arbitrary constants can be chosen to satisfy $m_0(c) = 0$. Finally, we comment that the formula $m(v) = m_0^*[1-(v/c)^2]^{-1/2}$ is one of many expressions showing a particular variation of mass with its velocity, and this expression has a long and extensive history involving many eminent scientists such as Abraham, Bücherer, Lorentz, Ehrenfest, Kaufmann and of course Einstein, who first grappled with the notion that the 'transverse and longitudinal' masses may be distinct. The story describing the development of the Einstein expression is fully detailed by Weinstein [10].

In the following section we present a brief summary of some of the basic equations of special relativity with particular reference to the derivation of the Lorentz invariant mass-momentum relations and force invariance. In the subsequent section we show how corresponding mass-momentum relations might be deduced without any assumptions on the variation of mass with velocity. Using these relations together with the force invariance and another invariance involving mass (see Eq. (2.9)) we deduce in the section thereafter the governing first order ordinary differential equation restricting the variation of mass with velocity (namely Eq. (3.7)). On solving this equation we eventually deduce new mass variation formulae, which include the Einstein expression as a special case. In the final section of the paper we make some brief concluding remarks. In this present part I, we deal exclusively with subluminal relative frame velocities $v < c$, and corresponding results are presented in part II for $v > c$.

Classical special relativity

We consider a rectangular Cartesian frame (X, Y, Z) and another frame (x, y, z) moving with constant velocity v relative to the first frame and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 1. We note that the coordinate notation adopted here is slightly different to that normally used in special relativity involving primed and unprimed variables. We do this purposely because it is convenient to view the relative velocity v as a parameter measuring the departure of the current frame (x, y, z) from the rest frame (X, Y, Z), and for this purpose the notation employed in nonlinear continuum mechanics is preferable. Time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t. Following normal practice, we assume that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest.

For $0 \leqslant v < c$, the standard Lorentz transformations are

$$X = \frac{x + vt}{[1-(v/c)^2]^{1/2}}, \quad T = \frac{t + vx/c^2}{[1-(v/c)^2]^{1/2}},$$

with the inverse transformation characterised by $-v$, thus

$$x = \frac{X - vT}{[1-(v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1-(v/c)^2]^{1/2}}, \tag{2.1}$$

and various derivations of these equations can be found in many standard textbooks such as Feynmann et al. [2] and Landau & Lifshitz [5], and other novel derivations are given by Lee & Kalotas [6] and Levy-Leblond [7]. The above equations reflect, of course, that the two coordinate frames coincide when the relative velocity v is zero, namely

$$x = X, \quad t = T, \quad v = 0.$$

With velocities $U = dX/dT$ and $u = dx/dt$, (2.1) yields the addition of velocity law

$$u = \frac{U - v}{(1 - Uv/c^2)}, \tag{2.2}$$

which of course is well known and due to Einstein. As an immediate consequence of (2.2) is the identity

$$[1-(u/c)^2](1-Uv/c^2)^2 = [1-(v/c)^2][1-(U/c)^2], \tag{2.3}$$

which is not so well-known, but is nevertheless fundamental to the development of the formulation of special relativity. Another not so well-known formula arising from (2.2) is

$$\left(\frac{1+U/c}{1-U/c}\right) = \left(\frac{1+u/c}{1-u/c}\right)\left(\frac{1+v/c}{1-v/c}\right), \tag{2.4}$$

and both (2.3) and (2.4) apply for both sub and super luminal motion. These two formulae reveal that at least one of the velocities u, v or U must not exceed the speed of light, and clearly both formulae need re-arrangement depending upon the particular values of the three velocities. In this paper, we have in mind subluminal frame velocities v, which means that either both u and U are subluminal (see Section 3) or both are superluminal (see Section 5). In these sections we need to take the square root of (2.3) and the

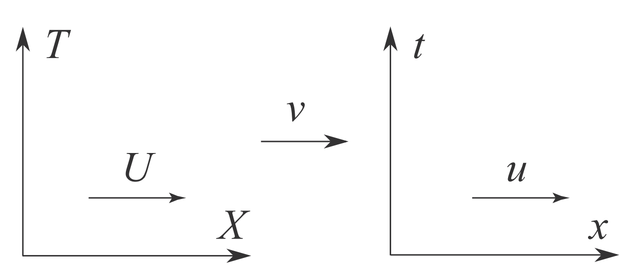

Fig. 1. Two inertial frames moving along x-axis with relative velocity v.

logarithm of (2.4) and in both cases the formulae need appropriate re-arrangement prior to making the operations.

By way of illustration, for $v, u, U < c$, and assuming the Einstein mass variation in both frames

$$m(u) = \frac{m_0}{\left[1-(u/c)^2\right]^{1/2}}, \quad M(U) = \frac{m_0}{\left[1-(U/c)^2\right]^{1/2}}, \tag{2.5}$$

and with momenta $P = MU$ and $p = mu$, we have on multiplication of (2.2) by $m_0^*\left[1-(u/c)^2\right]^{-1/2}$ and by using the appropriate square root identity from (2.3), we may readily deduce

$$p = \frac{P - Mv}{\left[1-(v/c)^2\right]^{1/2}}, \quad m = \frac{M - Pv/c^2}{\left[1-(v/c)^2\right]^{1/2}}, \tag{2.6}$$

where $(2.6)_2$ arises directly from $(2.5)_1$, and from (2.3). We comment that some authors [1] refer respectively to the above notions of mass m and momentum p as 'temporal' and 'spacial' momentum. We do not follow that distinction here and we refer to the relations (2.6) as the Lorentz invariant mass-momentum equations, but they are usually referred to as the Lorentz invariant energy–momentum relations in consideration of the formula $E = Mc^2$. From the above we have

$$p^2 - (mc)^2 = P^2 - (Mc)^2 = -m_0^2c^2,$$

and therefore taking the total derivative of this equation we have the differential relations

$$u\,dp = c^2\,dm, \quad U\,dP = c^2\,dM, \tag{2.7}$$

from which we may deduce $\mathscr{E} = mc^2$ or $E = Mc^2$ from the energy and momentum equations. Thus for example in the (x,t) frame we have

$$\frac{d\mathscr{E}}{dt} = fu, \quad f = \frac{dp}{dt}, \tag{2.8}$$

and therefore $d\mathscr{E} = udp = c^2dm$, from which we may deduce $\mathscr{E} = mc^2$.

Fundamental to special relativity is that the forces in the direction of relative motion as measured from the two frames coincide, that is $f = dp/dt$ and $F = dP/dT$ and $f = F$. This is an assumed consequence of the assumption that the two frames are moving with a constant relative velocity v. However, notice that formally the equation $f = F$ hinges on the assumption of the mass variation (2.5). Thus, for example, from $(2.6)_1$ and $(2.1)_1$ we have

$$f = \frac{dp}{dt} = \frac{dP - v\,dM}{dT - v\,dX/c^2} = \frac{dP(1 - Uv/c^2)}{dT(1 - Uv/c^2)} = \frac{dP}{dT} = F,$$

on using $(2.7)_2$. Alternatively,

$$f = \frac{d}{dt}\left(\frac{m_0^*u}{\left[1-(u/c)^2\right]^{1/2}}\right) = \frac{m_0^*\,du/dt}{\left[1-(u/c)^2\right]^{3/2}},$$

and on using $(2.1)_2$ and the velocity addition formula (2.2) we have

$$f = \frac{m_0^*\left[1-(v/c)^2\right]^{3/2}dU}{\left[1-(u/c)^2\right]^{3/2}\left[1 - vU/c^2\right]^3 dT} = \frac{m_0^*\,dU/dT}{\left[1-(U/c)^2\right]^{3/2}},$$

where the final step follows from (2.3) and again gives $f = F$. Thus although non-accelerating frames are fundamental to special relativity, the force equality $f = F$ in the direction of relative motion formally hinges on the Einstein mass variation (2.5).

