Int. J. Production Economics 158 (2014) 359–365

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Optimizing mining rates under financial uncertainty in global mining complexes

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ARTICLE INFO

Article history:
Received 7 December 2013
Accepted 11 August 2014
Available online 28 August 2014

Keywords:
Mining rate tracking
Stochastic optimal control
Dynamic programming
Financial uncertainty
Distributed control

ABSTRACT

This paper presents a distributed and dynamic programming framework to the mining production rate target tracking of multiple metal mines under financial uncertainty. A single mine's target tracking is stated as a stochastic optimization problem and the solution is obtained by solving the dynamic program which gives the optimal production rate schedule of each mine as a Markovian feedback control on the price process. The global solution is distributed on multiple mines by a policy iteration method, and this iterative method is shown to provide the unique equilibrium among Markovian strategies. Numerical results confirm the efficacy of the proposed global method when compared to individual optimization of mining rate target tracking.

1. Introduction

A mining complex is composed of multiple mines, material types, and several processing streams including stockpiles. A global optimization framework for a mining complex should take into account the dynamics and mutual constraints of the overall complex. In this paper, we investigate mining production rate target tracking of multiple metal mines in a mining complex over the life of operations from which life of mine schedules are generated. It is important to maintain a steady mining rate during the life of mine since moving mining equipment and relocating personnel is costly. However, in changing market dynamics, the trade-off between following the planned mining rate and cost of rate change forms a dynamic stochastic optimization problem, which is termed mining rate tracking problem.

Three fundamental properties affect mining rate planning in a mining complex: metal uncertainty, financial uncertainty and inter-dependence of mines in a mining complex. First, since the metal content of each mining block is not known, the associated financial value of a block is stochastic. Traditionally, to overcome this stochasticity, scenario methodologies are applied (Ramazan and Dimitrakopoulos, 2013; Boland et al., 2008; Meagher et al., 2009), the cost function is assumed to be linear and stochastic mixed integer programming (SIP) based solutions are adopted. Since the number of mining blocks is usually very large, heuristic approaches have been applied (Lamghari and Dimitrakopoulos, 2012). An alternative is to develop sequential models to form a complete plan, as per Lerchs and Grossmann (1965) and Whittle (1988). We take a similar approach here and extend this sequential approach to multiple mines in a single mine complex with the novelty that (i) it is dynamic programming based and (ii) it takes global dependences into account in an iterative manner.

Secondly, the price of the metal is a stochastic process. Since mining rate tracking is a horizon optimization problem and the price is observed progressively on the horizon, this introduces feedback controls to the tracking problem. Lastly, there are mutual constraints that have to be addressed by all mines such as stockpiles and processing destinations that are common parts of a mining complex. Therefore, a global optimization framework is needed.

Even though financial uncertainty has been addressed less than geological uncertainty in the mining literature (Godoy, 2003), there has been progress in the recent years. Simulation-based approaches have been presented by Abdel Sabour and Dimitrakopoulos (2011), and a methodology to quantify the effect of price uncertainty within reserve estimates has been given by Evatt et al. (2012). A graph-based parametric maximum flow algorithm for developing ultimate pit limit and phase design under metal and financial uncertainty has been presented by Asad and Dimitrakopoulos (2013).

The problem discussed herein may be seen as a sub-problem under the larger problem named production scheduling of a mining complex under financial uncertainty. Ideally, this problem should be solved globally in a single stochastic mixed integer

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http://dx.doi.org/10.1016/j.ijpe.2014.08.009
0925-5273/

program. However, the computational complexity is insurmountable. In the work presented here, a distributed approach rather than a centralized framework is taken. The problem is divided into four phases, the necessary and salient dependences between the phases are established, and then solved iteratively. Phase 1 is the mining pit limit calculation of each mine and phase 2 is the calculation of individual mining production rate target functions which includes identifying independent and dependent constraints. In phase 3, mining rate trackings are solved globally for all mines considering financial uncertainty. Phase 4 includes the calculation of individual production schedules considering metal uncertainty. Note that metal uncertainties are independent of each other and independent of the financial uncertainty, which allows parallel computation in phase 4. It should be stressed that without this iterative approach it would not be possible to employ parallelization in the scheduling phase, and the resulting single stochastic mixed integer program would have an enormous number of variables and constraints, which necessarily would lead to intractability. This paper proposes a solution to the phase 3 described above. The proposed framework provides a significant reduction in computation of decision making in an environment where every decision is dependent on volatile prices. Once the extraction rates are calculated the production scheduling can be done individually for each mine taking into account each mine's individual geological uncertainties, which are in principle independent of each other.

