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Local Convergence of Inexact Newton-Like-Iterative Methods and Applications

I. K. ARGYROS
Department of Mathematics, Cameron University
Lawton, OK 73505, U.S.A.

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Abstract—We provide local convergence results in affine form for inexact Newton-like as well as quasi-Newton iterative methods in a Banach space setting. We use hypotheses on the second or on the first and second Fréchet-derivative of the operator involved. Our results allow a wider choice of starting points since our radius of convergence can be larger than the corresponding one given in earlier results using hypotheses on the first-Fréchet-derivative only. A numerical example is provided to illustrate this fact. Our results apply when the method is, for example, a difference Newton-like or update-Newton method. Furthermore, our results have direct applications to the solution of autonomous differential equations. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Banach space, Radius of convergence, Inexact Newton-like method, Nonlinear equation, Fréchet-derivative, Quasi-Newton method, Autonomous differential equation.

1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a twice differentiable nonlinear operator defined on an open convex subset D of a Banach space E_1 with values in a Banach space E_2.

In order to approximate a solution x^* of equation (1), we use inexact Newton methods of the form

$$x_{i+1} = x_i + s_i, \qquad (i \geq 0), \tag{2}$$

where $x_i \in E_1$ $(i \geq 0)$ satisfy the equation

$$F'(x_i)s_i = -F(x_i) + r_i, \qquad (i \geq 0) \tag{3}$$

for some residual sequence $\{r_i\} \subseteq E_1$ $(i \geq 0)$. In [1–4], the local convergence of iteration $\{x_i\}$ $(i \geq 0)$ to x^* was studied, under the assumption that for some $\lambda_1 \in [0,1]$ and some $\sigma_1 > 0$,

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$$F \in \mathcal{P}_{\lambda_1}(\sigma_1) = \Big\{F: U(x^*,\sigma_1) \to \mathbf{R}^j,\ j \in N, \text{ with } U(x^*,\sigma_1) = \{x \in \mathbf{R}^j / \|x - x^*\| < \sigma_1\} \subseteq D, \text{ where } F(x^*) = 0, F \text{ is Fréchet differentiable on } U(x^*,\sigma) \text{ and } F'(x)^{-1} \text{ exists for all } x \in U(x^*,\sigma_1), F' \text{ is continuous on } U(x^*,\sigma_1) \text{ and there exists } \mu_{\lambda_1} \geq 0 \text{ such that for all } y,z \in U(x^*,\sigma_1) \left\|F'(x^*)^{-1}(F'(y) - F'(z))\right\| \leq \mu_{\lambda_1}\|y - z\|^{\lambda_1}\Big\}. \tag{4}$$

In [5], we proved similar results (see Theorem 1 that follows) for a different class of operators. In particular we considered for some $\lambda \in [0,1]$ and some $\sigma > 0$,

$$F \in P_\lambda(\sigma) \equiv \{F : U(x^*,\sigma) \to E_1, U(x^*,\sigma) \subseteq D, \text{ where } F(x^*) = 0, F \text{ is twice Fréchet-differentiable on } U(x^*,\sigma), F'(x)^{-1} \text{ exists for all } x \in U(x^*,\sigma) \text{ and there exist } m_\lambda(x^*),\ b(x^*) \text{ such that for all } x \in U(x^*,\sigma) \left\|F'(x^*)^{-1}(F"(x) - F"(x))\right\| \leq \lambda(x^*)\|x - x^*\|^\lambda \text{ and } \left\|F'(x^*)^{-1}F'(x^*)\right\| \leq b(x^*)\}. \tag{5}$$

Moreover, we proved similar results for operators

$$F \in P_{\lambda,\lambda_1}(\sigma_0) = P_\lambda(\sigma) \cap \mathcal{P}_{\lambda_1}(\sigma_1), \qquad \text{where } \sigma_0 = \min\{\sigma,\sigma_1\}.$$

Here, we arrive at the conclusions derived in [4] but for the classes of operators $P_\lambda(\sigma)$ and $P_{\lambda,\lambda_1}(\sigma_0)$.

