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Impulsive Displacement of a Quasi-Linear Viscoelastic Material through Accurate Numerical Inversion of the Laplace Transform

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Abstract—An analytical model for the one-dimensional, impulsive displacement of a quasi-linear viscoelastic material has been developed. The quasi-linear model of Fung [1] has been used successfully for a wide range of soft biological tissues. Due to the integral definition of linear viscoelastic materials, solutions are conveniently performed in the Laplace transform plane. Complex kernels like the quasi-linear model are challenging to invert back to the real plane. Here, the method of Gaver [2] and Stehfest [3] is used to numerically carry Laplace space solutions to the real plane. Parametric results for a basic impulsive disturbance problem are presented. Results indicate that stress wave propagation is weakly dependent on the fast time, slow time ratio and more strongly dependent on the logarithmic damping parameter. Limitations of the numerical inversion method in the face of discontinuities are discussed as well using asymptotic methods. As an alternative to the numerical/polynomial-based Gaver-Stehfest method, a semianalytical regularization function useful near large gradient regions method is developed. A composite method that utilizes both the fully numerical and semianalytical convolution-based method is also described. The composite model provides improved results in terms of reducing computational undershoot and overshoot (wiggles) which limit both the fully numerical and the semianalytical models alone. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Numerical inversion, Laplace transform, Quasi-linear model, Displacement of material.

NOMENCLATURE

c	logarithmic damping term (quasi-linear viscoelastic model parameter)	$I_1(x)$	modified Bessel function of the first kind
E	relaxation function	P	pressure
$H(t)$	Heaviside unit step function	r	radial spatial location

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s	Laplace transform parameter
t	time
v	radial velocity
w	axial velocity
z	axial spatial location
η	dynamic viscosity
λ	time relaxation parameter
ν	kinematic viscosity
ρ	density
σ	stress
τ	temporal term or deviatoric stress
$\tau_1 \tau_2$	slow time, fast time parameters (quasi-linear viscoelastic model parameter)

1. INTRODUCTION

The mechanical behavior of biological tissues is a time dependent phenomenon. This time dependence is caused by the complexity of the tissues which are characterized by a range of structures at multiple length scales [4]. Rheological models that map this complex microstructure into a global continuum equation are of great practical and theoretical value. The quasi-linear model of Fung [1] has been a particularly successful rheological closure. Fung's model is a three parameter (two time scales and a damping constant) linear viscoelastic kernel. The viscoelastic function has been used to model embryonic myocardium [5], tendon [6], ligament [7], cartilage [8], skeletal muscle [9], spine [10], and artery [11]. To better understand the problem, the one-dimensional relationship for a general linear viscoelastic material [12,13]

$$\tau = \int_{-\infty}^{t} G(t-t')\, \frac{d\gamma}{dt}\, dt', \tag{1}$$

where $\frac{d\gamma}{dt}$ is the strain rate and the kernel $G(t)$ is the quasi-linear model of [1]

$$G(t) = \frac{1 + cE_1\left(t/\tau_2\right) - cE_1\left(t/\tau_1\right)}{1 + c\ln\left(\tau_2/\tau_1\right)}, \tag{2}$$

and $E1(x)$ is the exponential integral [14] defined by

$$E1(x) \equiv \int_{x}^{\infty} \frac{e^{-t}}{t}\, dt. \tag{3}$$

Substitution of equations (1) and (2) into a force balance relationship would yield a (linear) but complex integrodifferential equation which is difficult to solve. Equation (1) is a convolution form (see [15] or [16]) that may be converted to a much simpler problem in the Laplace plane. Though this simplifies the forward solution, we are then faced with the formidable task of inverting the transform and carrying the solution back to the real plane. For simple kernels $G(t)$ this inversion is possible analytically, while for more complex and realistic models, such as the quasi-linear model above, the inversion cannot be achieved in closed form. Here, the method of Gaver and Stehfest is used to numerically carry Laplace space solutions to the real plane.

The problem of specific interest to us is the one-dimensional, impulsive displacement of a finite plug of quasi-linear viscoelastic material. Consider the finite cylinder of material as shown in Figure 1.