Along with $dp/dt = dP/dT$, another special relativistic invariance which is not so well known is $dm/dx = dM/dX$ which arises from $(2.1)_1$ and $(2.6)_2$ as follows,

$$\frac{dm}{dx} = \frac{dM - dP\,v/c^2}{dX - v dT} = \frac{dM(1 - v/U)}{dX(1 - v/U)} = \frac{dM}{dX},$$

on using $U = dX/dT$ and $(2.7)_2$, namely $dP = c^2dM/U$. In the following development of special relativity we adopt the above two invariances, namely

$$\frac{dp}{dt} = \frac{dP}{dT}, \quad \frac{dm}{dx} = \frac{dM}{dX}, \tag{2.9}$$

and in this sense we claim that the resulting new mass variation formulae carry a corresponding status to the Einstein expression. The new formulae involve two arbitrary constants, while the Einstein expression involves only the rest mass as a single arbitrary constant. We comment that together the above two invariances imply that the energy–mass rate is the same in both frames, namely

$$\frac{d\mathscr{E}}{dm} = \frac{dE}{dM},$$

which evidently applies for conventional special relativity and both energy–mass rates are then equal to c^2.

In the following section we extend the above development assuming only the validity of the Lorentz transformations (2.1) and their consequences (2.2), (2.3) and (2.4), but not making any assumptions of the mass variation with velocity. We do however assume that $m(u)$ and $M(U)$ have the particular structure of the forms,

$$m(u) = \frac{m_0(u)}{\left[1-(u/c)^2\right]^{1/2}}, \quad M(U) = \frac{M_0(U)}{\left[1-(U/c)^2\right]^{1/2}}, \tag{2.10}$$

where $m_0(u)$ and $M_0(U)$ are referred to as the residual masses, and denote arbitrary functions to be determined subsequently. The particular assumed structure (2.10) is not restrictive in any sense and merely facilitates the mathematical analysis. Together with the formal identities (2.3) and (2.9) it is sufficient to duplicate the essential structure of the Lorentz invariant mass-momentum relations (2.6). In particular, we may introduce the variable parameter $\lambda(u,U) = m_0(u)/M_0(U)$ to extend the above formalism and we derive new mass variation formulae based on maintaining the two invariances (2.9), and noting that the Einstein mass variation arises from $\lambda \equiv 1$.

From (2.4) we may introduce new velocity variables defined by

$$\xi = \log\left(\frac{1+U/c}{1-U/c}\right), \quad \eta = \log\left(\frac{1+u/c}{1-u/c}\right), \quad \gamma = \log\left(\frac{1+v/c}{1-v/c}\right), \tag{2.11}$$

in which case (2.4) becomes simply the translation

$$\xi = \eta + \gamma, \tag{2.12}$$

and the velocities are given by

$$U = c\tanh(\xi/2), \quad u = c\tanh(\eta/2), \quad v = c\tanh(\gamma/2). \tag{2.13}$$

We comment that the above transformations hinge only on the velocity addition formula (2.2) and its consequences (2.3) and (2.4) applying irrespective of any assumed mass variation.

If we introduce the notation $m_0(u) = n_0(\eta)$ and $M_0(U) = N_0(\xi)$ then from $dE/dT = FU$ and $F = dP/dT$ we may by integration show that

$$E + \int_0^U WM(W)\,dW = M(U)U^2 + m_0^*c^2,$$

assuming that $m_0^*c^2$ corresponds to zero energy. On making the substitution $W = c\tanh(\zeta/2)$ in the integral and performing an integration by parts, we may, on simplification deduce the interesting formula

$$E = M(U)c^2 - c^2\int_0^{\xi} \operatorname{sech}(\zeta/2)\frac{dN_0(\zeta)}{d\zeta}d\zeta, \quad (2.14)$$

where ξ is defined above, and by a similar argument we may deduce from (2.8).

$$\mathscr{E} = m(u)c^2 - c^2\int_0^{\eta} \operatorname{sech}(\psi/2)\frac{dn_0(\psi)}{d\psi}d\psi. \quad (2.15)$$

In the above special relativistic preliminaries we have presented illustrative formulae, and in the following section, we obtain new mass variation formulae for which (ξ, η, γ) are assumed to be defined by (2.11) and for which the relations (2.12) and (2.13) apply.

Generalised Lorentz invariant mass-momentum relations

In this section, $m_0(u)$ and $M_0(U)$ are assumed to be defined by (2.10) and $\lambda = m_0(u)/M_0(U)$. Following the procedure described in the previous section, the generalised Lorentz invariant mass-momentum relations and their inverses can be shown to be given by

$$p = \frac{\lambda(P - Mv)}{\left[1-(v/c)^2\right]^{1/2}}, \quad m = \frac{\lambda(M - Pv/c^2)}{\left[1-(v/c)^2\right]^{1/2}}, \quad (3.1)$$

$$P = \frac{(p + mv)}{\lambda\left[1-(v/c)^2\right]^{1/2}}, \quad M = \frac{(m + pv/c^2)}{\lambda\left[1-(v/c)^2\right]^{1/2}}, \quad (3.2)$$

and from (3.1) and the assumed invariances (2.9) we may deduce

$$\begin{aligned}&\lambda(dP - v\,dM) + d\lambda(P - Mv) = dP(1 - Uv/c^2),\\ &\lambda(dM - v\,dP/c^2) + d\lambda(M - vP/c^2) = dM(1 - v/U).\end{aligned} \quad (3.3)$$

These two equations for $Q = dP/dM$ and $\mu = d\lambda/dM$ can be solved to eventually deduce

$$\begin{aligned}&M\frac{dU}{dM} = \frac{\left[1-(U/c)^2\right]\left[v+(\beta-1)c^2/U\right]}{\left(\beta\lambda-1+Uv/c^2\right)},\\ &M\frac{d\lambda}{dM} = -\frac{(\lambda-1)(\beta\lambda-1+v/U)}{\left(\beta\lambda-1+Uv/c^2\right)},\end{aligned} \quad (3.4)$$

where the quantity β is defined by

$$\beta = \frac{1-(v/c)^2}{1-Uv/c^2}, \quad (3.5)$$

and $\beta - 1 = uv/c^2$. In deriving (3.4), we have used $P = MU$ and $Q = MdU/dM + U$. On dividing $(3.4)_1$ by $(3.4)_2$ and making use of (3.5), the final equation for the determination of λ as a function of U becomes

$$(\lambda-1)\frac{dU}{d\lambda} = \frac{v^2\left[1-\left(\frac{U}{c}\right)^2\right]\left[\left(\frac{U}{c}\right)^2 - \frac{2U}{v} + 1\right]}{\left\{(\lambda-1)\left[1-\left(\frac{v}{c}\right)^2\right]U + v\left[\left(\frac{U}{c}\right)^2 - \frac{2Uv}{c^2} + 1\right]\right\}}. \quad (3.6)$$

The substitution $\sigma = 1/(\lambda - 1)$ reduces (3.6) to a standard first order ordinary differential equation, and the transformations (2.11) and (2.13) eventually yield

$$\frac{d\sigma}{d\xi} - \frac{\sigma}{2}\frac{\cosh(\xi-\gamma/2)}{\sinh(\xi-\gamma/2)} = \frac{\sinh\xi}{4\sinh(\gamma/2)\sinh(\xi-\gamma/2)}. \quad (3.7)$$

Multiplication by the integrating factor $(\sinh(\xi-\gamma/2))^{-1/2}$ and writing $\sinh\xi$ as $\sinh[(\xi-\gamma/2)+\gamma/2]$ and expanding, we may readily deduce

$$\sigma = -\frac{1}{2} + \frac{\left[\sinh\left(\xi-\frac{\gamma}{2}\right)\right]^{1/2}}{4\tanh(\gamma/2)}\int_0^{\xi-\frac{\gamma}{2}}\frac{d\rho}{(\sinh\rho)^{1/2}} + C_1\left[\sinh\left(\xi-\frac{\gamma}{2}\right)\right]^{1/2}, \quad (3.8)$$

where C_1 denotes the arbitrary constant of integration. From $\sigma = (\lambda-1)^{-1}$ we find

$$\lambda = m_0(u)/M_0(U) = \frac{\frac{1}{2} + \frac{[\sinh(\xi-\gamma/2)]^{1/2}}{4\tanh(\gamma/2)}\int_0^{\xi-\gamma/2}\frac{d\rho}{(\sinh\rho)^{1/2}} + C_1[\sinh(\xi-\gamma/2)]^{1/2}}{-\frac{1}{2} + \frac{[\sinh(\xi-\gamma/2)]^{1/2}}{4\tanh(\gamma/2)}\int_0^{\xi-\gamma/2}\frac{d\rho}{(\sinh\rho)^{1/2}} + C_1[\sinh(\xi-\gamma/2)]^{1/2}}. \quad (3.9)$$