Our conditions differ from the ones in [4] unless if $m_\lambda(x^*) = 0$ and condition (4) holds for x^* replacing y. Hence, our results have theoretical as well as practical value. In particular, we show using a numerical example that our results allow a wider range of starting points since the convergence radius for operators in classes $P_\lambda(\sigma)$ or $P_{\lambda,\lambda_1}(\sigma_0)$ can be larger. Hence, it is also true that there is a wider choice for the numerical sequences involved (see sequences $\{v_i\}$, $\{\alpha_i\}$, $\{\theta_i\}$, $\{\gamma_i\}$, $(i \geq 0)$ that follow).

Our results find applications, if for example, the inexact Newton method is for example, a difference Newton-like or update Newton method. Our results find applications to the solution of autonomous differential equations (see the end of this study for an example).

2. CONVERGENCE ANALYSIS

We state the following semilocal result for inexact Newton iterates $\{x_i\}$ $(i \geq 0)$. The proof can be found in [5].

THEOREM 1. *Let $F \in P_\lambda(\sigma)$. Assume inexact Newton iterates $\{x_i\}$ $(i \geq 0)$ generated by (2),(3) satisfy*

$$\frac{\left\|s_i + F'(x_i)^{-1}F(x_i)\right\|}{\|F'(x_i)^{-1}F(x_i)\|} \equiv \frac{\left\|F'(x_i)^{-1}r_i\right\|}{\|F'(x_i)^{-1}F(x_i)\|} \leq v_i, \qquad (i \geq 0), \tag{6}$$

for some sequence $\{v_i\} \subseteq E_1$ $(i \geq 0)$. If $x_i \in U(x^*,\sigma)$, then the following hold for all $i \geq 0$:

$$\|x_{i+1} - x^*\| \leq \alpha_i \|x_i - x^*\|, \tag{7}$$

where,

$$\alpha_i = v_i + \frac{(1+v_i)\left[(1/(\lambda+1)(\lambda+2))m_\lambda(x^*)\|x_i - x^*\|^\lambda + (1/2)b(x^*)\right]\|x_i - x^*\|}{1 - b(x^*)\|x_i - x^*\| - (m_\lambda(x^*)/(\lambda+1))\|x_i - x^*\|^{\lambda+1}}. \tag{8}$$

Moreover, assume $v_i \leq v < 1$ $(i \geq 0)$. Then, iteration $\{x_i\}$ $(i \geq 0)$ converges to x^* from any $x_0 \in U(x^*,\sigma)$ satisfying

$$\|x_0 - x^*\| \leq \delta_0, \tag{9}$$

where δ_0 is the minimum positive zero of the function

$$f(t) = \frac{(1+v)+(1-v)(\lambda+2)}{(\lambda+1)(\lambda+2)} m_\lambda(x^*)t^{\lambda+1} + \frac{3-v}{2} b(x^*)t - (1-v). \tag{10}$$

Furthermore, the following estimate holds for all $i \geq 0$:

$$\|x_{i+1} - x^*\| \leq \alpha \|x_i - x^*\|, \tag{11}$$

where

$$\alpha = v + \frac{(1+v)\left[(1/(\lambda+1)(\lambda+2))m_\lambda(x^*)\|x_0 - x^*\|^\lambda + (1/2)b(x^*)\right]\|x_0 - x^*\|}{1 - b(x^*)\|x_0 - x^*\| - (m_\lambda(x^*)/(\lambda+1))\|x_0 - x^*\|^{\lambda+1}} \tag{12}$$

and $\alpha \in (0,1)$.

Finally, if $\{x_i\}$ $(i \geq 0)$ converges to x^*, then $\{x_i\}$ converges

(a) Q-superlinearly if $\lim_{i\to\infty} v_i = 0$;

(b) with R-order at least $1+\lambda$ if $\lim_{i\to\infty} v_i^{(1+\lambda)^{-1}} < 1$;

where these rates of convergence are as defined in [2,6].

As in Theorem 4.1 in [4, p. 245] we can show the following local convergence result for operators $F \in P_\lambda(\sigma)$. Note that the importance of inexact Newton-like method (13),(14) that follows was explained in great detail in [4, p. 243].