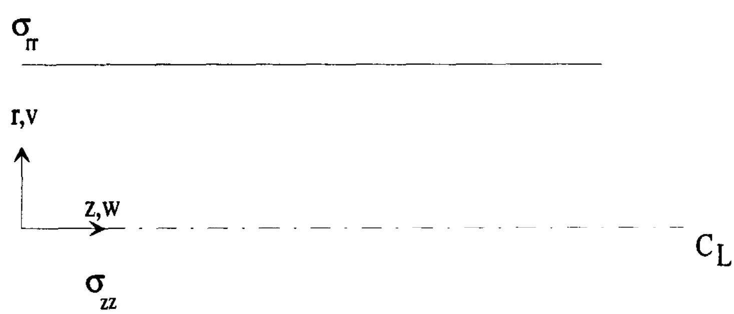

Figure 1. Flow field definitions for a finite cylinder of material.

The goal of this research is to model the velocity of the stress wave in the material due to an instantaneous finite displacement and to understand the effect of the measured parameters in the quasi-linear model. Further, we demonstrate the utility of formulating the linear viscoelastic problem using transforms. Additionally, we demonstrate that the numerical inversion of the Laplace transform solution is an effective solution method.

2. ANALYSIS

For the present small disturbances the convective acceleration, i.e., $\frac{\partial w}{\partial z} \ll 1$, can be neglected so that the equation of motion associated with incompressible, unsteady, one-dimensional axial flow [17] is

$$\rho \frac{\partial w}{\partial t} = \frac{\partial \sigma}{\partial z}, \tag{4}$$

where

$$\sigma = -p + \tau_{zz}. \tag{5}$$

Continuity (mass conservation) may be written as

$$\frac{\partial \nu}{\partial r} + \frac{1}{2} \frac{\partial w}{\partial z} = 0. \tag{6}$$

The radial momentum equation is expressed as

$$\sigma_{rr} = -p + \tau_{rr} = -p + \int_{-\infty}^{t} G(t - t') \frac{\partial \nu}{\partial r} \, dt' = 0. \tag{7}$$

Laplace transforming equation (7) yields

$$\tilde{p} = \tilde{G} \frac{d\tilde{\nu}}{dr}, \tag{8}$$

and from the transform of equation (6), we obtain

$$\tilde{p} = -\frac{\tilde{G}}{2} \frac{d\tilde{w}}{dz}, \tag{9}$$

so that the axial stress is written (using the transformed axial form of equation (1) and equation (9))

$$\tilde{\sigma}_{zz} = \frac{3}{2} \tilde{G} \frac{d\tilde{w}}{dz}. \tag{10}$$

Finally, introducing equation (10) into axial momentum relationship, equation (4) yields the governing differential equation

$$\frac{d^2\tilde{w}}{dz^2} - \frac{2}{3} \frac{\rho}{\tilde{G}} s\tilde{w} = 0. \tag{11}$$

Boundary conditions for equation (11) are chosen by introducing an initial condition for the disturbance of the cylinder. Considering a step velocity impulse of strength w_0, we transform $\tilde{w}(0) = w_0/s$. Equation (11) may be solved by requiring a bounded solution for $z \to \infty$ and applying the initial condition as

$$\tilde{w} = \frac{w_0}{s} e^{-z(2\rho s/3\tilde{G})^{1/2}}. \tag{12}$$

At this point, it becomes necessary to choose viscoelasticity kernels. Before employing the quasi-linear model of interest, we consider the several simple closures. Solutions using these equations

Table 1. Solution of the impulsive cylinder for several invertible viscoelastic kernels.

