Partial Fractions and Trigonometric Identities

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Trigonometric summations over the angles equally divided on the upper half plane are investigated systematically. Their generating functions are established by expansions of trigonometric polynomials in partial fractions. The explicit formulas are displayed and their proofs are presented in brief through the formal power series method.

For an elementary trigonometric function \( T(\theta) \), defines its power sums

\[
\sum_{k} (\pm 1)^k T^{(p)}(\theta_k)
\]

over the angles equally divided on the upper half plane. The evaluation of such sums arises in number theory and discrete Fourier series. Two elegant examples from [10, p. 234] may be reproduced as

\[
\sum_{k=1}^{n-1} \cot^2 \left( \frac{k\pi}{n} \right) = \frac{(n-1)(n-2)}{3} \quad (0.1a)
\]

\[
\sum_{k=1}^{n-1} \csc^2 \left( \frac{k\pi}{n} \right) = \frac{(n-1)(n+1)}{3} \quad (0.1b)
\]

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For the related formulas with small values of $p$, refer to the handbook [7, Section 4.4.6-7].

This article will pursue the closed formulas of such sums. By expanding the rational functions of the trigonometric polynomials in partial fractions, we will establish the generating functions. Then the formal power series method will be used to find the explicit summation formulas.

To facilitate our presentation, we will recall some facts about trigonometric expansions and binomial identities.

Recalling the Euler formulas

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

we can express trigonometric functions in terms of exponential functions

$$\sec n\theta = \frac{2e^{ni\theta}}{1 + e^{2ni\theta}} \quad (0.2a)$$

$$\csc n\theta = \frac{-2ie^{ni\theta}}{1 - e^{2ni\theta}} \quad (0.2b)$$

$$\tan n\theta = \frac{1 - e^{2ni\theta}}{1 + e^{2ni\theta}} \quad (0.2c)$$

$$\cot n\theta = -\frac{1 + e^{2ni\theta}}{1 - e^{2ni\theta}} \quad (0.2d)$$

which will be applied in combination with power series expansions

$$y = \sin \theta \Rightarrow e^{i\theta} = yi + \sqrt{1 - y^2} \quad (0.3a)$$

$$\Re\left(e^{\beta i\theta}\right) = \sum_{\ell \geq 0} (-1)^\ell \left(\frac{\beta}{2\ell}\right) y^{2\ell} \left(\sqrt{1 - y^2}\right)^{\beta - 2\ell} \quad (0.3b)$$

$$\Im\left(e^{\beta i\theta}\right) = \sum_{\ell \geq 0} (-1)^\ell \left(\frac{\beta}{1 + 2\ell}\right) y^{1 + 2\ell} \left(\sqrt{1 - y^2}\right)^{\beta - 1 - 2\ell} \quad (0.3c)$$

and

$$y = \tan \theta \Rightarrow e^{i\theta} = (1 + yi) / \sqrt{1 + y^2} \quad (0.4a)$$

$$\Re\left(e^{\beta i\theta}\right) = \sum_{\ell \geq 0} (-1)^\ell \left(\frac{\beta}{2\ell}\right) y^{2\ell} \left(\sqrt{1 + y^2}\right)^{\beta} \quad (0.4b)$$

$$\Im\left(e^{\beta i\theta}\right) = \sum_{\ell \geq 0} (-1)^\ell \left(\frac{\beta}{1 + 2\ell}\right) y^{1 + 2\ell} \left(\sqrt{1 + y^2}\right)^{\beta} \quad (0.4c)$$
For two indeterminates \( x \) and \( y \), we have binomial expansions

\[
(1 + x)^a = \sum_{k \geq 0} \binom{a}{k} x^k 
\]

(0.5a)

\[
(1 - y)^{-1 - y} = \sum_{k \geq 0} \binom{y + k}{k} y^k
\]