We observe that λ is a function of $\xi - \gamma/2 = \eta + \gamma/2$, and that under the interchange $v \leftrightarrow -v, \gamma \leftrightarrow -\gamma$, the quantity $\xi - \gamma/2$ is invariant. Accordingly, we can divide the numerator and the denominator of (3.9) by $[\sinh(\xi-\gamma/2)]^{1/2}$ and the formula (3.9) suggests that

$$\begin{aligned}M_0(U) &= C_2\left\{C_1 + \frac{1}{4\tanh(\gamma/2)}\int_0^{\xi-\gamma/2}\frac{d\rho}{(\sinh\rho)^{1/2}} - \frac{1}{2[\sinh(\xi-\gamma/2)]^{1/2}}\right\},\\ m_0(u) &= C_2\left\{C_1 + \frac{1}{4\tanh(\gamma/2)}\int_0^{\eta+\gamma/2}\frac{d\rho}{(\sinh\rho)^{1/2}} + \frac{1}{2[\sinh(\eta+\gamma/2)]^{1/2}}\right\},\end{aligned} \quad (3.10)$$

where C_2 denotes an arbitrary constant and the variables (ξ, η, γ) are defined by (2.11). We comment that formally for the interchange $v \leftrightarrow -v$ we may accommodate the sign switching of the final terms simply by multiplying the numerator and the denominator of (3.9) by γ, since γ may then become absorbed into the arbitrary constant C_2. We further comment that the Einstein formulae arise from the above Eq. (3.9), namely $\lambda = 1$, in the limiting case $C_1 \to \infty$.

To verify that (3.10) indeed constitutes the correct choice for $m_0(u)$ and $M_0(U)$, with λ defined by (3.9), we may formally integrate $(3.4)_1$ as follows

$$\begin{aligned}\frac{dM}{M} &= \frac{U\,dU}{c^2\left[1-(U/c)^2\right]}\left\{1 + \frac{\beta(\lambda-1)}{(\beta-1+Uv/c^2)}\right\}\\ &= \frac{U\,dU}{c^2\left[1-(U/c)^2\right]}\left\{1 - \frac{\left[1-(v/c)^2\right]}{\sigma(v/c)^2\left[(U/c)^2-2U/v+1\right]}\right\}\\ &= \frac{1}{2}\tanh(\xi/2)\,d\xi + \frac{\sinh\xi\,d\xi}{4\sigma\sinh(\gamma/2)\sinh(\xi-\gamma/2)}.\end{aligned} \quad (3.11)$$

However, the final term of this equation also appears in (3.7), and with the aid of (3.7), it is not difficult to show that (3.11) becomes

$$\frac{dM}{M} = \frac{1}{2}\frac{\sinh(\xi/2)}{\cosh(\xi/2)}d\xi + \frac{d\left(\sigma/[\sinh(\xi-\gamma/2)]^{1/2}\right)}{\sigma/[\sinh(\xi-\gamma/2)]^{1/2}},$$

and therefore on integration we obtain

$$M(U) = \frac{C_2\cosh(\xi/2)\sigma}{[\sinh(\xi-\gamma/2)]^{1/2}}, \quad (3.12)$$

where C_2 can be shown to denote the arbitrary constant introduced above.

From $\lambda = m_0(u)/M_0(U)$ and $\sigma = (\lambda-1)^{-1}$ we may readily deduce from (3.12) the remarkably simple result

$$m_0(u) = M_0(U) + \frac{C_2}{[\sinh(\xi-\gamma/2)]^{1/2}}, \quad (3.13)$$

and which is entirely consistent with the suggested expressions (3.10). Using the variables defined by (2.11), Eq. (3.13) becomes

$$m_0(u) - M_0(U) = \frac{C_2\left[1-(U/c)^2\right]^{1/2}\left[1-(v/c)^2\right]^{1/4}}{\left(2\frac{U}{c} - \left[1+\left(\frac{U}{c}\right)^2\right]\frac{v}{c}\right)^{1/2}}.$$

We comment that by alternately placing the particle at the origin of either frame, so that we have either

$$U = v, \quad u = 0, \quad \xi = \gamma, \quad \eta = 0, \tag{3.14}$$

or

$$U = 0, \quad u = -v, \quad \xi = 0, \quad \eta = -\gamma, \tag{3.15}$$

we may deduce from this equation the two constraints

$$m_0^* - M_0(v) = \frac{C_2}{[\sinh(\gamma/2)]^{1/2}} = C_2\left(\frac{c}{v}\right)^{1/2}\left[1-(v/c)^2\right]^{1/4},$$
$$m_0(-v) - m_0^* = \frac{C_2}{[\sinh(-\gamma/2)]^{1/2}} = C_2\left(\frac{c}{-v}\right)^{1/2}\left[1-(v/c)^2\right]^{1/4}.$$

Assuming that the relations $m_0(-v) = m_0(v) = M_0(v)$ hold, then for real outcomes, it is clear that these two equations are not consistent unless $C_2 = 0$ and in which case, $m_0(v) = M_0(v) = m_0^*$, which of course is simply the Einstein mass variation. Accordingly, the more general mass variation appears not to apply

$$\frac{dE}{dM} = \frac{\frac{1}{2}[\cosh(\gamma/2)\cosh(\xi-\gamma/2)-1] + [\sinh(\xi-\gamma/2)]^{3/2}\left\{\frac{\cosh(\gamma/2)}{4}\int_0^{\xi-\gamma/2}\frac{d\rho}{(\sinh\rho)^{1/2}} + C_1\sinh(\gamma/2)\right\}}{\frac{1}{2}[\cosh(\gamma/2)\cosh(\xi-\gamma/2)+1] + [\sinh(\xi-\gamma/2)]^{3/2}\left\{\frac{\cosh(\gamma/2)}{4}\int_0^{\xi-\gamma/2}\frac{d\rho}{(\sinh\rho)^{1/2}} + C_1\sinh(\gamma/2)\right\}}.$$

in a straightforward manner as is the case for Einstein's formula. In the following section, we show that care must be exercised in ensuring that attention is restricted to an appropriate region of (U, u) space. In the above example, assuming that $v > 0$, it turns out that the first set of conditions (3.14) are permissible while the second (3.15) are not, and vice-versa if $v < 0$. We show in the following section that while the above two conditions (3.14) and (3.15) do not apply simultaneously, they may possibly apply separately but with restrictions on the sign of $\xi - \gamma/2$.

A related issue arises when we need to specify the datum energies corresponding to the solutions (3.10). Recall that by definition $m_0(u) = n_0(\eta)$ and $M_0(U) = N_0(\xi)$ and from the expressions (3.10) we may deduce

$$\frac{dN_0(\xi)}{d\xi} = \frac{C_2\sinh\xi}{4\sinh(\gamma/2)[\sinh(\xi-\gamma/2)]^{3/2}},$$
$$\frac{dn_0(\eta)}{d\eta} = \frac{C_2\sinh\eta}{4\sinh(\gamma/2)[\sinh(\eta+\gamma/2)]^{3/2}}, \tag{3.16}$$

which on respective substitution into (2.14) and (2.15) yield

$$E = M(U)c^2 - \frac{C_2c^2}{\sinh(\gamma/2)}\left\{\frac{\sinh(\xi-\gamma)/2}{[\sinh(\xi-\gamma/2)]^{1/2}} - [\sinh(-\gamma/2)]^{1/2}\right\},$$
$$\mathscr{E} = m(u)c^2 - \frac{C_2c^2}{\sinh(\gamma/2)}\left\{\frac{\sinh(\eta+\gamma)/2}{[\sinh(\eta+\gamma/2)]^{1/2}} - [\sinh(\gamma/2)]^{1/2}\right\}, \tag{3.17}$$

noting that the above expressions have been normalised such that $E = m_0^*c^2$ when $U = 0$ and $\mathscr{E} = m_0^*c^2$ when $u = 0$. Although to a certain extent these datum energy levels are arbitrary, we observe that for this particular normalisation, one or other of the additive constants becomes pure imaginary depending upon whether $v > 0$ or $v < 0$. We have purposely included this to highlight the issue that the region of validity of the solutions (3.10) requires careful consideration, and even the seemingly natural choices of datum energy levels may not be applicable. This issue is analysed at length in the following section.