Kernel	Real Space Solution	Ref.
Viscous fluid $G(t) = \eta_0\delta(t)$	$w = w_0 \operatorname{erf} c\left[\left(\frac{1}{6\nu t}\right)^{1/2} z\right]$	[13]
Elastic solid $G(t) = GH(t)$	$w = w_0 H\left(t - \left(\frac{2\rho}{3G}\right)^{1/2} z\right)$	[13]
Maxwell liquid $G(t) = \frac{\eta_0}{\lambda} e^{-t/\lambda}$	$w = w_0 H\left(t - \left(\frac{2}{3}\frac{\lambda}{\nu_0}\right)^{1/2} z\right) e^{-t/\lambda} + \frac{r}{2}\sum_0^{t/\lambda} \frac{e^{-\xi/2}}{(\xi^2 - r^2)} I_1\left(\frac{1}{2}(\xi^2 - r^2)\right) H(r - \xi)\, d\xi$ $r \equiv z\left(\frac{2}{3}\frac{1}{\lambda\nu_0}\right)^{1/2}$	[12], [13], [18]

have been tabulated by Tanner [12,13] for the related problem of an impulsively started plate in a semi-infinite fluid field. As such, we merely list the solutions in Table 1.

Consider the more complex quasi-linear kernel of Fung [1], i.e., equation (1). Transforming equation (2) yields

$$\tilde{G}(s) = \frac{1 + c\ln(\tau_2 s + 1) - cE_1\left((\tau_2/\tau_1)s + 1\right)}{1 + c\ln\left(\tau_2/\tau_1\right)}. \tag{13}$$

It is convenient to introduce the dimensionless time, space, and velocity variables $s^* = \tau_2 s$, $z^* = z[(2/3)\rho/(\eta\tau)]^{1/2}$, $w^* = w/(w_0\tau_0)$ to yield the dimensionless problem

$$\tilde{w}(s^*, z^*) = \frac{1}{s^*}\exp\left[-s^* z^*\left(\frac{1 + c\ln\left(\tau_2/\tau_1\right)}{1 + c\ln(s^* + 1) - c\ln\left((\tau_2/\tau_1)s^* + 1\right)}\right)^{1/2}\right]. \tag{14}$$

Equation (14) is difficult to carry back to the real plane. The generalized inverse theorem or Bromwich contour [15] can be used with singular points at $s^* = 0$, $s^* = -1$, $s^* = -\tau_1/\tau_2$, however like the equation obtained for the Maxwell liquid (see [12]) will be a generalized integral that cannot be evaluated in closed form. Since a seminumerical solution will be required in any case to evaluate an infinite series, special functions, and numerical quadratures, it is more efficient to introduce a numerical inversion method. Here, we use the method of Gaver [2] and Stehfest [3].

The method of Gaver [2] and Stehfest [3] is an approximate inversion technique based upon an expectation integral relationship, asymptotic expansion methods, and convergence acceleration through linear recombination. Stehfest [3] gives a lucid summary of this methodology. It is of value, however, to briefly outline the inversion method. We start by considering the "expectation" integral

$$\begin{aligned} f_n(a, t) &= a\frac{(2n)!}{n!(n-1)}\left(1 - e^{-at}\right)e^{-nat}, \\ \bar{F}_n &\equiv \int F(t) f(a, t)\, dt = a\frac{(2n)!}{n!(n-1)}\sum_{i=0}^{n}\binom{n}{i}(-1)^i \tilde{F}((n + i)a), \end{aligned} \tag{15}$$

where $\bar{F}$ is the approximate inverse of $\tilde{F}$, the exact Laplace transform of $F(t)$. Following [3], the asymptotic expansion for $\bar{F}$ is written

$$\bar{F}_n \simeq F\left(\frac{\ln 2}{a}\right) + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots. \tag{16}$$

Rather than merely accept the $O(1/n)$ convergence implied by equation (16), a convergence acceleration technique is employed. Gaver [2] originally proposed accelerating convergence through

Aitken's "del" squared procedure [19], also known as extrapolation to the limit. Stehfest [3] demonstrates that an alternative linear combination of equation (16) for multiple samples is more effective. Defining the coefficient $x(N)$, Stehfest writes

$$F\left(\frac{\ln 2}{a}\right) = \sum_{i=1}^{N/2} x_i\left(\frac{N}{2}\right)\bar{F}_{N/2+1-i},$$
$$x_i\left(\frac{N}{2}\right) = \frac{(-1)^{i-1}}{(N/2)!}\binom{N/2}{i} i\left(\frac{N}{2}+1-i\right)^{N/2-1}, \tag{17}$$

so that using equation (15) and defining the parameter $a = \ln 2/t$, Stehfest obtains

$$\bar{F}_n = \frac{\ln 2}{t}\sum_{i=0}^{n} V_i \tilde{F}\left(\frac{\ln 2}{t}\right),$$
$$V_i = (-1)^{N/2+i}\sum_{k=(i+1)/2}^{\min(i,N/2)} \frac{k^{N/2+1}(2k)!}{(N/2-k)!(k-1)!(i-k)!(2k-i)!}. \tag{18}$$

