JOURNAL OF DIFFERENTIAL EQUATIONS 18, 103–110 (1975)

Existence of Solutions of Two Point Boundary Value Problems for Nonlinear Systems

STEPHEN R. BERNFELD

Department of Mathematics, Memphis State University, Memphis, Tennessee 38152

G. S. LADDE

Department of Mathematics, State College of New York, Potsdam, New York 13676

AND

V. LAKSHMIKANTHAM

Department of Mathematics, University of Texas at Arlington, Arlington, Texas 76010

Received July 24, 1972

1. INTRODUCTION

Let us consider the differential system

$$x'' = f(t, x, x'), \tag{1.1}$$

where $f \in C[[0, 1] \times R^d \times R^d, R^d]$, subject to the boundary conditions

$$x(0) - A_0 x'(0) = 0, \tag{1.2}$$

$$x(1) + A_1 x'(1) = 0, \tag{1.3}$$

A_0, A_1 being $d \times d$ matrices.

Recently Lasota and Yorke [4] studied the existence of solutions of the boundary value problem (1.1), (1.2) and (1.3) utilizing Leray–Schauder's alternative, while Hartman [1, 2] employed the modified function approach. Since the proofs in [4] are based on different geometric ideas from those of Hartman [1, 2], as stated in [4], it was possible to omit Nagumo's condition and to assume less restrictive conditions.

In this paper, we wish to show that whatever is achieved by the application of Leray–Schauder's alternative, can also be realized by the modified function technique under the same set of assumptions. Furthermore, our results are presented in a more general setup employing Lyapunov-like functions and the theory of differential inequalities.

2. Basic Lemmas

The proofs of our results utilize the following lemmas.

Lemma 1. *Let* $f \in C[[0, 1] \times R^d \times R^d, R^d]$, $h \in C[R^+, (0, \infty)]$ *satisfy*

$$\|f(t, x, y)\| \leqslant h(\|y\|), \qquad (t, x, y) \in [0, 1] \times R^d \times R^d,$$

and

$$\int_0^\infty \frac{s\, ds}{h(s)} = \infty.$$

Suppose that $\gamma: R^+ \to R^+$ *is the function defined by*

$$\int_\theta^{\gamma(\theta)} \frac{s\, ds}{h(s)} = \theta, \qquad \theta \in [0, \infty).$$

Let $x(t)$ *be any solution of* (1.1) *defined on* [0, 1] *and let* θ_1 *be the arc length of* $x(t)$; *that is*

$$\theta_1 = \int_0^1 \| x'(s)\| \, ds.$$

Then

$$\| x'(t)\| \leqslant \gamma(\theta_1), \qquad t \in [0, 1].$$

For proof of Lemma 1 see Lasota and Yorke [4].

Lemma 2. *Assume that*

(i) $u \in C^{(2)}[[0, 1], R^+]$, $g \in C[[0, 1] \times R^+ \times R, R^-]$, $g(t, u, v)$ *is nonincreasing in* u *for each* (t, v), *and*

$$u'' \geqslant g(t, u, u'); \tag{2.2}$$

(ii) $u'(1) \leqslant 0$ *and* $u(0) \leqslant \alpha u'(0)$ *for some* $\alpha \geqslant 0$;

(iii) $G \in C[[0, 1] \times R^+, R]$ *and there exists an* $L > 0$ *such that for* $u \geqslant L$, $t \in [0, 1]$,

$$(1/u)\, g(t, u, v) - (v/u)^2 \geqslant G(t, v/u), \tag{2.3}$$

and for any $\tau \in (0, 1]$, *the left maximal solution* $r(t, \tau, 0)$ *of*

$$z' = G(t, z), \qquad z(\tau) = 0, \tag{2.4}$$

satisfies the estimate $r(t, \tau, 0) < \alpha_0$, $t \in [0, \tau]$, *where*

$$\alpha_0 = \min(\tfrac{1}{2}, 1/\alpha), \qquad (\alpha_0 = \tfrac{1}{2} \text{ if } \alpha = 0);$$

(iv) *the left maximal solution* $r(t, 1, 0)$ *and the right minimal solution* $\rho(t, 0, 0)$ *of*

$$v' = g(t, 2L, v) \tag{2.5}$$

exists on $[0, 1]$.