As previously noted the two invariances (2.9) imply that the energy–mass rates are the same in both frames, namely $d\mathscr{E}/dm = dE/dM$, which may be formally proved from the above solutions (3.10) as follows. From the relations $dE = UdP, P = MU$, (2.10) and the new velocity variables (2.11), we may deduce

$$\frac{dE}{dM} = \frac{C_2[\sinh(\xi/2)]^2 + N_0(\xi)\sinh(\gamma/2)[\sinh(\xi-\gamma/2)]^{3/2}}{C_2[\cosh(\xi/2)]^2 + N_0(\xi)\sinh(\gamma/2)[\sinh(\xi-\gamma/2)]^{3/2}},$$
$$\frac{d\mathscr{E}}{dm} = \frac{C_2[\sinh(\eta/2)]^2 + n_0(\eta)\sinh(\gamma/2)[\sinh(\eta+\gamma/2)]^{3/2}}{C_2[\cosh(\eta/2)]^2 + n_0(\eta)\sinh(\gamma/2)[\sinh(\eta+\gamma/2)]^{3/2}},$$

and the equality of these two expressions follows from the relation $\xi = \eta + \gamma$, the integral (3.13) and on noting relations such as

$$\sinh^2(\xi/2) - \sinh^2(\eta/2) = \cosh^2(\xi/2) - \cosh^2(\eta/2) = \sinh(\gamma/2)\sinh(\xi-\gamma/2).$$

On exploiting the explicit form of the mass solutions (3.10), we may for example deduce expressions such as

Finally in this section, we observe that on using (3.13) to eliminate the quantity $C_2/[\sinh(\xi-\gamma/2)]^{1/2}$ from Eq. (3.17), we may simplify (3.17) to become

$$E = \left\{M(U)U - m(u)u\left[1-(v/c)^2\right]^{1/2}\right\}\frac{c^2}{v} + E_0,$$
$$\mathscr{E} = -\left\{m(u)u - M(U)U\left[1-(v/c)^2\right]^{1/2}\right\}\frac{c^2}{v} + \mathscr{E}_0, \tag{3.18}$$

for certain constants E_0 and $\mathscr{E}_0$, and we may provide an independent check on the above relations as arising directly from the energy and momentum Eqs. (2.8) and $dE/dT = FU$ and $F = dP/dT$ as follows. In consideration of

$$\frac{d}{dT}\left\{\mathscr{E} - \frac{(E-Pv)}{\left[1-(v/c)^2\right]^{1/2}}\right\} = \frac{f}{\left[1-(v/c)^2\right]^{1/2}}\left\{u\left(1-\frac{Uv}{c^2}\right) - (U-v)\right\} = 0,$$

since $u = (U-v)/(1-Uv/c^2)$ and noting that we have used force invariance $f = F$ and the Lorentz transformation $(2.1)_2$, and

$$\frac{d}{dT}\left\{E - \frac{(\mathscr{E}+pv)}{\left[1-(v/c)^2\right]^{1/2}}\right\} = \frac{f}{\left[1-(v/c)^2\right]}\left\{U\left[1-\left(\frac{v}{c}\right)^2\right] - (u+v)\left(1-\frac{Uv}{c^2}\right)\right\}$$
$$= \frac{f}{\left[1-(v/c)^2\right]}\left\{U\left[1-\left(\frac{v}{c}\right)^2\right] - (U-v) - v\left(1-\frac{Uv}{c^2}\right)\right\} = 0,$$

we may deduce the Lorentz invariant energy–momentum relations

$$\mathscr{E} = \frac{(E-Pv)}{\left[1-(v/c)^2\right]^{1/2}} + \text{const}, \quad E = \frac{(\mathscr{E}+pv)}{\left[1-(v/c)^2\right]^{1/2}} + \text{const}, \tag{3.19}$$

which apply for all mass variations, and the constants are determined by prescribed initial data. Now on re-writing (3.18) as,

$$E = \left(P - p\left[1-(v/c)^2\right]^{1/2}\right)\frac{c^2}{v} + E_0,$$
$$\mathcal{E} = -\left(p - P\left[1-(v/c)^2\right]^{1/2}\right)\frac{c^2}{v} + \mathcal{E}_0, \tag{3.20}$$

it is a simple matter to show that (3.19) are indeed satisfied by (3.20), which evidently admit the following interesting identity

$$(\mathcal{E}-\mathcal{E}_0)^2 - (E-E_0)^2 = (p^2-P^2)c^2. \tag{3.21}$$

We further comment that although (3.20) has been derived within the context of a particular mass variation, these relations can be verified directly for any mass variation since

$$\frac{d}{dT}\left\{E - \frac{c^2}{v}\left(P - p\left[1-(v/c)^2\right]^{1/2}\right)\right\} = f\left\{U - \frac{c^2}{v} + \frac{c^2}{v}\left(1 - \frac{Uv}{c^2}\right)\right\} = 0,$$

and

$$\frac{d}{dT}\left\{\mathcal{E} - \frac{c^2}{v}\left(p - P\left[1-(v/c)^2\right]^{1/2}\right)\right\}$$
$$= \frac{f}{\left[1-(v/c)^2\right]^{1/2}}\left\{(U-v) + \frac{c^2}{v}\left(1 - \frac{Uv}{c^2}\right) - \frac{c^2}{v}\left[1 - \left(\frac{v}{c}\right)^2\right]\right\} = 0,$$

assuming only the Lorentz transformations (2.1), the energy and momentum equations such as (2.8) and force invariance $f = F$. Together the relations (3.19) and (3.20) yield

$$\mathcal{E}-\mathcal{E}_0 = \frac{(E-E_0) - Pv}{\left[1-(v/c)^2\right]^{1/2}}, \quad p = \frac{P - (E-E_0)v/c^2}{\left[1-(v/c)^2\right]^{1/2}},$$

along with the inverse relations

$$E - E_0 = \frac{(\mathcal{E}-\mathcal{E}_0) + pv}{\left[1-(v/c)^2\right]^{1/2}}, \quad P = \frac{p + (\mathcal{E}-\mathcal{E}_0)v/c^2}{\left[1-(v/c)^2\right]^{1/2}}.$$

Analysis and application of new mass variation formulae

Eqs. (3.10) along with the integral (3.13) describe a two-parameter family for the variation of mass. However, the formal analysis leading to the solutions (3.10) and their applicability, is critically based on the assumption that the quantity $\xi - \gamma/2$ remains positive in the (U, u) region of interest. The formulae (3.10) and (3.13) only remain real provided we confine ourselves to the regime $\xi - \gamma/2 = \eta + \gamma/2 > 0$ which may be shown to correspond to either $\left(2U - \left[1 + (U/c)^2\right]v\right) > 0$ for U or to $\left(2u + \left[1 + (u/c)^2\right]v\right) > 0$ for u, so that for $v > 0, U_{min} < U < U_{max}$ and for u, either $u < -U_{max}$ or $u > -U_{min}$ where for $v > 0$ we adopt U_{min} and U_{max} to be the two positive numbers that are defined respectively by

$$U_{min} = \frac{c^2}{v}\left(1 - \left[1-(v/c)^2\right]^{1/2}\right),$$
$$U_{max} = \frac{c^2}{v}\left(1 + \left[1-(v/c)^2\right]^{1/2}\right), \tag{4.1}$$

noting that in Figs. 2 and 3, we utilise the same two positive numbers as defined above for both $v > 0$ and $v < 0$. We comment that these two values of U formally arise from the condition $u = -U$. For $v > 0$, both v and c always lie in the interval (U_{min}, U_{max}), and there are corresponding constraints for the case $v < 0$. Further, the imposed conditions on the appropriate (U, u) region, effectively means that we are not at liberty to arbitrarily impose any two conditions, and this must be done with care ensuring the prescribed data lies within an appropriate interval, as shown schematically in

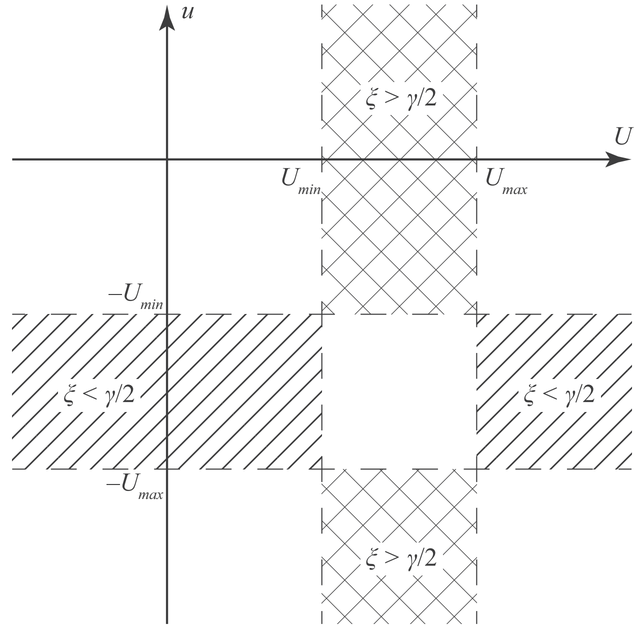

Fig. 2. Allowable (U, u) regions for solutions (3.10) and (4.2) for $v > 0$.