In this paper, an optimal control framework is developed to address financial uncertainty and global optimization in mining production rate tracking for multiple mines in a mining complex. Note that, in previous work, receding control has been applied to mine production scheduling by Goodwin et al. (2006) and Rojas et al. (2007) in a deterministic framework. Herein a stochastic optimization framework is presented where individual target tracking on the horizon is shown to be in the class of Markovian feedback controls, where the price process is progressively measured, but only the instantaneous price is employed to calculate the mining rate. The stochastic properties of the price process are handled in a dynamic program. Since the individual dynamic optimizations of the mines are coupled a distributed policy iteration method is provided, and it is shown that successive iterations converge to a unique fixed point which represents the unique Nash equilibrium.

The assumptions below are made in the present work.

A1: The existence of a target extraction rate function is assumed for each mine parameterized by the price process and the extraction rates of other mines, denoted as $\Psi^k(p_t, \bar{x}_t), t \geq 0$ where $p_t, t \geq 0$ is the price process and $\bar{x}_t, t \geq 0$ denotes the average extraction rate of all mines. The target extraction rate function is strictly increasing in price and strictly decreasing in the average extraction rate of all mines. Several other parameters can be injected into this function such as the overall estimated value of the mining complex, the relative complexity of the transportation for each mine, etc. The key idea here is to group dependences into a single dynamics function where dynamic optimization can be applied. The determination of the structure of the target extraction rate function precisely requires a sensitivity analysis with respect to the selected parameters and is beyond the scope of the paper.

The mines are not independent; for instance, if all the mines increase their extraction rates, even though mines respect their individual constraints, the global constraints could be violated or stockpile capacities that are commonly used could be exceeded. It is to be noted that the results of the paper hold in the case when **A1** is generalized to a more general functional where (a) the parameter set is finite and (b) Markovian property is not violated. **A1** has been established for notational brevity.

A2: It is assumed that the metal price follows a stochastic differential equation (Schwartz, 1997) which is subject to Brownian increments, nowhere differentiable, Markovian and given by

$$dp_t = f^p(p_t; \mu)dt + \sigma(p_t)dw_t, \tag{1}$$

where $f^p(p;\mu)$ is the drift and $\sigma(p)$ is the volatility, whereas w is a standard Wiener process (Brownian motion).

The time evolution of the probability density function $\zeta(t, p_t)$ of the metal price that is modeled through (1) is given by the Fokker–Planck equation which in physics provides the evolution of the probability density function of the velocity of a particle given by

$$\partial_t \zeta(t, p_t) + \partial_p [f^p(p_t; \mu)\zeta(t, p_t)] - \frac{1}{2}\partial^2_{pp}\sigma(p_t)^2 \zeta(t, p_t) = 0, \tag{2}$$

where a closed form solution may exist depending on the properties of f^p and σ. Since the time varying distribution of the metal price is explicitly stated through a partial differential equation (PDE), stochastic mining rate target tracking can be simply formulated as a stochastic mixed integer program with recourse. However, despite its simple model, the solution would be hit by the curse of dimensionality in the uncountable and unbounded state space, and it would be computationally intractable to provide Monte Carlo solutions even if the distribution (2) is very roughly sampled. In this paper, optimal mining rate tracking is solved via a dynamic program formulation solvable in closed form; therefore the approach offers a significant complexity advantage.

Classical optimization and control theory studies problems with a single decision maker and offers tools and algorithms that can guarantee a certain performance and robustness. Decentralization of a global system immediately poses new problems to be solved such as those raised by the well-known Witsenhausen counterexample (Witsenhausen, 1968), or the stability issues which arise for systems subject to communication constraints (Nair and Evans, 2004). Viewed from this perspective, attention is needed for the utilization of parallelization. There are cases in which an equilibrium may not exist where no unilateral deviations are profitable. Even if an equilibrium exists, iterations of the distributed sub-problems might not converge to this equilibrium. In this paper it is shown that for the distributed mining rate target tracking, there exists a unique equilibrium, where no unilateral deviation is profitable, and a policy iteration method is shown to converge to this equilibrium.

The remainder of this paper is organized as follows. In Section 2 the mathematical model is introduced, where each individual optimization problem is formulated as a mining rate target tracking problem. In Section 2.1, the dynamic program is solved and the closed form solutions that generate the optimal mining rate of each mine are presented. In Section 3 the distributed algorithm is given and the convergence of the algorithm toward the equilibrium is given, where profitable unilateral deviations do not exist. In Section 4, the maximum likelihood method to calibrate the stochastic price process parameters is briefly discussed and simulation results are provided. Conclusions follow.

2. Optimal target control

The mathematical model for the mining rate tracking optimization is introduced in this section. Each mine tries to track the planned extraction trajectory in order to fulfill its planned schedule. The optimization is computed on a horizon through a cost function where both deviations from the target and change in the rate of mining are penalized. Moreover, these plans are dependent on the stochastic process $p_t, t \geq 0$, the price process.