Then there exists a $B_0 > 0$ *such that*

$$u(t) \leqslant B_0 \qquad \text{and} \qquad |u'(t)| \leqslant B_0, \qquad 0 \leqslant t \leqslant 1. \tag{2.6}$$

Proof. Assume that the maximum of $u(t)$ occurs at a point t_1. From the condition (ii), $u'(0) \geqslant 0$ and $u'(1) \leqslant 0$, and it follows that $u'(t_1) = 0$. Clearly $t_1 > 0$. For otherwise we would have $u(t_1) \leqslant \alpha u'(t_1) = 0$ and consequently $u(t) \equiv 0$.

We shall show that $u(t) \leqslant 2L$, $0 \leqslant t \leqslant 1$. If not, let $u(t_1) > 2L$. Define $t_0 = 0$, if $u(t) > L$ for $t \in [0, t_1]$. If not, define

$$t_0 = \sup[t \in [0, t_1]: u'(t) \geqslant \tfrac{1}{2}u(t)].$$

Since $u(t_1) > 2L$, by the mean value theorem, t_0 is well defined. It is then easily seen that

$$u'(t_0) \geqslant \alpha_0 u(t_0), \qquad L \leqslant u(t), \qquad t \in [t_0, t_1]. \tag{2.7}$$

Setting $z(t) = u'(t)/u(t)$ for $t \in [t_0, t_1]$ and using the assumption (2.3), we readily deduce that

$$z'(t) \geqslant G(t, z(t)), \qquad t \in [t_0, t_1].$$

Notice that $z(t_1) = 0$ and $z(t_0) \geqslant \alpha_0 > 0$. By the theory of differential inequalities [3], we then infer that

$$z(t) \leqslant r(t, t_1, z(t_1)), \; t \in [t_0, t_1],$$

where $r(t, t_1, z(t_1))$ is the left maximal solution of (2.4) with $\tau = t_1$. Since $z(t_1) = 0$, we see that $r(t, t_1, 0) < \alpha_0$ on $[t_0, t_1]$, and as a result, we are lead to the contradiction

$$\alpha_0 \leqslant z(t_0) \leqslant r(t_0, t_1, 0) < \alpha_0 .$$

This proves that $u(t) < 2L$ on $[0, 1]$.

Using this inequality and the nonincreasing nature of $g(t, u, v)$ we obtain

$$u'' \geqslant g(t, 2L, u').$$

Again, using the fact $u'(0) \geqslant 0$, $u'(1) \leqslant 0$ and the theory of differential inequalities [3], we get

$$u'(t) \leqslant r(t, 1, 0), \qquad 0 \leqslant t \leqslant 1,$$

and

$$u'(t) > \rho(t, 0, 0), \qquad 0 \leqslant t \leqslant 1,$$

where $r(t, 1, 0)$, $\rho(t, 0, 0)$ are, respectively, the left maximal and right minimal solutions of (2.5) which are assumed to exist on [0, 1]. Thus, we can find a $B > 0$ such that $|u'(t)| \leqslant B$, for $0 \leqslant t \leqslant 1$, where

$$B = \max[|\max_{0 \leqslant t \leqslant 1} r(t, 1, 0)|, |\min_{0 \leqslant t \leqslant 1} \rho(t, 0, 0)|].$$

The conclusion of the Lemma now follows by choosing $B_0 = \max[2L, B]$. The proof is complete.

Remark. The functions $g(t, u, v) = -k[1 + (2u)^{1/2} + |v|]$, $k > 0$, $G(t, z) = -(a + k|z| + z^2)$, are admissible in Lemma 2 where

$$La = k\left(1 + \frac{1}{2h} + h\right), \qquad L = \frac{8}{\alpha_0^2} e^{3(k+1)}, \qquad h = \frac{\alpha_0}{4} e^{-3/2(k+1)}$$

(see also Ref. [4]).