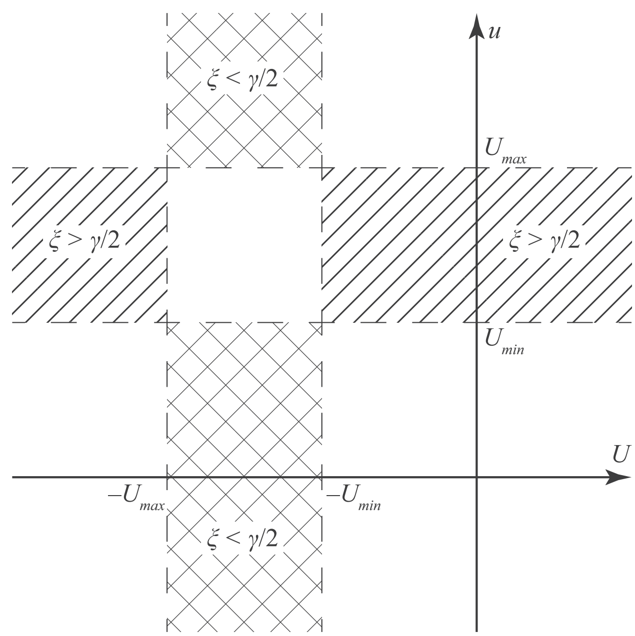

Fig. 3. Allowable (U, u) regions for solutions (3.10) and (4.2) for $v < 0$.

Figs. 2 and 3. For U in the interval $U_{min} < U < U_{max}$ and close to the end-points, the solutions (3.10) exhibit singularities; thus for example

$$M(U) \sim -\frac{C_2 c^{3/2}\left[1-(v/c)^2\right]^{1/4}}{2v^{1/2}[(U_{max}-U)(U-U_{min})]^{1/2}},$$

and with a similar asymptotic expression for $m(u)$.

As an aside, it is interesting to observe that with the above terminology (4.1), the relations (3.20) become simply

$$E + \mathcal{E} = (P - p)U_{max} + E_0 + \mathcal{E}_0,$$
$$E - \mathcal{E} = (P + p)U_{min} + E_0 - \mathcal{E}_0,$$

so that we may re-affirm the identity (3.21) noting the identity $U_{max}U_{min} = c^2$. In the event that $\xi - \gamma/2 < 0$, then in place of (3.10) we need to solve the differential equation

$$\frac{d\sigma}{d\xi}+\frac{\sigma}{2}\frac{\cosh(\gamma/2-\xi)}{\sinh(\gamma/2-\xi)}=\frac{\sinh((\gamma/2-\xi)-\gamma/2)}{4\sinh(\gamma/2)\sinh(\gamma/2-\xi)}.$$

and a straightforward integration yields

$$\sigma=-\frac{1}{2}-\frac{[\sinh(\gamma/2-\xi)]^{1/2}}{4\tanh(\gamma/2)}\int_0^{\frac{\gamma}{2}-\xi}\frac{d\rho}{(\sinh\rho)^{1/2}}+c_1\left[\sinh\left(\frac{\gamma}{2}-\xi\right)\right]^{1/2},$$

where c_1 denotes the arbitrary constant of integration, which is possibly distinct from C_1. The solutions corresponding to (3.10) can be shown to become

$$\begin{aligned} M_0(U)&=c_2\left\{c_1-\tfrac{1}{4\tanh(\gamma/2)}\int_0^{\gamma/2-\xi}\tfrac{d\rho}{(\sinh\rho)^{1/2}}-\tfrac{1}{2[\sinh(\gamma/2-\xi)]^{1/2}}\right\},\\ m_0(u)&=c_2\left\{c_1-\tfrac{1}{4\tanh(\gamma/2)}\int_0^{-(\eta+\gamma/2)}\tfrac{d\rho}{(\sinh\rho)^{1/2}}+\tfrac{1}{2[\sinh(-(\eta+\gamma/2))]^{1/2}}\right\}, \end{aligned} \tag{4.2}$$

where c_2 denotes a further arbitrary constant, again possibly distinct from C_2. The variables (ξ,η,γ) are defined by (2.11), and noting that if $\xi-\gamma/2<0$, then $-(\eta+\gamma/2)>0$. The various (U,u) regions of applicability for these solutions and the solutions (3.10) are shown schematically in Fig. 2 for $v>0$ and Fig. 3 for $v<0$. Thus for $v>0$, if $U_{\min}<U<U_{\max}$ then either $u<-U_{\max}$ or $u>-U_{\min}$, while if $-U_{\max}<u<-U_{\min}$ then either $U<U_{\min}$ or $U>U_{\max}$, where $U_{\min}$ and $U_{\max}$ are as defined by (4.1).

It is possible to determine the constant C_1 in such a manner that both $m(u)$ and $M(U)$ remain finite in the limit $u,U\to c$. If C_1 is chosen so that

$$C_1=-\frac{1}{4\tanh(\gamma/2)}\int_0^{\infty}\frac{d\rho}{[\sinh\rho]^{1/2}}=-\frac{1}{8\tanh(\gamma/2)}B\left(\frac{1}{4},\frac{1}{2}\right), \tag{4.3}$$

noting that $B(1/4,1/2)=5.244115\ldots$ is the usual beta function, and then both $m_0(u)$ and $M_0(U)$ tend to zero in the limit as $u,U\to c$, and therefore for example

$$M_c=\lim_{U\to c}\frac{M_0(U)}{\left[1-(U/c)^2\right]^{1/2}}=\lim_{\xi\to\infty}\frac{N_0(\xi)}{\operatorname{sech}(\xi/2)}=\lim_{\xi\to\infty}\frac{dN_0(\xi)/d\xi}{\frac{-\tanh(\xi/2)}{2\cosh(\xi/2)}},$$

and on using (3.16), we may eventually deduce

$$M_c=-\frac{C_2e^{3\gamma/4}}{2^{3/2}\sinh(\gamma/2)}=-\frac{C_2[1+(v/c)]^{5/4}}{2\sqrt{2}(v/c)[1-(v/c)]^{1/4}}.$$

By a completely analogous calculation using (3.16) we may deduce

$$m_c=-\frac{C_2e^{-3\gamma/4}}{2^{3/2}\sinh(\gamma/2)}=-\frac{C_2[1-(v/c)]^{5/4}}{2\sqrt{2}(v/c)[1+(v/c)]^{1/4}},$$

and from which we may deduce the interesting relations

$$m_cM_c=\frac{C_2^2}{8}\frac{\left[1-(v/c)^2\right]}{(v/c)^2},\quad \frac{m_c}{M_c}=\left(\frac{1-(v/c)}{1+(v/c)}\right)^{3/2}.$$