(0.5b)

which lead us to the Vandermonde convolutions

\[
\binom{a + c}{m} = \sum_{k=0}^m \binom{a}{k} \binom{c}{m-k}
\]

(0.6a)

\[
\binom{a + c + m + 1}{m} = \sum_{k=0}^m \binom{a + k}{k} \binom{c + m - k}{m-k}.
\]

(0.6b)

In particular, we have two binomial summation formulas

\[
\sum_{m=k}^p \binom{m}{k} = \binom{p + 1}{k + 1}
\]

(0.7a)

\[
\sum_{m=k}^p \binom{m}{k} 2^{p-m} = \sum_{j=k}^p \binom{p + 1}{j + 1}
\]

(0.7b)

where the former is a special case of (0.6b) and the latter may be derived, by means of the same convolution formula, as follows:

\[
\sum_{m=k}^p \binom{m}{k} 2^{p-m} = \sum_{m=k}^p \binom{m}{k} \sum_{\ell=m}^p \binom{p-m}{p-\ell} = \sum_{\ell=k}^p \sum_{m=\ell}^p \binom{m}{k} \binom{p-m}{p-\ell} = \sum_{\ell=k}^p \binom{1+p}{\ell-k} = \sum_{j=k}^p \binom{1+p}{1+j}.
\]

From the Vandermonde convolution formula (0.6a), we may further derive two binomial identities of James Moriarty

\[
\sum_{k=0}^n \binom{1 + 2x - \delta}{\delta + 2k} \binom{x - k - \delta}{n - k} = 2^{\delta + 2n} \binom{x + n}{\delta + 2n}
\]

(0.8a)

\[
\sum_{k=0}^n \binom{2x - \delta}{\delta + 2k} \binom{x - k - \delta}{n - k} = 2^{\delta + 2n} \frac{x - \delta / 2}{x + n} \binom{x + n}{\delta + 2n}
\]

(0.8b)
\[ \sum_{\ell \geq 0} \chi(m - 2\ell) \binom{x + \ell - m/2}{\ell} = \begin{cases} \frac{2m}{x + m/2} \binom{x + m/2}{m}, & \delta = 0 \\ \frac{1}{2}m \binom{x + m/2}{m}, & \delta = 1 \end{cases} \]  

(0.8c)

where \( \delta = 0, 1 \) is the Kronecker delta and (0.8c) is the unified form of the first two binomial identities.

In fact, denote the shifted factorial by \((x)_\ell = 1 \) and \((x)_\ell = x(x - 1) \cdots (x - n + 1) \) for \( n = 1, 2, \ldots \). Then by means of (0.6a), we can deal with the following binomial summation:

\[
\sum_{k=0}^{n} \binom{1 + 2x - \delta}{\delta + 2k} \binom{x - k - \delta}{n - k} = \sum_{k=0}^{n} \frac{(2 - 2\delta + 2x - 2k)_{\delta + 2k} (1 - \delta + x - n)_{n - k}}{\delta + 2k)!(n - k)!} = \frac{(1 - \delta + x - n)_{\delta + n}}{(1/2)_{\delta + n}} \sum_{k=0}^{n} \binom{\frac{1}{2} - \delta + x}{k} \binom{-\frac{1}{2} + \delta + n}{n - k} = \frac{(1 - \delta + x - n)_{\delta + n}}{(1/2)_{\delta + n}} \binom{x + n}{n} 
\]

which may be reduced to the right hand side of (0.8a) without difficulty.