Equation (18) provides us with the necessary inversion relationship. In the next section, solutions obtained using the numerical inversion technique of Gaver and Stehfest for the quasi-linear viscoelastic impulse cylinder problem are presented.

3. RESULTS

In this section, solutions obtained using the numerical inversion technique of Gaver and Stehfest for the quasi-linear viscoelastic impulse cylinder problem are presented. We start by considering the influence of the logarithmic damping parameter "c". The logarithmic damping parameter serves to control the magnitude of time-dependent behavior, i.e., the viscous or rate dependent portion of the solution. As shown subsequently, for $c = 0$ the problem becomes an elastic material.

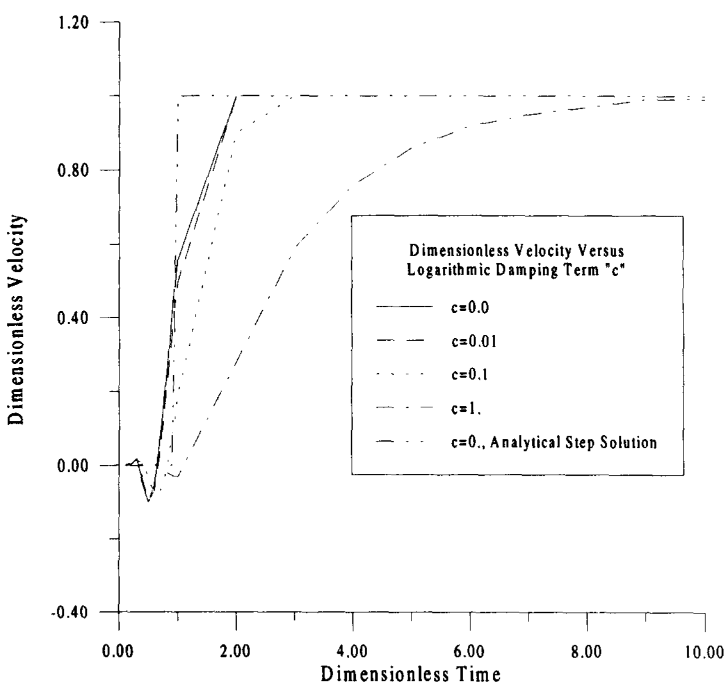

Figure 2. Dimensionless velocity versus dimensionless time for a range of logarithmic damping parameter values. The exact step function solution is shown for $c = 0$.

As a useful test of the fidelity of the numerical solver we consider the asymptotic problem with the $c \to 0$, i.e., the logarithmic function small. Under these conditions the viscoelasticity kernel collapses to unity and we recover the elastic problem given by the Heaviside step function shown in Table 1,

$$\tilde{w}(s^*, z^*) = \frac{1}{s^*} \exp[-s^* z^*], \tag{19}$$

which may be inverted to yield

$$w^* = H(t^* - z^*). \tag{20}$$

Thus, at $t^* = z^* = 1$ we have a step response. As shown by the solid line $c = 0$, the numerical Gaver-Stehfest solution in Figure 2 cannot resolve the discontinuous nature of the step-function solution. In the next section, we discuss methods to improve the solution procedure in the presence of the strong gradient. In any case, however, the solver is recovering the overall features of the solutions, including the early time and late time solutions. The location of the jump can be adequately estimated by location of the rapid transition.