3. Modified Function Approach

Our aim is to prove the following result.

Theorem 1. *Assume that*

(a) $f \in C[[0, 1] \times R^d \times R^d, R^d]$ *and* A_0, A_1 *are positive definite or identically zero;*

(b) $V \in C^{(2)}[[0, 1] \times R^d, R^+]$, $V(t, x) \to \infty$ *as* $\|x\| \to \infty$ *uniformly in* $t \in [0, 1]$, $g \in C[[0, 1] \times R^+ \times R, R^-]$, $g(t, u, v)$ *is nonincreasing in u for each* (t, v); *and for* $(t, x, x') \in [0, 1] \times R^d \times R^d$,

$$V_f''(t, x) \geqslant g(t, V(t, x), V'(t, x)) + \sigma\|f(t, x, x')\|, \qquad \sigma > 0, \quad (3.1)$$

$$U(t, x, x') \equiv V_{tt}(t, x) + 2V_{tx}(t, x) \cdot x' + V_{xx}(t, x) \cdot x' \cdot x' \geqslant 0, \quad (3.2)$$

where $V'(t, x) = V_t(t, x) + V_x(t, x) \cdot x'$, *and*

$$V''_f(t, x) = V_{tt}(t, x) + 2V_{tx}(t, x) \cdot x' + V_{xx}(t, x) \cdot x' \cdot x' + V_x(t, x) \cdot f(t, x, x'); \tag{3.3}$$

(c) *the boundary conditions* (1.2), (1.3) *imply, for some* $\alpha \geqslant 0$, *that*

$$V'(1, x(1)) \leqslant 0 \qquad and \qquad V(0, x(0)) \leqslant \alpha V'(0, x(0)); \tag{3.4}$$

(d) $G \in C[[0, 1] \times R^+, R]$ *and there exists an* $L > 0$ *such that for* $u \geqslant L$, $t \in [0, 1]$

$$(1/u)\, g(t, u, v) - (v/u)^2 \geqslant G(t, v/u);$$

and for any $\tau \in (0, 1]$, *the left maximal solution* $r(t, \tau, 0)$ *of*

$$z' = G(t, z), \qquad z(\tau) = 0$$

satisfies the inequality $r(t, \tau, 0) < \alpha_0$, $t \in [0, \tau]$, *where* $\alpha_0 = \min(\frac{1}{2}, 1/\alpha)$, ($\alpha_0 = \frac{1}{2}$ *if* $\alpha = 0$);

(e) *the left maximal solution* $r(t, 1, 0)$ *and the right minimal solution* $\rho(t, 0, 0)$ *of*

$$v' = g(t, 2L, v)$$

exist on [0, 1].

Then there exists a solution $x \in C^{(2)}[[0, 1], R^d]$ *of the boundary value problem* (1.1), (1.2), (1.3).

Proof. Let $\delta(u, v) \in C[R^+ \times R^+, [0, 1]]$ have compact support with $\delta(u, v) \leqslant 1$ and $\delta(u, v) \equiv 1$ for $0 \leqslant u, v \leqslant B$, where B is a constant to be specified below.

Next define the modified function F of f on $[0, 1] \times R^d \times R^d$ by

$$F(t, x, x') = \delta(\| x \|, \| x' \|)\, f(t, x, x'). \tag{3.5}$$

Clearly the function F is continuous and bounded on $[0, 1] \times R^d \times R^d$. Hence [see 2, Chapter 12] there exists a solution $x \in C^{(2)}[[0, 1], R^d]$ of the boundary value problem

$$\begin{gathered} x'' = F(t, x, x'), \\ x(0) - A_0 x'(0) = 0, \qquad x(1) + A_1 x'(1) = 0. \end{gathered} \tag{3.6}$$