We might for example fix C_2 by positioning a particle at the origin of the (x,t) frame and imposing the conditions

$$U=v,\quad u=0,\quad \xi=\gamma,\quad \eta=0,\quad m(0)=m_0^*, \tag{4.4}$$

where m_0^* is the assumed known rest mass. From this condition, using both (3.10) and (4.3) we may deduce the equation

$$m_0^*=C_2\left\{\frac{1}{2[\sinh(\gamma/2)]^{1/2}}-\frac{1}{4\tanh(\gamma/2)}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}\right\}, \tag{4.5}$$

as the determining equation for the constant C_2. From (3.10), (4.3) and (4.5), we may deduce the expressions for $M_0(U)$ and $m_0(u)$ respectively

$$\begin{aligned} M_0(U)=N_0(\xi)&=m_0^*\frac{\left\{\frac{1}{4\tanh(\gamma/2)}\int_{\xi-\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}+\frac{1}{2[\sinh(\xi-\gamma/2)]^{1/2}}\right\}}{\left\{\frac{1}{4\tanh(\gamma/2)}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}-\frac{1}{2[\sinh(\gamma/2)]^{1/2}}\right\}},\\ m_0(u)=n_0(\eta)&=m_0^*\frac{\left\{\frac{1}{4\tanh(\gamma/2)}\int_{\eta+\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}-\frac{1}{2[\sinh(\eta+\gamma/2)]^{1/2}}\right\}}{\left\{\frac{1}{4\tanh(\gamma/2)}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}-\frac{1}{2[\sinh(\gamma/2)]^{1/2}}\right\}}, \end{aligned} \tag{4.6}$$

which evidently, if necessary, admits some minor cancellations. We observe from $(4.6)_1$ with $\xi=\gamma$ that the actual mass $M(v)$ as a function of the velocity v is given by

$$M(v)=\frac{m_0^*}{\left(1-(v/c)^2\right)^{1/2}}\frac{\left\{\frac{1}{2(v/c)^{1/2}\left(1-(v/c)^2\right)^{1/4}}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}+1\right\}}{\left\{\frac{1}{2(v/c)^{1/2}\left(1-(v/c)^2\right)^{1/4}}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}-1\right\}}, \tag{4.7}$$

which is the comparison formula to be contrasted with the Einstein expression $M(v)=m_0^*[1-(v/c)^2]^{-1/2}$, noticing that (4.7) tends properly to m_0^* in the limit v tending to zero. Notice that with C_2 defined by (4.5) and the assumed conditions (4.4), Eq. (4.7) also arises directly from either (3.9) or (3.13). We also observe that while both $M_0(U)$ and $m_0(u)$ are well-defined and finite when $U=u=c$, these expressions remain singular as the relative frame speed v approaches c. Indeed, the denominator in (4.6) vanishes at $v=c$, which is most easily seen by taking the limit $\cosh(\gamma/2)\int_{\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}/2[\sinh(\gamma/2)]^{1/2}$ as γ tends to infinity. Using L'Hôpital's rule, this limit can be shown to be equal to unity, and therefore the expression (4.7) involves a singularity at $v=c$ which is stronger than the Einstein expression.

We note that alternatively, we might position the particle at the origin of the (X,T) frame and impose the conditions

$$U=0,\quad u=-v,\quad \xi=0,\quad \eta=-\gamma,\quad M(0)=m_0^*,$$

where again m_0^* denotes the assumed known rest mass. From this condition, using both (4.2) and (4.4) we may deduce the equation

$$m_0^*=c_2\left\{c_1-\frac{1}{4\tanh(\gamma/2)}\int_0^{\gamma/2}\frac{d\rho}{(\sinh\rho)^{1/2}}-\frac{1}{2[\sinh(\gamma/2)]^{1/2}}\right\},$$

as one condition for the determination of the constants c_1 and c_2. We note however from the solutions (4.2), that if we have in mind fixing the constant c_1 by imposing finite mass at the speed of light then necessarily this must be done at $u=U=-c$ since both U and u must remain negative in order that both ξ and η are negative. The resulting solution can be shown to become

$$\begin{aligned} M_0(U)=N_0(\xi)&=m_0^*\frac{\left\{\frac{1}{4\tanh(\gamma/2)}\int_{\gamma/2-\xi}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}-\frac{1}{2[\sinh(\gamma/2-\xi)]^{1/2}}\right\}}{\left\{\frac{1}{4\tanh(\gamma/2)}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}-\frac{1}{2[\sinh(\gamma/2)]^{1/2}}\right\}},\\ m_0(u)=n_0(\eta)&=m_0^*\frac{\left\{\frac{1}{4\tanh(\gamma/2)}\int_{-(\eta+\gamma/2)}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}+\frac{1}{2[\sinh(-(\eta+\gamma/2))]^{1/2}}\right\}}{\left\{\frac{1}{4\tanh(\gamma/2)}\int_{\gamma/2}^{\infty}\frac{d\rho}{(\sinh\rho)^{1/2}}-\frac{1}{2[\sinh(\gamma/2)]^{1/2}}\right\}}, \end{aligned} \tag{4.8}$$

noting that this solution applies only for $\xi<\gamma/2$ and that (4.8) is formally the same solution as that given by (4.6) but applying in the opposite direction; that is negative velocities as opposed to positive velocities.

From (4.1) it is clear that $U_{\max}U_{\min}=c^2$, so that $U_{\min}$ is always subluminal and $U_{\max}$ is always superluminal. We have chosen the constants such that the residual masses vanish at $u=U=c$, so that the actual masses remain finite there. Now the value $U=c$ lies in the region for which the solutions (3.10) apply, and for subluminal frame velocities v, either both u and U are subluminal or both are superluminal. This means that we need to determine the corresponding new solutions for $u,U>c$, and this is done in the following section. By choosing appropriate new variables ξ and η, and redefining residual masses, we find that the major formulae of this section also apply for $u, U > c$, and specifically the residual masses for both $u, U < c$ and $u, U > c$ are formally precisely as given by (4.6).

Superluminal motion ($v < c$ and $u, U > c$)

In this section for superluminal motion we follow the development of the present authors [3,4] and that of Vieira [9], and we detail the key equations arising for superluminal motion with $u, U > c$ and $v < c$ and for which (ξ, η, γ) are defined by

$$\xi = \log\left(\frac{U/c+1}{U/c-1}\right), \quad \eta = \log\left(\frac{u/c+1}{u/c-1}\right), \quad \gamma = \log\left(\frac{1+v/c}{1-v/c}\right), \tag{5.1}$$

and for which $\xi = \eta + \gamma$ still holds while the inverses are given by

$$U = c\coth(\xi/2), \quad u = c\coth(\eta/2), \quad v = c\tanh(\gamma/2). \tag{5.2}$$

In this case the residual masses $m_0(u)$ and $M_0(U)$ are defined by

$$m(u) = \frac{m_0(u)}{\left[(u/c)^2 - 1\right]^{1/2}}, \quad M(U) = \frac{M_0(U)}{\left[(U/c)^2 - 1\right]^{1/2}}, \tag{5.3}$$

and the appropriate square root relation arising from (2.3) becomes

$$\left[(u/c)^2 - 1\right]^{1/2}\left[1 - Uv/c^2\right] = \left[(U/c)^2 - 1\right]^{1/2}\left[1 - (v/c)^2\right]^{1/2}.$$

With λ defined in the usual way as $\lambda = m_0(u)/M_0(U)$, the generalised Lorentz invariant mass-momentum relations are again given by (3.1) and (3.2). This means that the differential relations (3.3) arising from the two invariances (2.9) are identical with (3.3) and all the subsequent Eqs. (3.4)–(3.7) that apply in the previous section also apply to the present case. Thus the forms of equations such as (3.7) coincide even though the respective transformations for the two cases (namely (2.11) and (5.1)) are different and the corresponding inverses (2.13) and (5.2) have tanh replaced by coth for ξ and η.

As in the previous section, we find that $\sigma = (\lambda - 1)^{-1}$ satisfies (3.7) and is given by (3.8). Further, λ is given by (3.9) and the first two equations of (3.11) remain unchanged, but the final line has $\coth(\xi/2)$ in place of $\tanh(\xi/2)$, thus

$$\frac{dM}{M} = \frac{1}{2}\coth(\xi/2)\,d\xi + \frac{\sinh(\xi)\,d\xi}{4\sigma\sinh(\gamma/2)\sinh(\xi-\gamma/2)},$$

which as before simplifies to yield

$$\frac{dM}{M} = \frac{1}{2}\frac{\cosh(\xi/2)}{\sinh(\xi/2)}\,d\xi + \frac{d\left(\sigma/[\sinh(\xi-\gamma/2)]^{1/2}\right)}{\sigma/[\sinh(\xi-\gamma/2)]^{1/2}},$$

and on integration gives

$$M(U) = \frac{C_2\sinh(\xi/2)\sigma}{[\sinh(\xi-\gamma/2)]^{1/2}}, \tag{5.4}$$

where again C_2 denotes an arbitrary constant, and again (5.4) formally simplifies to give (3.13) so that the formal solutions (3.10) remain unchanged, but now of course ξ and η are defined by (5.1). Further, from Eq. (5.4) we may deduce

$$m_0(u) - M_0(U) = \frac{C_2\left[(U/c)^2 - 1\right]^{1/2}\left[1 - (v/c)^2\right]^{1/4}}{\left(2\frac{U}{c} - \left[1 + \left(\frac{U}{c}\right)^2\right]\frac{v}{c}\right)^{1/2}}.$$