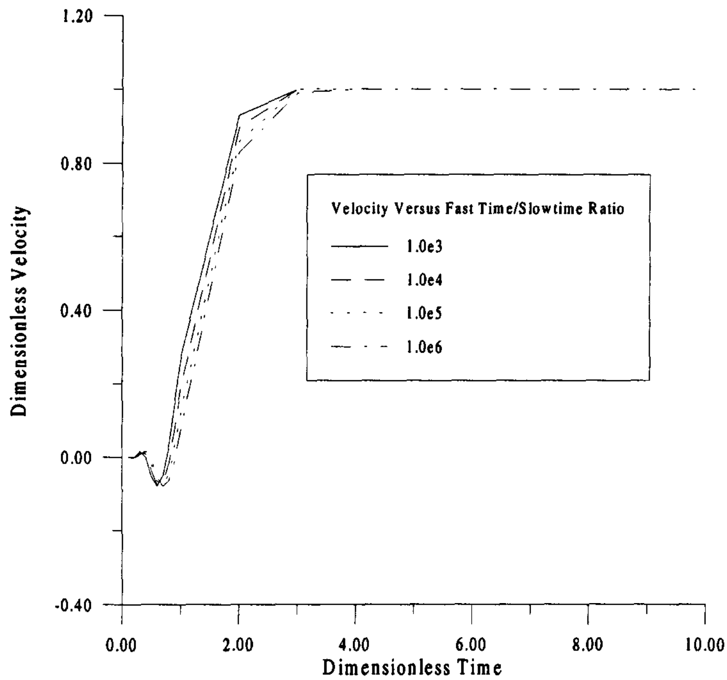

Figure 3. Dimensionless velocity versus dimensionless time for a range of fast time/slow time ratios. Notice the weak dependence of the dimensionless velocity over several decades of time ratios.

The second parameter grouping of interest is the fast time/slow time ratio, τ_2/τ_1, physically a measure of time constants associated with the viscoelastic relaxation of Figure 3, presents dimensionless velocity for a range of fast time/slow time ratios. The same numerical issues that were apparent with the log decay constant are shown here. Of more interest is the general insensitivity of the solution to the time ratio. Indeed, with over three decades of variation we see the minimal variation expressed by Figure 3. As such, we expect that measurement of the time constants associated with viscoelastic behavior is of secondary importance to identification of a model with appropriate time dependent behavior, such as a linear Maxwell liquid [12].

4. AN IMPROVED SOLUTION PROCEDURE

The previous section indicates that the Gaver-Stehfest inversion technique describes the behavior of the viscoelastic material in an adequate manner except or rapidly varying cases, i.e., damping $c \ll 1$. We attribute this behavior to the polynomial basis (consider, for example, well-known overshoot and oscillation associated with high-order Lagrange polynomial fits of data [19]) of the Gaver-Stehfest algorithm (see [20]).

Since the Gaver-Stehfest method is both accurate and widely applicable, we would like to use it as much as possible, but improve its performance in the face of large gradients. One approach to improving the current inversion method in the presence of large gradients is to reduce the strength of the gradient by introducing a "regularization function" in the Laplace plane, then inverting this function analytically back to the real plane.

Consider a rapidly varying Laplace function $w(s)$, e.g., equation (14). We can decompose $w(s)$ into a product (if only by multiplying and dividing by $u(s)$) such that we "weaken" the rapidly varying behavior of the original function. Applying this procedure to equation (14), we write

$$\tilde{w}(s^*, z^*) = \left\{ \frac{1}{s^*} \exp\left[-s^* z^* \left(\frac{1 + c\ln(\tau_2/\tau_1)}{1 + c\ln(s^*+1) - c\ln((\tau_2/\tau_1)s^* + 1)} \right)^{1/2} \right] \frac{1}{u(s)} \right\} u(s). \tag{21}$$

Experience with equation (14) indicates that equation (21) acts like a unit step-function, e.g., $H(t-z)$. For $z \ll 1$, a useful regularization function is proposed of the form

$$u(t) = u_0 + (u_\infty - u_0)\left(1 - e^{-t/a}\right), \tag{22}$$

where u_0, u_∞, and $1/a$ are free constants. Clearly, u_0 and u_∞ are chosen to match early and late time behavior of the step-like function, while $1/a$ describes the gradient of the step. For the problem considered here, $u_0 = 0$, and $u_\infty = 1$, while we choose $1/a = 1/z$ (the application of this model for the step-like function described by equation (14) is only weakly dependent on the choice for $1/a$).