Similarly, another identity (0.8b) may be established as follows:

\[
\sum_{k=0}^{n} \binom{2x - \delta}{\delta + 2k} \binom{x - k - \delta}{n - k} = \sum_{k=0}^{n} \frac{(1 - 2\delta + 2x - 2k)_{\delta + 2k} (1 - \delta + x - n)_{n - k}}{\delta + 2k)!(n - k)!} = \frac{(1/2 - \delta + x)_{\delta} (1 - \delta + x - n)_{n}}{(1/2)_{\delta + n}} \times \sum_{k=0}^{n} \binom{-\frac{1}{2} - \delta + x}{k} \binom{-\frac{1}{2} + \delta + n}{n - k} = \frac{(1/2 - \delta + x)_{\delta} (1 - \delta + x - n)_{n}}{(1/2)_{\delta + n}} \binom{x + n - 1}{n} = \frac{x - \delta/2}{x + n} \binom{x + n}{\delta + 2n} 2^{\delta + 2n}. 
\]
For complex numbers \( \lambda, \alpha, \beta, \gamma \) and two natural numbers \( n > p \geq 0 \), three useful binomial formulas may be displayed as

\[
0 = \sum_{m=0}^{n} \binom{n}{m} \left( \frac{-1}{\beta} \right)^{m} \sum_{k=0}^{m} \binom{m}{k} \frac{(\alpha + \beta k + m/2)}{p + m} \quad (0.9a)
\]

\[
0 = \sum_{m=0}^{n} \binom{n}{m} \left( \frac{-1}{\beta} \right)^{m} \times \sum_{k=0}^{m} \binom{m}{k} \frac{(\alpha + \beta k + m/2)}{p + m} \frac{(\alpha + \beta k - p/2)}{(\alpha + \beta k + m/2)} \quad (0.9b)
\]

\[
0 = \sum_{m=0}^{n} \binom{n}{m} \left( \frac{-1}{2\beta} \right)^{m} \sum_{k=0}^{m} \binom{m}{k} S_{m}(\alpha, \beta, \gamma, \lambda) \quad (0.9c)
\]

where

\[S_{m}(\alpha, \beta, \gamma, \lambda) = \sum_{\ell \geq 0} \binom{2\alpha + 2\beta k}{p + m - 2\ell} \binom{\alpha + \beta k + \gamma m + \lambda + \ell}{\ell}\]

and the first two identities are special cases of (0.9c) because of that

\[S_{m}(\alpha - (p - 1)/2, \beta, -1/2, -(p + 1)/2) = 2^{p+m} \left( \frac{\alpha + \beta k + m/2}{p + m} \right)\]

\[S_{m}(\alpha - p/2, \beta, -1/2, -p/2) = 2^{p+m} \left( \frac{\alpha + \beta k + m/2}{p + m} \right) \frac{\alpha + \beta k - p/2}{\alpha + \beta k + m/2}\]

in view of the binomial identities (0.8a) and (0.8b), respectively.

Let \([y^n]f(y)\) be the coefficient of \( y^n \) in the power series expansion for the function \( f(y) \) defined by

\[f(y) = \frac{e^{2ai\theta}}{(1 + y^2)^{1+a}} \left( 1 + \frac{1 - e^{2bi\theta}}{2biy(1 + y^2)} \right)^n\]

where \( \theta \) and \( y \) are two indeterminates related by (0.4a). Noting that \( f(0) = 0 \), we have

\[0 = [y^n]f(y), \quad (n > p \geq 0)\]

Now we will show that this trivial fact implies (0.9c), and so all the three multiple binomial summation formulas.
By means of (0.5a), we may reformulate \( f(y) \) in succession as

\[
f(y) = \frac{e^{2\alpha i\theta}}{(1 + y^2)^{1+\lambda}} \sum_{m=0}^{n} \left( \frac{n}{m} \right) \left( 1 - e^{2\beta i\theta} \right)^m / \left[ 2\beta iy(1 + y^2) \right]^m
\]

\[
= \sum_{m=0}^{n} \sum_{k=0}^{m} \left( \frac{n}{m} \right) \left( \frac{m}{k} \right) \frac{(-1)^k}{(2\beta)^m} a(\alpha + \beta k) i\theta / (1 + y^2)^{1+\lambda + \gamma m}
\]