THEOREM 2. *Let $F \in P_\lambda(\sigma)$. Also, let $A_i, B_i, C_i \in L(E_1, E_2)$ such that $A_i = B_i - C_i$ $(i \geq 0)$ with A_i, B_i nonsingular. Consider the iteration $\{x_i\}$ $(i \geq 0)$ generated by*

$$x_{i+1} = x_i + s_i^{k_i}, \qquad (i \geq 0), \tag{13}$$

$$B_i s_i^{j+1} = C_i s_i^j - F(x_i), \qquad j = 0, 1, \dots, k_i - 1, \tag{14}$$

where $\{k_i\}$ $(i \geq 0)$ is a sequence of positive integers and $\{s_i^0\} \subseteq E_1$ $(i \geq 0)$ is a sequence of starting values. Set $H_i = B_i^{-1}C_i$ and define

$$\gamma_i = \left(1 + \left\|H_i^{k_i}\right\|\right) \frac{\left\|\left[I - A_i^{-1}F'(x_i)\right]F'(x_i)^{-1}F(x_i)\right\|}{\|F'(x_i)^{-1}F(x_i)\|} + \left\|H_i^{k_i}\right\| \frac{\left\|s_i^0 + F'(x_i)^{-1}F(x_i)\right\|}{\|F'(x_i)^{-1}F(x_i)\|}. \tag{15}$$

If $x_i \in U(x^*, \sigma)$ $(i \geq 0)$, then the following hold for all $i \geq 0$:

$$\|x_{i+1} - x^*\| \leq \theta_i \|x_i - x^*\|, \tag{16}$$

where

$$\theta_i = \gamma_i + \frac{(1+\gamma_i)\left[(1/(\lambda+1)(\lambda+2))m_\lambda(x^*)\|x_i - x^*\|^\lambda + (1/2)b(x^*)\right]\|x_i - x^*\|}{1 - b(x^*)\|x_i - x^*\| - (m_\lambda(x^*)/(\lambda+1))\|x_i - x^*\|^{\lambda+1}}. \tag{17}$$

Moreover, suppose that

$$\gamma_i \leq v_i \leq v < 1, \qquad (i \geq 0) \tag{18}$$

for some sequence $\{v_i\} \subseteq [0,1)$ $(i \geq 0)$. Then, the sequence $\{x_i\}$ $(i \geq 0)$ converges to x^* from any $x_0 \in U(x^*, \sigma)$ which satisfies (9) and so that error estimates (11) hold for all $i \geq 0$.

PROOF. The result follows immediately from Theorem 1 and the approximation

$$\begin{aligned} s_i^{k_i} + F'(x_i)^{-1}F(x_i) = \left(I - H_i^{k_i}\right)\left[I - A_i^{-1}F'(x_i)\right]F'(x_i)^{-1}F(x_i) \\ + H_i^{k_i}\left[s_i^9 + F'(x_i)^{-1}F(x_i)\right], \qquad (i \geq 0). \end{aligned} \tag{19}$$

We finally show the following local convergence results for quasi-Newton-iterative methods.

THEOREM 3. *Let $F \in P_\lambda(\sigma)$ for some $\lambda \in (0,1]$. Assume the following.*

(a) *There exist sequences $\{A_i\} \in L(E_1, E_2)$ and $\{x_i\} \subseteq E_1$ $(i \geq 0)$ satisfying approximations (13),(14) and such that for some $\beta_1 > 0$ and $\beta_2 > 0$*

$$\left\|I - F'(x^*)^{-1}A_{i+1}\right\| \leq \left(1 + \beta_1\sigma_i^\lambda\right)\left\|I - F'(x^*)^{-1}A_i\right\| + \beta_2\sigma_i^\lambda, \tag{20}$$

where

$$\sigma_i = \max\{\|x_i - x^*\|, \|x_{i+1} - x^*\|\}, \qquad (i \geq 0). \tag{21}$$

(b) *There exist $\eta \in [0,1)$, $\beta^i \in R$, and $\beta < +\infty$ such that conditions*

$$\|H_i\| \leq n < 1, \qquad (i \geq 0) \tag{22}$$

and

$$\frac{\left\|s_i^0 + F'(x_i)^{-1}F(x_i)\right\|}{\|F'(x_i)^{-1}F(x_i)\|} \leq \beta^i \leq \beta, \qquad (i \geq 0) \tag{23}$$

hold.