Upon specification of the real form of the regularization function, the Laplace plane form can be specified. The Laplace transform of the regularization function used here, equation (22), is

$$\tilde{u}(s) = \frac{u_0}{s} + (u_\infty - u_0)\frac{1}{s(1+as)}. \tag{23}$$

With the regularization function specified, a semianalytical inversion can be performed. The semianalytical inversion to the Laplace is facilitated by the product form of the regularized step-like function since the inverse of functions in product form is determined by the well-known convolution theorem (see [16], for example). For a Laplace function of the form $F(s)G(s)$, the convolution theorem states that the inverse is

$$L^{-1}(F(s)G(s)) = \int_0^t g(\bar{t})\, f(t-\bar{t})\, d\bar{t}. \tag{24}$$

For the method developed here, the convolution form of the inverse is particularly well suited to a semianalytical approach, since the inverse of the regularization function, say $f(t) = u(t)$, is already known, while $g(t)$, here defined as the portion of equation (21) within brackets, written here as equation (25)

$$\begin{aligned} G(s) \equiv \frac{1}{s} \exp\left[-sz^* \left(\frac{1 + c\ln(\tau_2/\tau_1)}{1 + c\ln(s+1) - c\ln((\tau_2/\tau_1)s+1)} \right)^{1/2} \right] \\ \times \left[\frac{u_0}{s} + (u_\infty - u_0)\frac{1}{s(1+as)} \right]^{-1}, \end{aligned} \tag{25}$$

can be obtained numerically using the Gaver-Stehfest method. Obviously, it is assumed that the numerical inversion of equation (25) is better posed than the original relationship, equation (14).

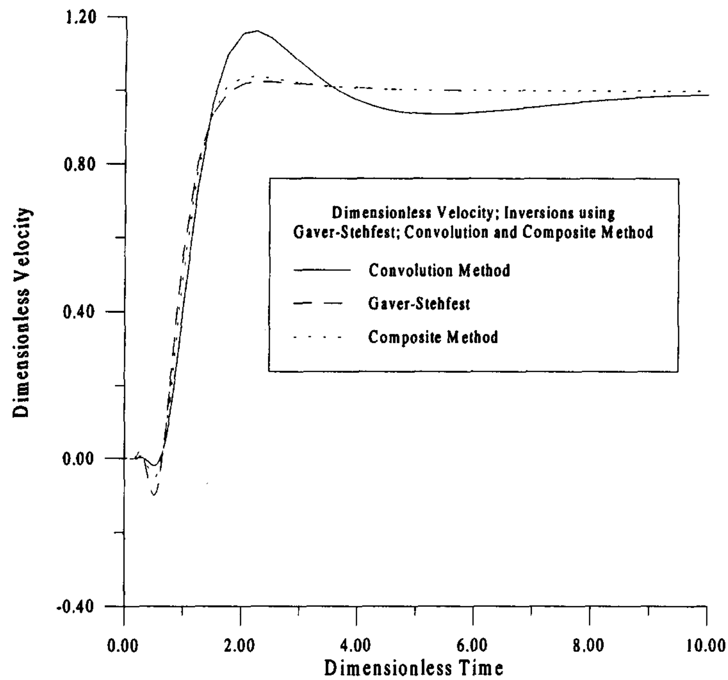

Figure 4. Velocity field obtained using: Gaver-Stehfest alone, semianalytical convolution, and composite methods for logarithmic damping factor, $c = 0.001$.