\[
= \sum_{m=0}^{n} \sum_{k=0}^{m} \left( \frac{n}{m} \right) \left( \frac{m}{k} \right) \times \sum_{\iota=0}^{\infty} \left( 2\alpha + 2\beta k \right) \frac{i^{\iota-m} y^{1-m}}{(1 + y^2)^{1+\lambda + \alpha + \beta k + \gamma m}}
\]

where the power series expansion (0.4b) and (0.4d) have been used. The last sum with respect to \( \iota \) may further be expressed, with the help of (0.5b), as a formal power series

\[
\sum_{p=0}^{\infty} \left( y^p \right)^p \sum_{\iota=p+m(\mod 2)} \left( 2\alpha + 2\beta k \right) \\
\times \left( \lambda + \alpha + \beta k + \gamma m + (p + m - \iota)/2 \right)
\]

It is easy to check that the inner sum is in fact equal to \( S_m(\alpha, \beta, \gamma, \lambda) \). Therefore we find an explicit expression

\[
[y^p]f(y) = i^p \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{(-1)^k}{(2\beta)^m} \left( \frac{n}{m} \right) \left( \frac{m}{k} \right) S_m(\alpha, \beta, \gamma, \lambda)
\]

which completes the proof of (0.9c).

A. SUMMATIONS OVER \( (k\pi/n) \) WITH \( n \equiv 1(\mod 2) \)

Throughout this section, \( n \) will denote an odd natural number. Then the trigonometric function \( \sin \theta \sin n\theta \) may be considered as a polynomial of degree \( n + 1 \) in \( \cos \theta \), whose \( n + 1 \) distinct zeros are \( \{\alpha_k\}_{k=0}^{n+1} \) with \( \alpha_k = k\pi/n \). If \( P(\theta) \) is a polynomial of degree \( \leq n \) in \( \cos \theta \), we have two
expansions in partial fractions

\[
\frac{n P(\theta)}{\sin \theta \sin n \theta} = \sum_{k=0}^{n} \frac{\lambda_k}{\cos \theta - \cos \alpha_k} \tag{A.0.1a}
\]

\[
\frac{n P(\pi - \theta)}{\sin \theta \sin n \theta} = \sum_{k=0}^{n} \frac{-\lambda_k}{\cos \theta + \cos \alpha_k} \tag{A.0.1b}
\]

where the coefficients \(\{\lambda_k\}\) are determined by

\[
\lambda_k = n P(\alpha_k) \lim_{\theta \to \alpha_k} \frac{\cos \theta - \cos \alpha_k}{\sin \theta \sin n \theta}
\]

\[
= \begin{cases} 
(-1)^{k+1} P(k \pi/n), & k = 1, 2, \ldots, n-1 \\
(-1)^k P(k \pi/n)/2, & k = 0, n. 
\end{cases}
\]

From (A.0.1a) \(\pm\) (A.0.1b), we get

\[
\frac{n \{P(\pi - \theta) + P(\theta)\}}{2 \sin \theta \sin n \theta} = \frac{P(\pi) - P(0)}{2 \sin^2 \theta} - \sum_{k=0}^{n-1} (-1)^k \frac{P(k \pi/n) \cos(k \pi/n)}{\cos^2 \theta - \cos^2(k \pi/n)} \tag{A.0.2a}
\]

\[
\frac{n \{P(\pi - \theta) - P(\theta)\}}{\sin(2 \theta) \sin n \theta} = \frac{P(\pi) + P(0)}{2 \sin^2 \theta} + \sum_{k=0}^{n-1} (-1)^k \frac{P(k \pi/n)}{\cos^2 \theta - \cos^2(k \pi/n)}. \tag{A.0.2b}
\]