In the remaining of the paper, mine k is denoted as agent $\mathcal{A}_k$ and K agent systems $\mathcal{A}_k, 1 \le k \le K$, and a single stochastic process $p_t, t \ge 0$, are considered, where the dynamics are defined by

$$dp_t = \mu p_t \, dt + \sigma p_t \, dw_t,$$
$$dx^k = u^k \, dt, \tag{3}$$

$t \ge 0, 1 \le k \le K$. Here μ is the drift and σ is the volatility for the metal price. The terms $x_t^k \in R$ and $u_t^k \in R$ represent, respectively, the mine extraction rate and rate change for mine k, $1 \le k \le K$, at time t. We use u to denote "control" throughout the rest of the paper since it controls the mining rate on the horizon. The process w denotes a standard Wiener process in R on a sufficiently large underlying probability space $(\Omega, \mathcal{F}, P)$ such that w is progressively measurable with respect to $\mathcal{F}^w := (\mathcal{F}_t^w, t \ge 0)$; thus, the history of the Wiener process is available at time t.

The initial states are deterministic and known by all agents. It is assumed that $\mathbb{E} ww^T = 1$, and $p_0^2 < \infty$. Denote the state configuration by $x = (x^1, \cdots, x^K)^T$ and average state by $\bar{x} = (1/K)\sum_{k=1}^K x^k$.

The individual discounted quadratic cost function that penalizes a mine's deviation from its target and control action is given by

$$J_k(u^k, u^{-k}) = \mathbb{E} \int_0^\infty e^{-\delta t} \left\{ q^k \left(x_t^k - \Psi^k(p_t; \bar{x}_t) \right)^2 + r^k (u_t^k)^2 \right\} dt,$$
$$:= \mathbb{E} \int_0^\infty e^{-\delta t} l(x_t^k, u_t^k) dt. \tag{4}$$

Note that even though the horizon extends until ∞, the target Ψ^k determines the life of mine. The coefficients $q^k, r^k \in R$ will be called the cost parameters. The function $u^k(\cdot)$ is the control input of the agent $\mathcal{A}_k, 1 \le k \le K$, and u^{-k} denotes the control inputs of the complementary set of agents $\mathcal{A}_{-k} = \{\mathcal{A}_j : j \ne k, 1 \le j \le K\}$. Note that each agent's target Ψ^k is a dynamic quantity changing in time. Each mine wants to track its optimal target; however, changing mining rate is costly and this is reflected through the penalization of action in the cost function.

2.1. Control actions of individual agents

In this section, the dynamic program that solves optimal mining rate tracking for system (3) with the cost function (4) is presented. The existence of a solution to the dynamic program is shown, its particular form is provided, and the optimal control action is obtained in analytical form. For the optimality analysis, first introduce the admissible control set $\mathcal{U}$ which consists of all feedback controls adapted to $\mathcal{F}_t^w; t \ge 0$. In other words, at time t, action u_t is allowed to use all the histories of the disturbance process up until time t.

In the framework in this paper, the agents are in a relation such that each agent's behavior evolves as a function of other agents' states as well as the stochastic process x. Since $p_t, t \ge 0$ is a stochastic process, the interaction is dynamic. Without a loss of generality, for a system where $K = 3$, the dynamics of $\mathcal{A}_1$, the evolution of variables that affect its mining rate, may be written in the form

$$\begin{bmatrix} dx_t^1 \\ dx_t^2 \\ dx_t^3 \\ dp_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \\ p_t \end{bmatrix} dt + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_t^1 dt + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_t^1 \\ u_t^2 \\ u_t^3 \\ 0 \end{bmatrix} dt + \sigma p_t \begin{bmatrix} 0 \\ 0 \\ 0 \\ dw_t \end{bmatrix}$$
$$:= Ax_t \, dt + B^1 u_t^1 \, dt + M^1 u_t \, dt + \sigma p_t (0_{1\times 3}, dw_t)^T \tag{5}$$

Note that $x_t := [x_t^1, x_t^2, x_t^3, p_t]^T, u_t := [u_t^1, u_t^2, u_t^3, 0]^T$ and the states of other agents as well as the process $p_t, t \ge 0$ are augmented to the individual state of the agent $\mathcal{A}_k$.

A3: The methodology in this paper accommodates all linear functions of p and $\bar{x}$ with a finite number of parameters. Without a loss of generality it is assumed that $\Psi^k(p, \bar{x})$ is in the form

$$\Psi^k(p, \bar{x}) = \chi + \alpha^k p - \beta \bar{x}, \tag{6}$$

where $\chi, \alpha^k, \beta \in R$.