Then, there exist $\delta > 0$, $\varepsilon > 0$ and a positive integer $k = k(\delta, \varepsilon)$ such that if

$$\|x_0 - x^*\| < \delta, \quad \left\|I - F'(x^*)^{-1}A_0\right\| < \varepsilon, \tag{24}$$

$$k_i \geq k = k(\delta, \varepsilon), \qquad (i \geq 0), \tag{25}$$

then the iteration $\{x_i\}$ $(i \geq 0)$ converges to x^*. Moreover, if $B_i = A_i$ $(i \geq 0)$, then we can take

$$k_i = k(\delta, \varepsilon) = 1, \qquad (i \geq 0).$$

REMARK 1. Note that as in [4, p. 248] we are assuming $B_i \in L(E_1, E_2)$ $(i \geq 0)$ is nonsingular, whereas the invertibility of $A_i \in L(E_1, E_2)$ $(i \geq 0)$ will follow from the proof by using mathematical induction on the integer i.

PROOF. Fix $v \in (0,1)$. Let $\varepsilon, \varepsilon_1$ be such that $\varepsilon_1 > \varepsilon$ and $\varepsilon \in (0, v/(\varepsilon_1 + 2v))$. Define real functions g, h by

$$g(t) = \frac{m_\lambda(x^*)}{\lambda + 1}t^{\lambda+1} + b(x^*)t - 1 \tag{26}$$

and

$$h(t) = \frac{m_\lambda(x^*)}{\lambda + 1}t^{\lambda+1} + b(x^*)t + \varepsilon_1\varepsilon - (1 - 2\varepsilon)v. \tag{27}$$

Since $g(0) = -1 < 0$, $h(0) = \varepsilon$, $\varepsilon - (1 - 2\varepsilon)v < 0$, and for $t \to +\infty$, $g(t)$, $h(t) \to +\infty$ by the intermediate value theorem, there exist minimum positive numbers δ_1, δ_2 such that

$$g(\delta_1) = h(\delta_2) = 0, \tag{28}$$

$$g(t) < 0, \qquad t \in [0, \delta_1), \tag{29}$$

and

$$h(t) < 0, \qquad t \in [0, \delta_2). \tag{30}$$

Set $\delta_3 \in (0, \min\{\delta_0, \delta_1\})$ and define function

$$q(t) = (2\beta_1\varepsilon + \beta_2)\frac{t^\lambda}{1 - \theta_0^\lambda} - \varepsilon, \tag{31}$$

where

$$\overline{\theta}_0 \equiv v + (1 + v)\frac{(1/(\lambda+1)(\lambda+2))m_\lambda(x^*)\delta_3^{\lambda+1} + (1/2)b(x^*)\delta_3}{1 - b(x^*)\delta_3 - (m_\lambda(x^*)/(\lambda+1))\delta_3^{\lambda+1}}. \tag{32}$$

By the choice of δ_3, (10), (29), and (32), it follows that $\overline{\theta}_0 \in (0,1)$. Moreover, we have $q(0) = -\varepsilon < 0$ and for $t \to +\infty$, $q(t) \to +\infty$. Hence, there exists a minimum positive number δ_4 such that

$$q(\delta_4) = 0, \tag{33}$$

and

$$q(t) < 0, \qquad t \in [0, \delta_4). \tag{34}$$

Define $\delta \in (0, \delta_5)$, where $\delta_5 = \min\{\sigma, \delta_2, \delta_3, \delta_4\}$, and

$$p = \frac{(m_\lambda(x^*)/(\lambda+1))\delta^{\lambda+1} + b(x^*)\delta + \varepsilon_1\varepsilon}{1-2\varepsilon}. \tag{35}$$

By the choice of δ and (35), we have

$$p < v. \tag{36}$$

Define θ as θ_0 but replace δ_3 by δ. Note that for $x_0 \in U(x^*, \delta)$, it follows from (12) and the choice of θ that $\alpha \le \theta$. Let $k \equiv k(\delta, \varepsilon)$ be such that

$$\left(1 + \eta^k\right) p + \eta^k \beta \le v. \tag{37}$$

Assume that conditions (34) hold.