Set $m(t) = V(t, x(t))$ so that, because of the assumption (c), we have the relations

$$m'(1) \leqslant 0 \qquad \text{and} \qquad m(0) \leqslant \alpha m'(0). \tag{3.7}$$

Since

$$V''_F(t, x) = \delta(\|x\|, \|x'\|)\, V''_f(t, x) + [1 - \delta(\|x\|, \|x'\|)]\, U(t, x, x'), \quad (3.8)$$

we get, in view of the facts $0 \leqslant \delta(u, v) \leqslant 1$, $g(t, u, v) \leqslant 0$, $U(t, x, x') \geqslant 0$, and the assumption (3.1), the inequality

$$V''_F(t, x) \geqslant g(t, V(t, x), V'(t, x)) + \sigma \|F(t, x, x')\|,$$

which leads to the further inequality

$$m''(t) \geqslant g(t, m(t), m'(t)) + \sigma \|x''(t)\|. \quad (3.9)$$

Hence, by Lemma 2 it follows that there exists a $B_0 > 0$ such that

$$m(t) \leqslant B_0 \quad \text{and} \quad |m'(t)| \leqslant B_0, \quad 0 \leqslant t \leqslant 1.$$

As a result, setting

$$-N = [\min g(t, u, v)\colon 0 \leqslant t \leqslant 1, u \leqslant B_0, |v| \leqslant B_0],$$

we have from (3.9)

$$m''(t) \geqslant -N + \sigma \|x''(t)\|.$$

Thus, for $0 \leqslant s \leqslant t \leqslant 1$,

$$2B_0 \geqslant m'(t) - m'(s) \geqslant -N(t - s) + \sigma \left\| \int_s^t x''(\xi)\, d\xi \right\|$$
$$\geqslant -N + \sigma \|x'(t) - x'(s)\|.$$

Integrating this from 0 to 1, we obtain

$$\frac{(2B_0 + N)}{\sigma} \geqslant \int_0^1 \|x'(t) - x'(\xi)\|\, d\xi \geqslant \left\| \int_0^1 (x'(t) - x'(\xi))\, d\xi \right\|$$
$$\geqslant \|x'(t)\| - \|x(1)\| - \|x(0)\|.$$

Since $V(t, x) \to \infty$ as $\|x\| \to \infty$ uniformly in $t \in [0, 1]$, it follows, from the estimate $V(t, x(t)) = m(t) \leqslant B_0$, $0 \leqslant t \leqslant 1$, that $\|x(t)\| \leqslant B^*$, $0 \leqslant t \leqslant 1$, for some $B^* > 0$. Consequently, we deduce that

$$\|x'(t)\| \leqslant 2B^* + [(2B_0 + N)/\sigma] \equiv B, \quad 0 \leqslant t \leqslant 1.$$

Evidently, this implies that

$$\|x(t)\| \leqslant B, \quad \text{and} \quad \|x'(t)\| \leqslant B, \quad 0 \leqslant t \leqslant 1. \quad (3.10)$$

This, in view of the definition of F, assures that $x(t)$ is actually a solution of the boundary value problem (1.1), (1.2), (1.3). The proof is complete.

If f satisfies a Nagumo's condition, the assumption (3.1) may be changed as the next theorem shows.

Remark. Theorem 2 in Ref. [4] is a special case of Theorem 1 if we let

$$V(t, x) = \| x \|^2/2.$$

THEOREM 2. *Let the hypotheses* (a), (b), *and* (c) *of Theorem* 1 *hold except that the inequalities* (3.1) *and* (3.2) *be replaced by*

$$V_f''(t, x) \geqslant g(t, V(t, x), V'(t, x)) + \sigma \| x' \|, \qquad \sigma > 0, \tag{3.11}$$

$$U(t, x, x') + \tau \geqslant \sigma \| x' \|, \qquad \tau > 0. \tag{3.12}$$

Assume moreover that hypotheses (d) *and* (e) *of Theorem* 1 *hold with g and G replaced by $g_0 \equiv g - \tau$ and $G_0 \equiv G - (\tau/L)$ respectively. Suppose that $\| f(t, x, x') \| < h(\| x' \|)$ for $(t, x, x') \in [0, 1] \times R^d \times R^d$, where $h \in C[R^+, (0, \infty)]$ and*

$$\int_0^\infty \frac{s\, ds}{h(s)} = \infty.$$

Then there exists a solution $x \in C^{(2)}[[0, 1], R^d]$ of the boundary value problem (1.1), (1.2), (1.3).