The previous assumption is made for technical reasons in order to obtain a closed form solution. Note that the dynamic program is valid without this function; however the following closed form solutions are not attainable, and numerical PDE analyses are required. In the remainder of the paper, linearity of the tracking function will be employed to find a closed form solution to the dynamic program. For this system, the following definitions are made:

$$Q^1 := \left(1 + \frac{\beta}{3}, \frac{\beta}{3}, \frac{\beta}{3}, -\alpha^1\right)^T q^1 \left(1 + \frac{\beta}{3}, \frac{\beta}{3}, \frac{\beta}{3}, -\alpha^1\right),$$
$$\eta^1 := \left(1 + \frac{\beta}{3}, \frac{\beta}{3}, \frac{\beta}{3}, -\alpha^1\right)^T q^1 \chi.$$

For an initial state x_0 and a given $u^{-k} := u_\tau^{-k}, 0 \le \tau \le \infty$, the value function is defined as

$$V(x_0; u_0^{-k}) := \inf_{u \in \mathcal{U}} J_k(u^k, u^{-k}). \tag{7}$$

Lemma 2.1. *For all $0 \le t \le \infty$, the value function (7) is the solution to the dynamic program and is given by the Hamilton–Jacobi–Bellman equation*

$$\delta V(x_t, u_t^{-k}) - \inf_{u_t^k \in \mathcal{U}} \left\{ l(x_t, u_t^k) + \left(\partial_{x_t} V(x_t, u_t^{-k})\right)^T (Ax_t + B^k u_t^k + M^k u_t) \right\}$$
$$- \frac{1}{2} \sigma^2 \mathrm{Tr} \partial_{xx}^2 p_t V(x_t, u_t^{-k}) = 0. \tag{8}$$

The proof is given in Appendix A.

Lemma 2.2. *There exists a unique $\hat{u}^k \in \mathcal{U}$ such that $J_k(\hat{u}^k, u^{-k}) = \inf_{u^k \in \mathcal{U}} J_k(u^k, u^{-k})$, and if $\tilde{u}^k \in \mathcal{U}$ is another control such that $J_k(\tilde{u}^k, u^{-k}) = J_k(u^k, u^{-k})$, then $\mathbb{P}_\Omega(\tilde{u}^k \ne \hat{u}^k) > 0$ only on a set of times $s \in [0, \infty)$ of Lebesgue measure zero.*

The proof is generic and therefore is omitted (Fleming and Rishel, 1975).

Lemma 2.3. *For system (5) the minimum cost-to-go is quadratic in x; consequently, V can be written in the form*

$$V(x_t, u_t^{-k}) = x_t^T \Pi^k(t) x_t + 2x_t^T s^k(t; u^{-k}) + q^k(t). \tag{9}$$

The proof is given in Section 2.3 of Anderson and Moore (1989).

We inject (9) in (8) and obtain the optimal action $(u^k)^*$ together with (11) and (12):

$$(u_t^k)^* = -(r^k)^{-1} B^{kT} \left[\Pi^k x_t + s_t^k(u_t^{-k}) \right], \tag{10}$$

where

$$\delta \Pi^k = \Pi^k A + A^T \Pi^k - \Pi^k B^k (r^k)^{-1} B^{kT} \Pi^k + Q^k, \tag{11}$$

and

$$-\frac{ds_t^k}{dt} = (A - B^k (r^k)^{-1} B^{kT} \Pi^k - \delta I)^T s_t^k(u_t^{-k}) + \Pi M^k u_t - \eta^k. \tag{12}$$

Note that even though u^k is allowed to observe all the state space history up until t, the optimal control uses only the current state and current price; therefore, it is computationally easy to generate. The algebraic equation (11) is solved offline and at each time instant the optimal control needs to compute (12). Even though the solution to the dynamic program is a PDE, (8) and (12) are ordinary differential equations (ODE) and can be easily solved backwards in time. Note that even though a closed form exists for (10), there does not exist a closed form solution for (11) and it needs to be solved numerically.

3. Coupled systems

The solution to the optimal mining rate tracking was provided in the previous section; however, each agent's calculation requires all the rest of the agents' action data. In other words, the individual optimization problems are coupled; therefore the equations need to be solved altogether. The first question to answer is the existence of an action profile to the set of coupled equations. Here we prove the existence, and the action profile is unique under a set of conditions. The second question is the existence of a method to reach this equilibrium. We show in this section that the policy iteration methodology provides the unique equilibrium, where there does not exist a unilateral action that provides a smaller cost for any agent.