By the Banach Lemma in invertible operators [7,8] and the estimate $\|I - F'(x^*)^{-1}A_0\| < \varepsilon < 2\varepsilon < 1$, it follows that A_0 is nonsingular and

$$\left\|A_0^{-1}F'(x^*)\right\| \le \left[1 - \left\|I - F'(x^*)^{-1}A_0\right\|\right]^{-1}. \tag{38}$$

Using (5), (35), (36), and (38) we get from the approximations

$$\begin{aligned} A_0^{-1}F'(x_0) - I &= \left[A_0^{-1}F'(x^*)\right] F'(x^*)^{-1}\left[F'(x_0) - F'(x^*) + F'(x^*) - A_0\right], \\ F'(x_0) - F'(x^*) &= F'(x_0) - F'(x^*) - F''(x^*)(x_0 - x^*) + F''(x^*)(x_0 - x^*) \\ &= \int_0^1 \left[F''\left[x^* + t(x_0 - x^*)\right] - F''(x^*)\right](x_0 - x^*)\,dt + F''(x^*)(x_0 - x^*), \end{aligned}$$

that

$$\begin{aligned} \left\|A_0^{-1}F'(x_0) - I\right\| &\le \frac{(m_\lambda(x^*)/(\lambda+1))\delta^{\lambda+1} + b(x^*)\delta + \varepsilon}{1-\varepsilon} \\ &< \frac{(m_\lambda(x^*)/(\lambda+1))\delta^{\lambda+1} + b(x^*)\delta + \varepsilon_1\varepsilon}{1-2\varepsilon} = p < v. \end{aligned} \tag{39}$$

Let γ_i, $i \ge 0$ be given by (15), then

$$\gamma_0 \le \left(1 + \eta^{k_0}\right) p + \eta^{k_0}\beta \le v < 1. \tag{40}$$

By Theorem 2, we have $\|x_1 - x^*\| \le \theta\|x_0 - x^*\|$. Let us assume

$$\left\|I - F'(x^*)^{-1}A_i\right\| \le 2\varepsilon\, \|x_{i+1} - x^*\| \le \theta\, \|x_i - x^*\|, \qquad i = 0, 1, 2, \dots, m-1. \tag{41}$$

Hypothesis (20) can be rewritten as

$$\left\|I - F'(x^*)^{-1}A_{i+1}\right\| - \left\|I - F'(x^*)^{-1}A_i\right\| \le \sigma_i^\lambda \left(\beta_1 \left\|I - F'(x^*)^{-1}A_i\right\| + \beta_2\right). \tag{42}$$

Summing from $i = 0$ to $i = m-1$, estimate (42) gives

$$\begin{aligned} \left\|I - F'(x^*)^{-1}A_m\right\| &\le \left\|I - F'(x^*)^{-1}A_0\right\| + (2\beta_1 E + \beta_2)\sum_{i=0}^{m-1}\sigma_i^\lambda \\ &\le \varepsilon + (2\beta_1\varepsilon + \beta_2)\sum_{i=0}^{m-1}\theta^{i_\lambda}\delta^\lambda \le \varepsilon + (2\beta_1\varepsilon + \beta_2)\frac{\delta^\lambda}{1-\theta^\lambda} \le 2\varepsilon. \end{aligned} \tag{43}$$

As above it follows that A_m is nonsingular and

$$\left\|A_m^{-1}F'(x_m) - I\right\| < p < v.$$

That is $\gamma_m \leq v < 1$, and hence, $\|x_{m+1} - x^*\| \leq \theta \|x_m - x^*\|$. The induction is now complete.

That completes the proof of Theorem 2. ∎

As in [4], if $B_i = A_i$, then $C_i = 0$, so $A_i = 0$ and we can choose $\eta = 0$ in (22). Hence, (37) is satisfied for $k = 1$.

REMARK 2. For $F \in P_{\lambda,\lambda_1}(\sigma_0)$ and the rest of the hypotheses of Theorems 1–3 the conclusions also hold. Indeed the proofs can be carried out if we simply replace functions f, g by

$$\overline{f}(t) = \frac{(1+v)}{(\lambda+1)(\lambda+2)} m_\lambda(x^*) t^{\lambda+1} + \frac{(1+v)}{2} b(x^*) t + \mu_{\lambda_1}(1-v) t^{\lambda_1} - (1-v), \tag{44}$$

with $\overline{\delta}_0$ denoting the minimum positive zero of function $\overline{f}$,

$$\overline{g}(t) = 1 - \mu_{\lambda_1} \delta^{\lambda_1}, \qquad (\lambda_1 \in (0,1]), \tag{45}$$

sequence α_i by

$$\overline{\alpha}_i = v_i + \frac{(1+v_i)\left[(1/(\lambda+1)(\lambda+2)) m_\lambda(x^*) \|x_i - x^*\|^\lambda + (1/2) b(x^*)\right] \|x_i - x^*\|}{1 - \mu_{\lambda_1} \|x_i - x^*\|^{\lambda_1}}, \quad (i \geq 0), \tag{46}$$

point σ by $\overline{\sigma}$, and point α by

$$\overline{\alpha} = v + \frac{(1+v)\left[(1/(\lambda+1)(\lambda+2)) m_\lambda(x^*) \|x_0 - x^*\|^\lambda + (1/2) b(x^*)\right] \|x_0 - x^*\|}{1 - \mu_{\lambda_1} \|x_0 - x^*\|^{\lambda_1}}. \tag{47}$$