Proof. We proceed exactly as in the proof of Theorem 1 until we arrive at the inequalities (3.7).

From (3.8), by letting $U = U + \tau - \tau$ and using (3.11), (3.12) we get

$$\begin{aligned} V_F''(t, x) &\geqslant g(t, V(t, x), V'(t, x)) + [\delta\sigma + (1 - \delta)\sigma] \| x' \| - \tau(1 - \delta) \\ &\geqslant g(t, V(t, x), V'(t, x)) + \sigma \| x' \| - \tau \\ &\equiv g_0(t, V(t, x), V'(t, x)) + \sigma \| x' \|. \end{aligned} \tag{3.13}$$

Since by Lemma 2, we have,

$$m(t) \leqslant B_0, \qquad | m'(t) | \leqslant B_0, \qquad 0 \leqslant t \leqslant 1,$$

the inequality (3.13) leads to

$$m''(t) \geqslant -(N + \tau) + \sigma \| x'(t) \|$$

where, as before,

$$-N = [\min g(t, u, v) \colon 0 \leqslant t \leqslant 1, u \leqslant B_0, | v | \leqslant B_0].$$

Let $\theta(t) = \int_0^t \| x'(s) \| \, ds$. Then, the preceeding inequality gives

$$2B_0 \geqslant m'(1) - m'(0) \geqslant \int_0^1 [-(N+\tau) + \sigma \| x'(s) \|] \, ds$$

$$\geqslant -(N+\tau) + \sigma\theta(1).$$

It then follows that $\theta(1) \leqslant (2B_0 + N + \tau)/\sigma \equiv M$. From Lemma 1, we then have

$$\| x'(t) \| \leqslant \gamma(\theta(1)) \leqslant \gamma(M), \qquad 0 \leqslant t \leqslant 1.$$

Letting $B = \max[B^*, \gamma(M)]$, we obtain (3.10) which concludes the proof as before.

Remark. The functions $g(t, u, v) = -k(1 + (\tau/k) + (2u)^{1/2} + | v |], k > 0$ $G(t, z) = -(a + k | z | + z^2)$, where $La = k(1 + (\tau/k) + (1/2h) + h)$, $L = [4(1 + \tau)/\alpha_0^2] \, e^{3(k+1)}$ and $h = [\alpha_0/(2(1 + \tau))^{1/2}] \, e^{-3/2(k+1)}$ are admissible. By letting $\tau = 1$, $V(t, x) = \| x \|^2/2$, and $U(t, x) = \| x' \|^2$, we obtain Theorem 3 in Ref. [4]. The proof of Theorem 3 in Ref. [4] needs to be modified in the light of our proof of Theorem 2. As it stands the inequality $u'' \geqslant \xi + \sigma | x'(t) |$ (see [4, p. 517]) does not follow as stated in the proof of Theorem 3 in [4]. In particular by redefining g, L, h in Lemma 2 of [4], as above, the proof of Theorem 3 in [4] follows by using the inequality (3.13) with τ, U and V defined as above.

References

1. P. Hartman, On the boundary value problems for systems of ordinary nonlinear, second order differential equations, *Trans. Amer. Math. Soc.* **96** (1960), 493–509.
2. P. Hartman, Ordinary differential equations, Wiley, New York, 1964.
3. V. Lakshmikantham and S. Leela, Differential and integral inequalities, Vol. 1, Academic Press, New York, 1969.
4. A. Lasota and J. A. Yorke, Existence of solutions of two point boundary value problems for nonlinear systems, *J. Differential Equations* **11** (1972), 509–518.