The proposed solution is iterative. Agent 1 calculates its optimal action, then Agent 2 calculates its response, then Agent 3 and so on. The algorithm converges with probability 1 as the iterations size tends to infinity, and practically the algorithm can be stopped at some threshold. Note that due to the stochastic process' presence the trajectories cannot be obtained offline as the group evolves by reacting to the process $p_t, t \geq 0$, which is continuously subjected to disturbances. For the group, we shall now specify the coupled equation system:

$$dx_t^k = Ax_t dt + B^k u_t^k dt + M^k u_t dt \quad \forall k,$$
$$-\frac{ds_t^k}{dt} = (A - B^k r^{k-1} B^{kT} \Pi^k - \delta I)^T s_t^k + \Pi M^k u_t - \eta \quad \forall k,$$
$$u_t^k = -r^{k-1} B^{kT}\left[\Pi^k x_t^k + s_t^k(u_t^{-k})\right] \quad \forall k. \tag{13}$$

Here we introduce $\mathcal{T}: C_b[0,\infty) \to C_b[0,\infty)$, which is the operator for (13), where C_b denotes the set of all bounded continuous functions. Hence, we can write (13) as

$$u_t = \mathcal{T}(u_t), 0 \leq t \leq \infty.$$

3.1. Equilibrium

In this section, the equilibrium properties of (13) are analyzed. At each time instant and at each point in the state space each agent solves (13). Here we show that the system of equations regarding the optimal actions of all agents in the system has a unique solution. We also present the policy iteration procedure that leads to the unique solution of the system of equations when applied by all agents in the system. Due to the stochastic process $p_t, t \geq 0$, this procedure is repeated by each agent until the fixed point is obtained at each time instant.

Theorem 3.1. *The map $\mathcal{T}: C_b[0,\infty) \to C_b[0,\infty)$ has a unique fixed point which is continuous on $[0,\infty)$.*

Proof. The existence and uniqueness of a fixed point are guaranteed by Banach's fixed point theorem through the continuity of the operator, completeness of the space of bounded continuous functions on the infinite interval and provided by the contraction argument $|\mathcal{T}(x)| < \gamma |x|$, where $\gamma < 1$ is satisfied. □

The main result of this section immediately follows Theorem 3.1.

Corollary 1. *The equation system (13) admits a unique bounded solution.*

3.2. Policy iteration

The iterative policy of an agent is now considered. At time $t \in [0,\infty)$, for a fixed iteration number $n \geq 0$ and $\tau \in [t,\infty)$, suppose that there is a priori $u_\tau \in \mathbf{C}_b[0,\infty)$. Then, the best response action of each agent is in the form $u_\tau^k(n+1) = -(r^k)^{-1} B^{kT}[\Pi^k (x_\tau^{kT}, p_\tau)^T + s_\tau^k(u_\tau(n))]$. We get the recursion for u as $u_\tau(n+1) = \mathcal{T} u_\tau(n)$. The procedure can be applied for all $t \leq \tau \leq T$, and the recursion converges to a unique $u_\tau^*, t \leq \tau \leq T$. This procedure is independently performed by each agent in the system at each time instant.

Proposition 3.2. *$\lim_{n\to\infty} u_t(n) = u_t^*$ where $u_t^* \in \mathbf{C}_b[0,\infty)$ for all $0 \leq t < \infty$.*

Proof. It has been shown in Theorem 3.1 that $\mathcal{T}$ is a contraction. Therefore, for any $u(0) \in \mathbf{C}_b$, $\lim_{n\to\infty} u(n) = u^* \in \mathbf{C}_b$ follows. □

Before we present the subgame perfect equilibrium theorem, we employ the technical assumption below:

A4: Agents can use only Markovian strategies, that is, we rule out equilibria based on future punishments.

A Markovian strategy γ_i of a player is defined to be a strategy where for each $t, \gamma_i(t,x)$ depends on $\mathcal{F}_t$, the σ-field generated by the agents' trajectories and the price process $\{x_\tau^k, p_\tau; 0 \leq \tau \leq t, 1 \leq k \leq K\}$ only through $t, \{x_t^k, p_t; 1 \leq k \leq K\}$.

Let us define Γ to be the class of mappings $\gamma: [0,\infty) \times R^{K+1} \to R$ with the property that $u(t) = \gamma(t,x)$ is adapted to $\mathcal{F}$, the σ-field generated by the agents' trajectories and the price process $\{x_\tau^k, p_\tau; 0 \leq \tau \leq t, 1 \leq k \leq K\}$. Therefore, Γ is more general than Markovian strategies. A *subgame perfect equilibrium* of the dynamic game with the set of agents $\mathbf{K}$ with dynamics (3) and with the cost functions (4) is a strategy profile $\gamma^* \in \Gamma$ such that for any history h, the strategy profile $\gamma^*|_h$ is a Nash equilibrium of the subgame based on the history h.

The following corollary follows from Proposition 3.2.