For \(P(\theta) = 1\) and \(\cos n \theta\), we may specify (A.0.2a) and (A.0.2b), respectively, as

\[
\frac{-n}{\sin \theta \sin n \theta} = \sum_{k=0}^{n-1} (-1)^k \frac{\cos(k \pi/n)}{\cos^2 \theta - \cos^2(k \pi/n)} \tag{A.0.3a}
\]

\[
\frac{-n \cos n \theta}{\sin \theta \sin n \theta} = \sum_{k=0}^{n-1} \frac{\cos \theta}{\cos^2 \theta - \cos^2(k \pi/n)} \tag{A.0.3b}
\]

which will be used to derive trigonometric formulas in this section.
A1. Sums of $\sec^2p(k\pi/n)$

**Definition.**

$$A_{2p}^2(n) = \sum_{k=0}^{n-1} \sec^2p\left(\frac{k\pi}{n}\right)$$  \hspace{1cm} (A1.1)

**Generating Function.**

$$\sum_{p=1}^{\infty} A_{2p}^2(n) y^{2p} = \frac{ny}{\sqrt{1-y^2}} \tan(n \arcsin y)$$  \hspace{1cm} (A1.2)

**Explicit Formulae.**

$$A_{2p}^2(n) = n \sum_{k=1}^{2p-1} (-1)^{p+k}(p-1+kn) \sum_{j=k}^{2p-1} \left(\begin{array}{c} 2p-1 \\ j+1 \end{array}\right)$$  \hspace{1cm} (A1.3)

**Examples.**

- $A_2^2(n) = n^2$
- $A_4^2(n) = \frac{n^2}{3}(2 + n^2)$
- $A_6^2(n) = \frac{n^2}{15}(8 + 5n^2 + 2n^4)$
- $A_8^2(n) = \frac{n^2}{315}(144 + 98n^2 + 56n^4 + 17n^6)$
- $A_{10}^2(n) = \frac{n^2}{2835}(1152 + 820n^2 + 546n^4 + 255n^6 + 62n^8)$.

**Sketch of Proof.** The formal power series of (A0.3b) may be stated as

$$\frac{n \cos \theta \cos n\theta}{\sin \theta \sin n\theta} = \sum_{p=1}^{\infty} \sum_{k=0}^{n-1} \cos^{2p}k\pi/n$$

where the summation order has been changed. Then the generating function (A1.2) for $(A_{2p}^2(n))$ follows immediately from this expansion under the replacement $\cos \theta \rightarrow y$ and some trivial modification.
For \( y = \sin \theta \), applying (0.3c) to the power series expansion

\[
\frac{1 - e^{2ni\theta}}{1 + e^{2ni\theta}} = \frac{(1 - e^{2ni\theta})/2}{1 - (1 - e^{2ni\theta})/2}
\]

\[
= \sum_{m > 0} 2^{-m} (1 - e^{2ni\theta})^m
\]

we may reformulate the generating function (A1.2) as

\[
\frac{ny}{\sqrt{1 - y^2}} \tan(n \arcsin y) = \frac{nyi}{\sqrt{1 - y^2}} \frac{1 - e^{2ni\theta}}{1 + e^{2ni\theta}}
\]

\[
= n \sum_{m > 0} \sum_{k = 0}^m 2^{-m} \binom{m}{k} (-1)^k \times \sum_{j > 0} (-1)^j \binom{2kn}{2j - 1} y^2(1 - y^2)^{kn-j}
\]

where the last line may be expressed, through the binomial expansion (0.5a), as a formal power series in \( y \)

\[
\sum_{p \geq 0} (-1)^p y^{2p} \sum_{j = 0}^p \binom{2kn}{2j - 1} \binom{kn - j}{p - j}.
\]

Evaluating the binomial sum with (0.8a), we obtain an explicit formula

\[
\mathcal{A}^{(p)}_{2p}(n) = n \sum_{m > 0} 2^{2p - m - 1} \sum_{k = 0}^m (-1)^{p+k} \binom{m}{k} \binom{p - 1 + kn}{2p - 1}. \quad (A1.4)
\]