However, the two energy formulae now become

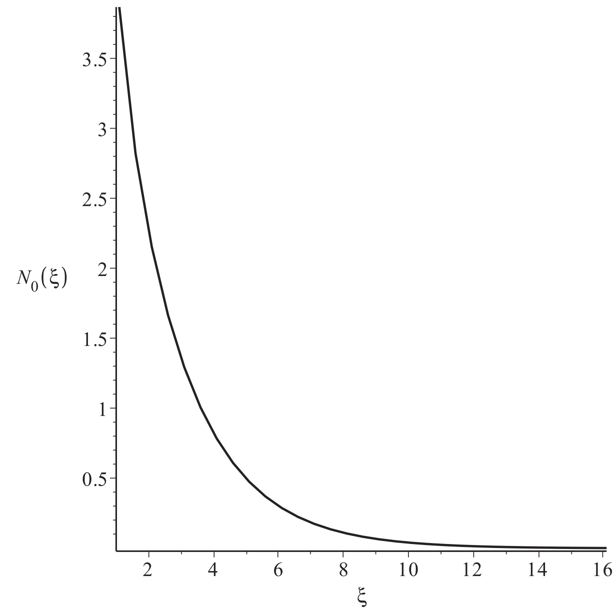

Fig. 4. Variation of residual mass $N_0(\xi)/m_0^*$ as given by $(4.6)_1$.

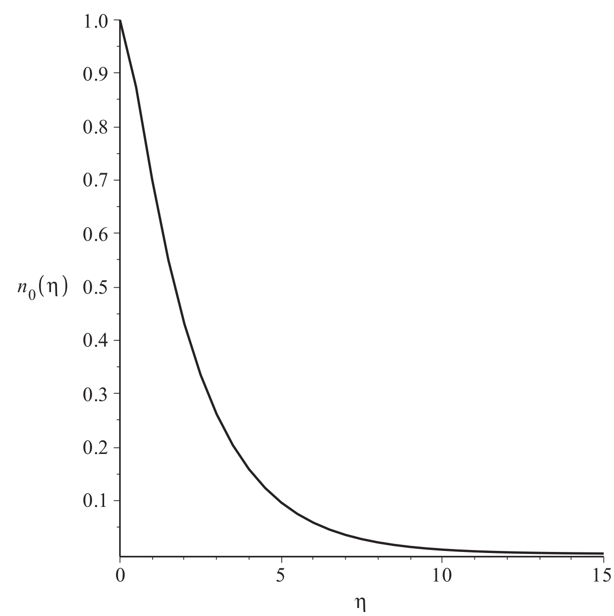

Fig. 5. Variation of residual mass $n_0(\eta)/m_0^*$ as given by $(4.6)_2$.

$$E = M(U)c^2 + c^2\int_0^{\xi}\operatorname{cosech}(\zeta/2)\frac{dN_0(\zeta)}{d\zeta}\,d\zeta + E_0,$$
$$\mathscr{E} = m(u)c^2 + c^2\int_0^{\eta}\operatorname{cosech}(\psi/2)\frac{dn_0(\psi)}{d\psi}\,d\psi + \mathscr{E}_0,$$

for certain constants E_0 and $\mathscr{E}_0$. For example, assuming the same datum energy level as arising from some limiting mass, say $m_\infty^* c^2$, assumed to apply respectively for $\xi = 0$ and $\eta = 0$, then in place of (3.17) we might have

$$E = M(U)c^2 - \frac{C_2c^2}{\sinh(\gamma/2)}\left\{\frac{\cosh(\xi-\gamma)/2}{[\sinh(\xi-\gamma/2)]^{1/2}} - \frac{\cosh(\gamma/2)}{[\sinh(-\gamma/2)]^{1/2}}\right\},$$
$$\mathscr{E} = m(u)c^2 - \frac{C_2c^2}{\sinh(\gamma/2)}\left\{\frac{\cosh(\eta+\gamma)/2}{[\sinh(\eta+\gamma/2)]^{1/2}} - \frac{\cosh(\gamma/2)}{[\sinh(\gamma/2)]^{1/2}}\right\},$$

so that again there is an apparent need for the requirement to ensure allowable initial data as arising from an appropriate (U, u) region as indicated in Figs. 2 and 3.

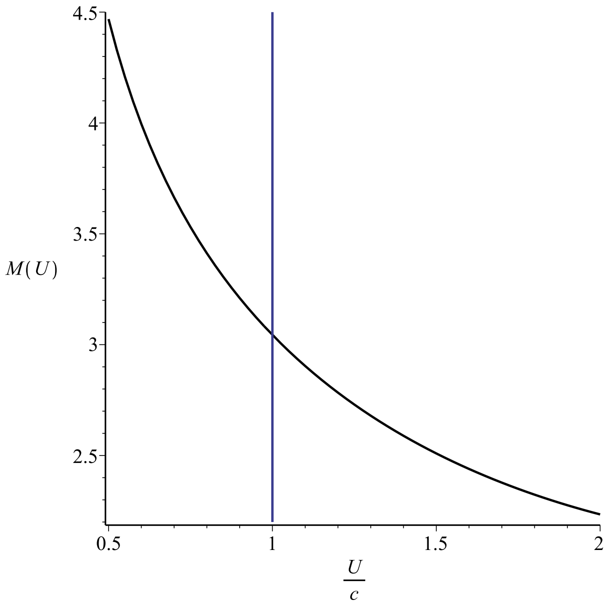

Fig. 6. Variation of actual mass $M(U)/m_0^*$ from (4.6)$_1$, (2.10)$_2$ and (5.3)$_2$.

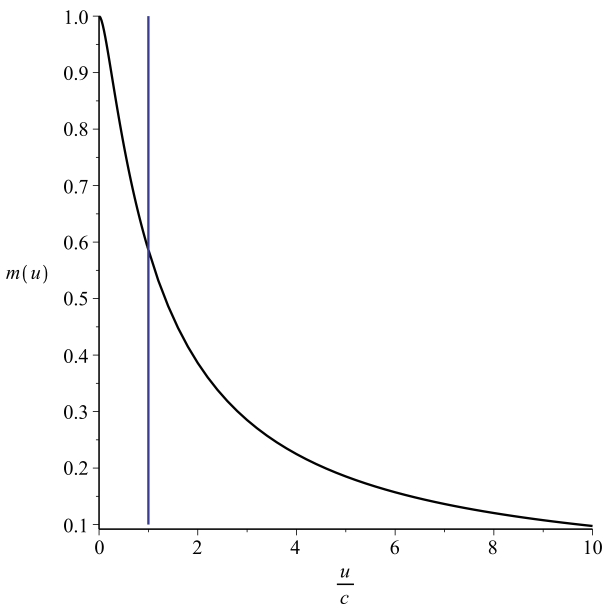

Fig. 7. Variation of actual mass $m(u)/m_0^*$ from (4.6)$_2$, (2.10)$_1$ and (5.3)$_1$.

On using the equivalent equation to (3.13) we find that these relations become identical with (3.18), namely

$$E = \left\{ M(U)U - m(u)u\left[1-(v/c)^2\right]^{1/2} \right\} \frac{c^2}{v} + E_0,$$
$$\mathscr{E} = -\left\{ m(u)u - M(U)U\left[1-(v/c)^2\right]^{1/2} \right\} \frac{c^2}{v} + \mathscr{E}_0,$$

but noting that the datum energy levels E_0 and $\mathscr{E}_0$ are different, and that when $U \to \infty, u \to -c^2/v$ and when $u \to \infty, U \to c^2/v$. The subsequent equations derived in the previous section, namely starting with (3.19) remain essentially unchanged for the case presented in this section, except of course for the particular values of the datum energy levels E_0 and $\mathscr{E}_0$. Accordingly, these details are not re-produced here.

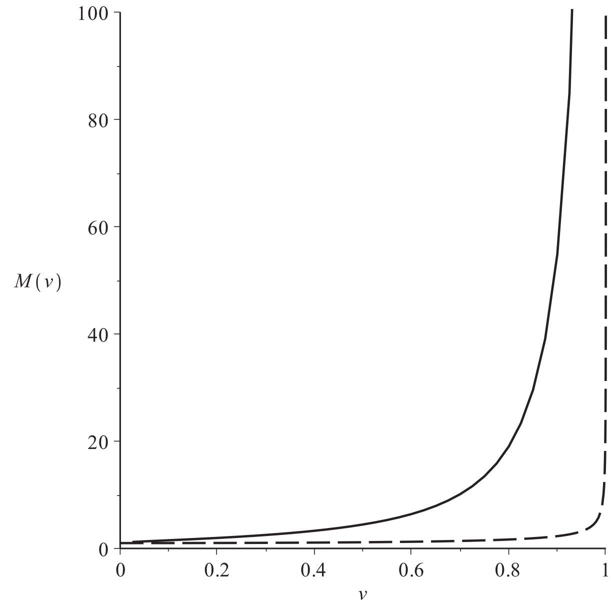

Fig. 8. Variation of actual mass $M(v)/m_0^*$ from (4.7) showing the Einstein formula (dashed).