Corollary 2. *Within Markovian strategies for agents with dynamics (3), the action profile obtained when all agents apply (10) at any $t \geq 0$ with the iterative procedure described above is the unique subgame perfect equilibrium of the game.*

The system has a unique equilibrium within Markovian strategies, and the action profile corresponding to the equilibrium can be obtained by an iterative algorithm applied by each agent. The equilibrium is shown by use of a fixed point argument, and the procedure is explained by a policy iteration methodology. At each time instant each agent considers future evolution, and calculates the best response action. This procedure leads to the unique best response profile, and the equilibrium is obtained.

4. Numerical results

4.1. Data calibration

In previous sections a framework for mining rate tracking under financial uncertainty was provided. For practical implementation in order to employ the particular function in (3) as the price model, function parameters need to be calibrated with real data. In this section maximum likelihood estimates are employed in order to estimate the parameters. The Black–Scholes formula uses the log-normal diffusion which is modeled by geometric Brownian motion (GBM)

$$dp_t = \mu p_t\, dt + \sigma p_t\, dw_t, \tag{14}$$

$p_0 > 0$, where μ and σ in (5) are the percentage drift and percentage volatility respectively.

Ito's lemma is employed to write

$$\begin{aligned} d \ln P_t &= \left(\mu - \frac{1}{2}\sigma 2\right) dt + \sigma\, dw_t \\ &= \alpha\, dt + \sigma\, dw_t. \end{aligned} \tag{15}$$

Maximum likelihood is employed to estimate the non-random parameters of the geometric Brownian motion in (15). The likelihood function is defined as

$$L(\theta) = f(x; \theta),$$

a function of θ for a fixed value of data x. The maximum likelihood estimator is defined as

$$\begin{aligned} \hat{\theta} &= \arg\ \max_{\theta \in \Theta} f(x; \theta) \\ &= \arg\ \max_{\theta} L(\theta) \\ &= \arg\ \max_{\theta} l(\theta) \end{aligned}$$

for any monotonic increasing function $l(\theta)$ of $L(\theta)$.

One may discretize (14) by the Euler–Maruyama method and for $h > 0$ write

$$\log\ P_k - \log\ P_{k-1} = \left(\mu - \frac{1}{2}\sigma^2\right) h + \sqrt{h}\sigma N(0,1).$$

Now, r_k is defined as

$$r_k(h) :=\ \log\left(\frac{P_k}{P_{k-1}}\right).$$

Note that the sequence (r_k) is independent and identically distributed with $N(\alpha h, \sigma^2 h)$. Recall that Gaussian distribution is specified with the first two orders. We define $\theta = (\alpha, \sigma^2)$ and the joint measurement distribution is written as

$$f(r; \theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2 h}\sum_{k=1}^{n}(r_k - \alpha h)^2\right\}.$$

Then, after taking the logarithm, it is

$$\begin{aligned} \log\ f(r; \Theta) &= \frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2 h}\sum_{k=1}^{n} r_k^2 + \frac{1}{2\sigma^2 h}\Sigma 2 r_k \alpha h \\ &\quad - \frac{1}{2\sigma^2 h}\sum_{k=1}^{n}\alpha^2 h^2. \end{aligned}$$

We take the first derivatives with respect to α and σ^2 and obtain

$$\hat{\alpha} = \frac{1}{nh}\sum_{k=1}^{n} r_k(h), \tag{16}$$

$$\hat{\sigma}^2 = \frac{1}{nh}\sum_{k=1}^{n}(r_k(h) - \hat{\alpha} h)^2. \tag{17}$$

4.2. Example

In this section a mining complex of three gold mines is simulated for a single scenario. The optimal mining rate schedule of each mine will be calculated under the uncertainty of gold price for 20 years. First, in order to properly model price stochasticity, maximum likelihood estimates will be obtained for the parameters μ and σ in (5). Then, using the distributed policy iteration method, the mining rate schedule is obtained for the three mines.

In Fig. 1 the ounce price of gold from 1978 till 2012 is shown. Then, (16) and (17) are employed to obtain estimates. In Fig. 2 50 simulations performed using the estimates are presented, and it is seen that simulations nicely cover the actual price history except for the very beginning of the history.

The target extraction rate (6) parameters are $\chi = 100$ and $\beta = 1$ for all three mines and $\alpha = \{1.25, 1.5, 1.75\}$. These mines also have different flexibilities in changing their mining rates; therefore, r parameters are selected as $r = \{150, 125, 100\}$. A single scenario for 20 years horizon is given in Fig. 3. We see that the simulated price is highly volatile. Each mine calculates its optimal mining rate using feedback on the gold price and after 20 iterations the algorithm is stopped. At this point the mines reach an equilibrium, where each mine establishes its optimal schedule. The optimal mining rates using the proposed distributed optimal control are plotted in Fig. 4. The extraction rate of mine 1 is plotted in blue, mine 2 is plotted in green and mine 3 is plotted in red. It is seen that mines do not react uniformly to the changes in price. Mines 1 and 2 maintain a more reactive scheduling while mine 3 keeps a more steady mining rate.