The inner sum in the last expression with respect to \( k \) is the \( m \)th difference operation on a polynomial with degree \( 2p - 1 \), which vanishes for \( m \geq 2p \). Therefore we may replace the summation limit for \( m \) by
2p - 1 and restate the formula as
\[
\mathcal{M}_{2p}^u(n) = n \sum_{k=1}^{2p-1} \left( \frac{p - 1 + kn}{2p - 1} \right) (-1)^{p+k} \sum_{m=k}^{2p-1} \frac{m}{k} 2^{2p-m-1} \tag{A.15}
\]
which is equivalent to (A.13) in view of (0.7b).

A2. **Alternating Sums of** \( \sec^{1+2p}(k\pi/n) \\
**Definition.**
\[
\mathcal{B}_{1+2p}^u(n) = \sum_{k=0}^{n-1} (-1)^k \sec^{1+2p} \left( \frac{k\pi}{n} \right) \tag{A.2.1}
\]

**Generating Function.**
\[
\sum_{p=0}^{\infty} \mathcal{B}_{1+2p}^u(n) y^{1+2p} = \frac{n y \sin(n\pi/2)}{\sqrt{1-y^2}} \sec(n \arcsin y) \tag{A.2.2}
\]

**Explicit Formulae.**
\[
\mathcal{B}_{1+2p}^u(n) = n \sin \left( \frac{n\pi}{2} \right) \sum_{k=0}^{2p} (-1)^{p+k} \left( p + \frac{n - 1}{2p} + kn \right) \sum_{j=k}^{2p} \left( 1 + 2p \right) \tag{A.2.3}
\]

**Examples.** Let \( \pm \) denote the alternating signs corresponding to \( n \equiv \pm 1 \pmod{4} \) respectively. Then we have
\[
\mathcal{B}_{1}^u(n) = \pm n
\]
\[
\mathcal{B}_{2}^u(n) = \pm \frac{n}{2} \{ 1 + n^2 \}
\]
\[
\mathcal{B}_{3}^u(n) = \pm \frac{n}{24} \{ 9 + 10n^2 + 5n^4 \}
\]
\[
\mathcal{B}_{4}^u(n) = \pm \frac{n}{720} \{ 225 + 259n^2 + 175n^4 + 61n^6 \}
\]
\[
\mathcal{B}_{5}^u(n) = \pm \frac{n}{40320} \{ 11025 + 12916n^2 + 9870n^4 + 5124n^6 + 1385n^8 \}.
\]
Sketch of Proof. The expansion of (A.0.3a) in formal power series reads as

\[
\frac{n \cos \theta}{\sin \theta \sin n\theta} = \sum_{p=0}^{\infty} \sum_{k=0}^{n-1} (-1)^k \frac{\cos^{1+2p} \theta}{\cos^{1+2p}(k\pi/n)}
\]

which results in the generating function (A.2.2) for \( \mathcal{A}_{n,2p}(n) \) under the replacement \( \cos \theta \to y \) and some trivial modification.

For \( y = \sin \theta \), the combination of

\[
\frac{2e^{ni\theta}}{1 + e^{2ni\theta}} = \frac{e^{ni\theta}}{1 - (1 - e^{2ni\theta})/2} = e^{ni\theta} \sum_{m \geq 0} 2^{-m}(1 - e^{2ni\theta})^m = \sum_{m \geq 0} 2^{-m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} e^{(2k+1)i\theta}
\]

and (0.3b) enables us to rewrite the generating function (A.2.2) as

\[
\frac{ny \sin(n\pi/2)}{\sqrt{1 - y^2}} \sec(n \arcsin y)
\]

\[
= \frac{ny \sin(n\pi/2)}{\sqrt{1 - y^2}} \frac{2e^{ni\theta}}{1 + e^{2ni\theta}}
\]