Note also that the results of Corollary 4.1, Theorem 4.2, and Theorem 4.3 in [4] hold also for classes of operators $F \in P_\lambda(\sigma)$ or $F \in P_{\lambda,\lambda_1}(\sigma_0)$.

3. APPLICATIONS

Consider operator $F\colon D \subseteq E_1 \to E_2$ which satisfies an autonomous differential equation of the form [7–9]

$$F'(x) = Q(F(x)), \qquad (x \in D), \tag{48}$$

where $Q\colon E_2 \to E_1$ is a known differentiable operator. Using (48) we get $F'(x^*) = Q(F(x^*)) = Q(0)$, and $F''(x^*) = F'(x^*)Q'(F(x^*)) = Q(0)Q'(0))$. That is, without actually knowing the solution x^*, we can use the results obtained in this study.

Below, we justify the claims made at the introduction with a numerical example.

EXAMPLE. Let $E_1 = E_2 = \mathbf{R}$, $D = U(0,1)$, and define functions F, Q by

$$F(x) = e^x - 1, \tag{49}$$

and

$$Q(x) = x + 1. \tag{50}$$

For $\sigma_0 = \sigma = \sigma_1 = 1$, $v = 0$, using (4), (5), (10), (44), (49), (50), and the radius of convergence given in [4, p. 243] by

$$\overline{\overline{\delta}}_0 = \left(\frac{(1+\lambda_1)(1-v)}{2+\lambda_1(1-v)} \mu_{\lambda_1}^{-1}\right)^{1/\lambda_1}, \tag{51}$$

we obtain $\lambda = \lambda_1 = 1$, $\mu_\lambda = m_\lambda(x^*) = e$, $b(x^*) = 1$, $\delta_0 = .5654448$, $\overline{\delta}_0 = .4364902$, and $\overline{\overline{\delta}}_0 = .245253$. That is,

$$\overline{\overline{\delta}}_0 < \overline{\delta}_0 < \delta_0. \tag{52}$$

Hence, our results provide a wider choice for initial guesses x_0 than the corresponding ones in [1–4]. This observation is important in numerical computations [1–5,7,9–11].

REFERENCES

1. P.N. Brown, A local convergence theory for combined inexact-Newton/finite-difference projection methods, *SIAM J. Numer. Anal.* **24** (2), 407–434, (1987).
2. R.S. Dembo, S.C. Eisenstat and T. Steihaug, Inexact Newton methods, *SIAM J. Numer. Anal.* **19** (2), 400–408, (1982).
3. T.J. Ypma, Local convergence of inexact Newton methods, *SIAM J. Numer. Anal.* **21** (3), 583–590, (1984).
4. T.J. Ypma, Convergence of Newton-like iterative methods, *Numer. Math.* **45**, 241–251, (1984).
5. I.K. Argyros, Relations between forcing sequences and inexact Newton iterates on Banach space, *Computing* **62**, (1999).
6. F.A. Potra, On Q-order and R-order of convergence, *SIAM J. Optimiz. Th. Appl.* **63** (3), 415–431, (1989).
7. I.K. Argyros and F. Szidarovszky, *The Theory and Application of Iteration Methods*, CRC, Boca Raton, FL, (1993).
8. L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Pergamon, Oxford, (1982).
9. I.K. Argyros, Comparing the radii of some balls appearing in connection to three local convergence theorems for Newton's method, *Southwest Journal of Pure Appl. Math.* **1**, (1998).
10. I.K. Argyros, On the convergence of some projection methods with perturbation, *J. Comput. Appl. Math.* **36**, 255–258, (1991).
11. J.M. Gutiérrez, A new semilocal convergence theorem for Newton's method, *J. Comput. Appl. Math.* **79**, 131–145, (1997).