In Fig. 5, for comparison we plot the optimal mining rate of each mine through individual independent optimization. In this figure, all three mines monotonically follow the changes in price. The effect of global distributed approach is clear. It treats the

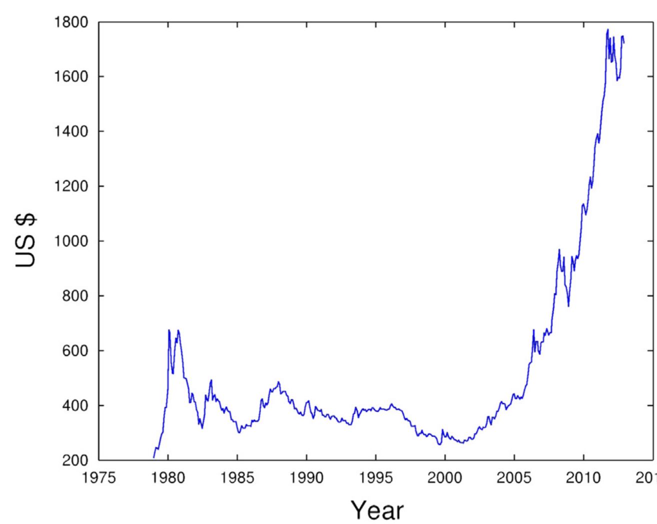

Fig. 1. Historical gold price chart.

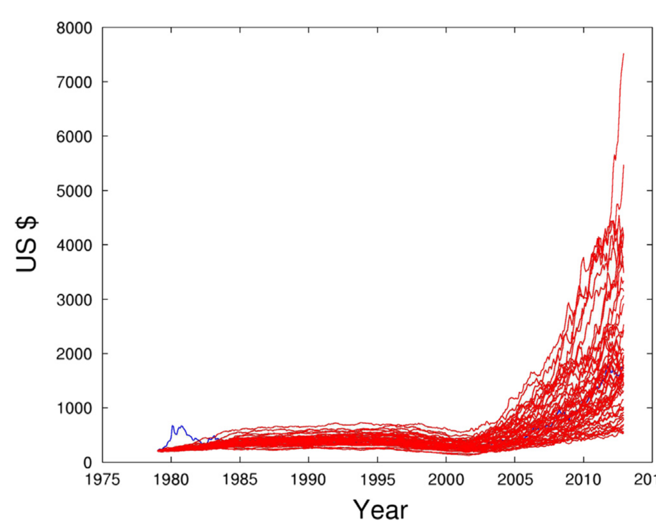

Fig. 2. Gold price simulations using the ML estimates.

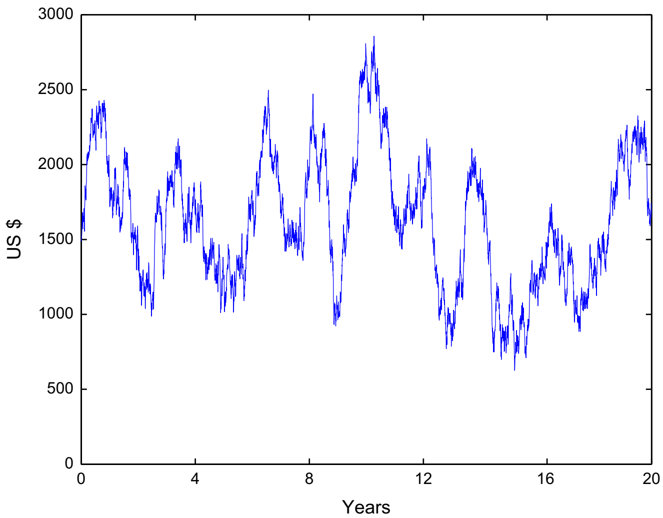

Fig. 3. Simulated fictional mine price trajectory.

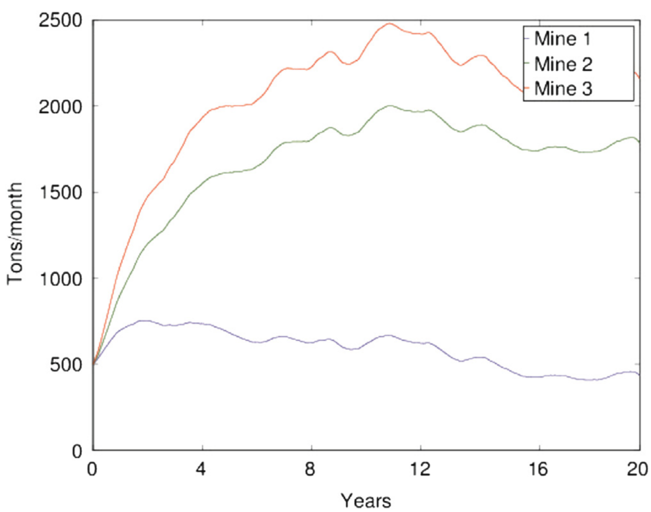

Fig. 4. Global mining rate tracking. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