\[
= n \sin(n\pi/2) \sum_{m \geq 0} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \times \sum_{j \geq 0} (-1)^j \left(\frac{n + 2kn}{2j}\right) y^{1+2j} (\sqrt{1 - y^2})^{n+2kn-2j-1}
\]

where the last line can be restated, with the help of the binomial expansion (0.5a) as a formal power series in \( y \)

\[
\sum_{p \geq 0} (-1)^p y^{1+2p} \sum_{j=0}^{p} \binom{n + 2kn}{2j} \left(\frac{n - 1}{2} + kn - j\right).
\]
In view of (0.8a), we derive an explicit formula for \( B_{1+2p}^n(n) \)

\[
B_{1+2p}^n(n) = n \sin \frac{n\pi}{2} \sum_{m=0}^{\infty} 2^{2p-m} \sum_{k=0}^{m} (-1)^{p+k} \binom{m}{k} \left( p + \frac{n-1}{2} + kn \right).
\]  
(A 2.4)

The last sum with respect to \( k \) is the \( m \)th difference operation on a polynomial with degree \( 2p \), which vanishes for \( m > 2p \). Therefore we may replace the summation limit for \( m \) by \( 2p \) and rewrite the last formula as

\[
B_{1+2p}^n(n) = n \sin \frac{n\pi}{2} \sum_{k=0}^{2p} \left( p + \frac{n-1}{2} + kn \right) (-1)^{p+k} \sum_{m=k}^{2p} \binom{m}{k} 2^{2p-m}
\]  
(A 2.5)

which is equivalent to (A 2.3) on account of (0.7b).

A 3. Sums of \( \csc^{2p}(k\pi/n) \)

**Definition.**

\[
\mathcal{C}_{2p}^n(n) = \sum_{k=1}^{n-1} \csc^{2p} \left( \frac{k\pi}{n} \right)
\]  
(A 3.1)

**Generating Function.**

\[
\sum_{p=1}^{\infty} \mathcal{C}_{2p}^n(n) y^{2p} = 1 - \frac{ny}{\sqrt{1-y^2}} \cot(n \arcsin y)
\]  
(A 3.2)

**Explicit Formulae.**

\[
\mathcal{C}_{2p}^n(n) = \delta_{0,p} - \frac{1}{2} \sum_{m=0}^{2p} \frac{(-4)^p}{n^m} \binom{1+2p}{1+m} \times \sum_{k=0}^{1+m} (-1)^k \binom{1+m}{k} \frac{1+m-2k}{1+m} \left( p + kn + \frac{m-1}{2} \right)^{2p+m}
\]  
(A 3.3a)

(A 3.3b)
Examples.

\[ E_2^u(n) = \frac{(n + 1)(n - 1)}{3} \]

\[ E_4^u(n) = \frac{(n + 1)(n - 1)}{45}(11 + n^2) \]

\[ E_6^u(n) = \frac{(n + 1)(n - 1)}{945}\{191 + 23n^2 + 2n^4\} \]

\[ E_8^u(n) = \frac{(n + 1)(n - 1)}{14175}(11 + n^2)(227 + 10n^2 + 3n^4) \]

\[ E_{10}^u(n) = \frac{(n + 1)(n - 1)}{93555}\{14797 + 2125n^2 + 321n^4 + 35n^6 + 2n^8\}. \]

Sketch of Proof. Rewrite the trigonometric identity (A.0.3b) as

\[ 1 - \frac{n \sin \theta \cos n\theta}{\cos \theta \sin n\theta} = \sum_{k=1}^{n-1} \frac{\sin^2 \theta}{\sin^2(k\pi/n) - \sin^2 \theta} = \sum_{\rho=1}^{\infty} \sum_{k=1}^{n-1} \frac{\sin^{2\rho} \theta}{\sin^{2\rho}(k\pi/n)}. \]