For the numerical results shown in Figs. 4–7, the variable η lies in the interval $(0, \infty)$ while the variable ξ lies in the interval (γ, ∞). The speed of light corresponds to $\xi = \eta = \infty$, and at infinity both residual masses vanish so that the actual masses remain finite there. We might visualise the solution as a symmetrically folded sheet with the fold corresponding to $\xi = \eta = \infty$ and any prescribed data on the residual mass at one edge of the sheet is automatically inherited at the other edge of the sheet. Thus for example, for $u, U < c$ and the conditions (4.4), the assumed initial condition $m_0(0) = m_0^*$ implies for $u, U > c$ that $m_0(\infty) = m_0^*$. There is a corresponding relation arising from the value of $M_0(v)$ for $u, U < c$, which generates the same value for $M_0(c^2/v)$ for $u, U > c$. This is because the curves for both $u, U < c$ and $u, U > c$ are the same curve, and it is only the re-interpretation to the velocity that changes for below and above c. Specifically, for $u, U < c$, the residual masses are defined by (2.10) and for $u, U > c$ by (5.3), but in both cases $M_0(U) = N_0(\xi)$ and $m_0(u) = n_0(\eta)$ are defined by (4.6). The two residual masses $M_0(U) = N_0(\xi)$ and $m_0(u) = n_0(\eta)$ are shown in Figs. 4 and 5 respectively for the particular value $v/c = 1/2$, so that $\gamma = \log 3$. Figs. 6 and 7 show the non-dimensionalised actual masses $M(U)/m_0^*$ and $m(u)/m_0^*$ as functions of the non-dimensionalised velocities U/c and u/c respectively. In both of these two figures we observe that there is a smooth transition through the speed of light as anticipated. Fig. 8 shows the variation of the actual mass $M(v)/m_0^*$ from (4.7) as compared with the conventional Einstein formula which is the dashed curve. Notice that the singularity at the speed of light is stronger than that predicted by the Einstein formula, which means that there is typically more mass than predicted by the Einstein expression.

Conclusions

In special relativity, mass as a function of its velocity is prescribed by a single arbitrary constant, termed its rest mass. In this paper we have posed the question as to whether there might exist other special relativistic mass variations, and we have determined new mass variations involving two arbitrary constants. As usual in special relativity we have considered two moving frames such that the (x, t) frame is moving with constant velocity v with respect to the (X, T) frame. We consider a moving particle having velocity u with respect to the (x, t) frame and U with respect to the (X, T) frame. We have assumed only that the standard Lorentz transformations (2.1) and that the Einstein variation of mass formula $m(v) = m_0^*\left[1-(v/c)^2\right]^{-1/2}$ does not apply. We have proposed the question of determining other mass variation formulae that preserve the structure of the Lorentz invariant mass-momentum relations as well as force invariance in the direction of relative motion and another invariance (see (2.9)) that is known to apply in special relativity. Together the two invariances imply that the energy–mass rates are the same in both frames, namely $d\mathscr{E}/dm = dE/dM$, which is clearly the case in conventional special relativity when both energy–mass rates are then equal to c^2.

Based only on these assumptions, we have determined new mass variation expressions given by (3.10) involving two arbitrary constants C_1 and C_2, and assuming $v > 0$. For $v < 0$ the corresponding expressions are given by (4.2). Further, we have termed $m_0(u)$ and $M_0(U)$ as the residual masses, being the actual mass with the Einstein factor removed. Specifically, for $u, U < c$ the residual masses $m_0(u)$ and $M_0(U)$ are defined by

$$m(u) = \frac{m_0(u)}{\left[1-(u/c)^2\right]^{1/2}}, \quad M(U) = \frac{M_0(U)}{\left[1-(U/c)^2\right]^{1/2}},$$

while for $u, U > c$ the residual masses $m_0(u)$ and $M_0(U)$ are defined by

$$m(u) = \frac{m_0(u)}{\left[(u/c)^2-1\right]^{1/2}}, \quad M(U) = \frac{M_0(U)}{\left[(U/c)^2-1\right]^{1/2}},$$

where $m(u)$ and $M(U)$ denote the actual masses from the two frames, and in both cases $M_0(U) = N_0(\xi)$ and $m_0(u) = n_0(\eta)$ are defined by (4.6).

The new formulae involve the integral $\int^x d\rho/(\sinh\rho)^{1/2}$ and two arbitrary constants C_1 and C_2 which can be determined by appropriate initial data. The integrals can, if necessary, be expressed in terms of standard elliptical functions, but the resulting expressions are not particularly helpful in terms of generating insight. It is clear from the above integral involving sinh that for negative values of the integration variable, imaginary numbers are generated and care must be exercised in proposing specific initial data for a boundary value problem to have real outcomes. For each of $v > 0$ and $v < 0$ there are two branches for the new solutions depending on the sign of $\xi - \gamma/2$, and the allowable (U, u) regions are shown schematically in Figs. 2 and 3. This apparent limitation tends to reinforce the robustness of the Einstein formula $m(v) = m_0^*[1-(v/c)^2]^{-1/2}$. However, the new mass variations permit finite mass solutions at the speed of light, for which the residual mass vanishes at $v = c$, namely $m_0(c) = 0$, and an illustrative boundary value problem is formulated. The existence of finite mass solutions at the speed of light is perhaps an important issue with interesting physical implications. One apparent consequence of this assumption is that the Einstein formula for mass underestimates that predicted by the new expressions.

Finally, we emphasise that throughout this paper we have only considered relative frame velocities $v < c$, and that there are corresponding formal solutions to those presented here that apply for relative frame velocities $v > c$. However, instead of involving the integral $\int^x d\rho/(\sinh\rho)^{1/2}$, the new expressions involve the integral $\int^x d\rho/(\cosh\rho)^{1/2}$ which is always well-defined for both positive and negative ρ and the region of validity is unrestricted. The solutions corresponding to superluminal relative frame velocities are presented in part II of this paper.

References

[1] Coleman Brian. Special relativity dynamics without a priori momentum conservation. Eur. J. Phys. 2005;26:647–50. http://dx.doi.org/10.1088/0143-0807/26/4/010.
[2] Richard P. Feynman, Robert B. Leighton, Matthew Sands, The Feynman Lectures on Physics, vol. 1, Addison-Wesley, 1964.
[3] Hill James M, Cox Barry J. Einstein's special relativity beyond the speed of light. Proc. R. Soc. A 2012;468:4174–92. http://dx.doi.org/10.1098/rspa.2012.0340.
[4] Hill James M, Cox Barry J. Dual universe and hyperboloidal relative velocity surface arising from extended special relativity. Z. Angew. Math. Phys. (ZAMP) 2014;65:1251–60. http://dx.doi.org/10.1007/s00033-013-0388-z.
[5] Lev D. Landau, Evgeny M. Lifshitz, Course of Theoretical Physics, vol. 2, Addison-Wesley, 1951.
[6] Lee Anthony R, Kalotas Tomas M . Lorentz transformations from the first postulate. Amer. J. Phys. 1975;43:434–7. http://dx.doi.org/10.1119/1.9807.
[7] Lévy-Leblond Jean-Marc. One more derivation of the Lorentz transformation. Amer. J. Phys. 1976;44:271–7. http://dx.doi.org/10.1119/1.10490.
[8] Saari Donald G. Mathematics and the dark matter puzzle. Amer. Math. Monthly 2015;122:407–27. http://dx.doi.org/10.4169/amer.math.monthly.122.5.407.
[9] Vieira Ricardo S. An introduction to the theory of tachyons. Rev. Bras. Ensino Fís. 2012;34:3306. http://dx.doi.org/10.1590/S1806-11172012000300006.
[10] Galina Weinstein, Variation of mass with velocity: kugeltheorie or relativtheorie. arXiv:1205.5951 [physics.hist-ph], 2012.