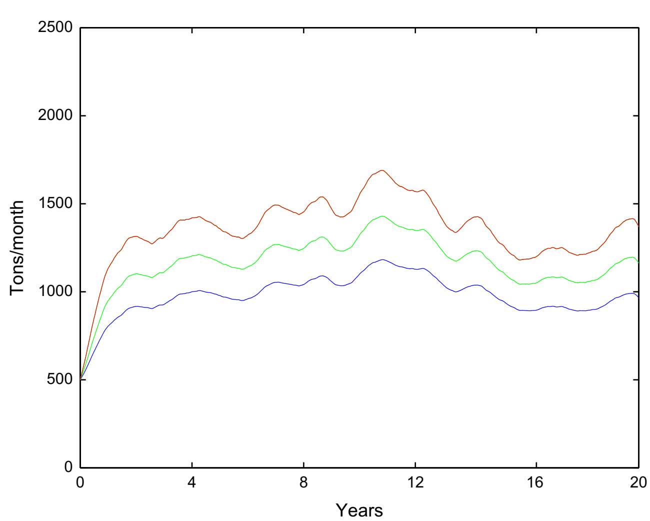

Fig. 5. Individual mining rate tracking.

mines in a non-uniform way, where mines with less penalty of deviation from their target are rewarded with more of the common resource, which leads to a smaller cost globally.

5. Conclusions

In this paper a mining rate target tracking problem has been studied, where each mine's extraction target is assumed to be a function of the stochastic price and other mines' extraction rates. Since the price process is modeled as a stochastic differential equation on a continuous horizon, a scenario method would lead to an optimization problem with a prohibitive number of variables. For individual tracking, a dynamic programming approach is employed and the problem on the continuous horizon is solved. For parallelization an iterative solution is used, and it is shown that the iterative method leads to a unique equilibrium.

This paper deals with complexity in two ways. First, we separate the individual mine scheduling problem from the global mine extraction rate target problem. The proposed methodology works on a higher level of hierarchy than the individual optimization of a single mine. This is a significant advantage because individual optimization is itself a stochastic optimization problem due to the metal content uncertainty. This way we are able to treat two uncertainties separately. Secondly we take a distributed approach and instead of trying to solve the target tracking globally, each mine extraction tracking is done in parallel in an iterative manner until the iterations converge.

Even though the paper assumes a simple tracking function class, the proposed method is easily, readily extendible to a more complicated function with a much higher number of parameters. How to characterize this function and how to calibrate these parameters with respect to the given data are subjects of future research.

Acknowledgments

The work in this paper was funded from the National Sciences and Engineering Research Council of Canada (NSERC) CDR Grant 411270 and NSERC Discovery Grant 239019; the industry consortium members of McGill's COSMO Lab: AngloGold Ashanti, Barrick Gold, BHP Billiton, De Beers, Newmont Mining, and Vale; as well as the Canada Research Chairs Program.

Appendix A

Proof of Lemma 2.1

Proof. For very small $\Delta t > 0$ we use the identity $e^{-\delta\,\Delta t} \approx 1 - \delta\,\Delta t$ and obtain

$$V(x_t, u_t^{-k}) = \inf_{u_t^k \in \mathcal{U}} \left\{ l(x_t, u_t^k)\Delta t + (1-\delta\,\Delta t)V(x_{t+\Delta t}, u_{t+\Delta t}^{-k}) \right\}. \tag{18}$$

For $V(x_{t+\Delta t}, u_{t+\Delta t}^{-k})$, Taylor series gives

$$V(x_{t+\Delta t}, u_{t+\Delta t}^{-k}) \approx V(x_t, u_t^{-k}) + \partial_t V(x_t, u_t^{-k})\Delta t + \partial_x V(x_t, u_t^{-k})\Delta x + \frac{1}{2}\mathrm{Tr}\,\partial_{xx}^2 p_t V(x_t, u_t^{-k})(\Delta x)^2,$$

which after applying Ito's rule can be written as

$$\approx V(x_t, u_t^{-k}) + \partial_t V(x_t, u_t^{-k})\Delta t + \partial_x V(x_t, u_t^{-k})(Ax_t + B^k u_t^k + M^k u_t)\Delta t + \frac{1}{2}\sigma^2 \mathrm{Tr}\,\partial_{xx}^2 p_t V(x_t, u_t^{-k})\Delta t. \tag{19}$$

We inject (19) into (18), take $\Delta t \to 0$, and obtain (8). □

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