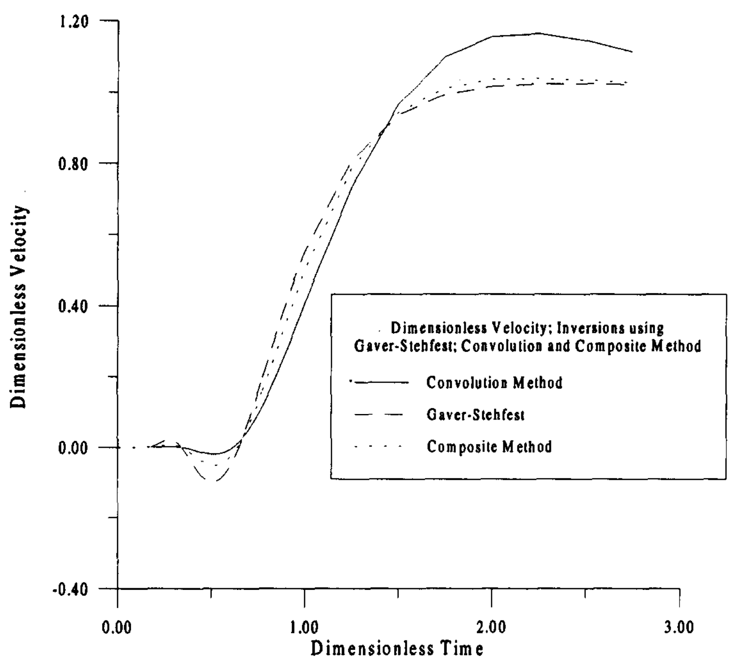

Figure 5. Velocity field obtained using: Gaver-Stehfest alone, semianalytical convolution, and composite methods for logarithmic damping factor, $c = 0.001$. Notice that the composite model yields an average between the two other methods near the step with improved undershoot and overshoot.

The integral structure of the convolution inverse is evaluated using numerical quadrature. To better handle the expected large gradients in the integral, the lower order (as compared to higher order Newton-Cotes methods, e.g., Simpsons rule) trapezoid rule is utilized. See [21] for a discussion of quadrature methods.

Though the regularization method should provide a better solution near the large gradient region, we expect that it will not improve the solution far from the step and that the full numerical method will work better. As such, it is useful to define a composite method which blends both the fully numerical Gaver-Stehfest model with the convolution model. For the viscoelasticity problems of interest in this paper, this is easily done by defining a composite model

$$w_{\text{composite}} = w_{\text{convol}}(t)e^{-t} + \left(1 - e^{-t}\right) w_{G-S}(t). \tag{26}$$

The procedure described here is applied to the viscoelastic problem shown in Figure 2 for a logarithmic damping factor, $c = 0.001$. As described, for $c \ll 1$ the inverse is virtually a step function. Figure 2 as computed using the Gaver-Stehfest algorithm alone performs well except for the oscillations preceding the step. Figure 4 presents the various solution methods: Gaver-Stehfest alone, semianalytical convolution, and composite methods.

As expected from Figure 2, the Stehfest method alone causes minor wiggles for $t < 1$ but does a good job resolving the step and the large time behavior. Figure 4 also shows that the convolution method does a better job reducing overshoot and undershoot (wiggles) previous to the step singularity, but causes overshoot and undershoot for $t > 1$. The composite method, from equation (26), combines both methods and yields an overall improved result throughout the time interval considered. Figure 5 better illustrates the value of the three methods near the singularity by focusing on the time near the step.

5. CONCLUSIONS

An analytical model for the one-dimensional, impulsive displacement of a quasi-linear viscoelastic material has been developed. The quasi-linear model of Fung [1] was chosen since it has been shown to be successful for a wide range of soft biological tissues. Due to the integral definition of linear viscoelastic materials, solutions were performed in the Laplace transform plane. In general, complex kernels like the quasi-linear model are challenging to invert back to the real plane. Here, the method of Gaver and Stehfest was used to numerically carry Laplace space solutions to the real plane. Parametric results for a basic impulsive disturbance problem were presented. Preliminary results indicate that stress wave propagation is weakly dependent on the fast time, slow time ratio and more strongly dependent on the logarithmic damping parameter. Limitations of the numerical inversion method in the face of discontinuities are discussed as well using asymptotic methods. As an alternative to the numerical/polynomial-based Gaver-Stehfest method, a semianalytical regularization function useful near large gradient regions method is developed. A composite method that utilizes both the fully numerical and semianalytical method is also described. The composite model provides improved results in terms of reducing computational undershoot and overshoot (wiggles) which limit both the fully numerical and the semianalytical models alone.

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