Then the substitution \( \sin \theta \rightarrow y \) leads us to the generating function (A.3.2). Noticing that \( 1 - e^{2ni\theta} = -2n i y \) for \( y = \sin \theta \), we can make, in succession, the power series expansions

\[ \frac{1 + e^{2ni\theta}}{1 - e^{2ni\theta}} = \frac{1 + e^{2ni\theta}}{-2ni y} \left(1 - \left(1 + \frac{1 - e^{2ni\theta}}{2ni y}\right)^\ell\right) \]

\[ = \frac{1 + e^{2ni\theta}}{-2ni y} \sum_{\ell \geq 0} \left(1 + \frac{1 - e^{2ni\theta}}{2ni y}\right)^\ell \]

\[ = \frac{1 + e^{2ni\theta}}{-2ni y} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \left(1 - \frac{e^{2ni\theta}}{2ni y}\right)^m \]

\[ = \frac{1 + e^{2ni\theta}}{-2ni y} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} e^{2nk i \theta} \left(\frac{1 - e^{2ni\theta}}{2ni y}\right)^m \]

\[ = \frac{-1}{2ni y} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \]

\[ \times \sum_{k=0}^{1+m} (-1)^k \frac{1 + m - 2k}{1 + m} \binom{m}{k} e^{2nk i \theta} \left(\frac{1 - e^{2ni\theta}}{2ni y}\right)^m. \]
Then we may reformulate, through the formal power series (0.3b) and (0.3c), the generating function (A.3.2) as

\[
1 - \frac{ny}{\sqrt{1 - y^2}} \cot(n \arcsin y)
\]

\[
= 1 + \frac{ny}{\sqrt{1 - y^2}} \frac{1 + e^{2n\Theta}}{1 - e^{2n\Theta}}
\]

\[
= 1 - \frac{1}{2} \sum_{\ell \geq 0} \sum_{m = 0}^{\ell+m} \left( m \right) \sum_{k = 0}^{1+m} (-1)^k \left( \frac{1+m}{k} \right) \frac{1+m-2k}{1+m}
\]

\[
\times \sum_{i = m \mod 2} \left( -1 \right)^{\left( m - i \right)/2} \frac{2kn}{\left( 2ny \right)^m} \left( \sqrt{1 - y^2} \right)^{2kn-1-i}
\]

where the last line can be restated, by means of the binomial expansion (0.5a), as a formal Laurent series in \( y \)

\[
\sum_p \frac{(-1)^p y^{2p}}{(2n)^m} \sum_{i = m \mod 2} \left( \frac{2kn}{i} \right) \left( \frac{kn - (1 + i)/2}{p + (m - i)/2} \right).
\]

Evaluating the inner sum by (0.8a), we get an explicit formula for \( C_{2p}^q(n) \)

\[
C_{2p}^q(n) = \delta_{0, p} - \frac{1}{2} \sum_{\ell \geq 0} \sum_{m = 0}^{\ell} \left( \ell \right) \frac{(-4)^p}{n^m}
\]

\[
\times \sum_{k = 0}^{1+m} (-1)^k \left( \frac{1+m}{k} \right) \frac{1+m-2k}{1+m} \left( \frac{p + kn + m - 1}{m + 2p} \right).
\]

According to (0.9a), the double sum with respect to \( m \) and \( k \) will vanish if \( \ell > 2p \). Therefore we may replace the upper limit for the summation variable \( \ell \) by \( 2p \) and simplify the summation through (0.7a), which result in (A.3.3).

A4. Alternating Sums of \( \cos(k \pi/n) \csc^2p(k \pi/n) \)

**Definition.**

\[
D_{2p}^q(n) = \sum_{k = 1}^{n-1} (-1)^k \cos \left( \frac{k \pi}{n} \right) \csc^2p \left( \frac{k \pi}{n} \right)
\]

(A4.1)
Generating Function.

\[
\sum_{p=1}^{\infty} \mathcal{D}_p^{n}(n) y^{2p} = 1 - ny \csc(n \arcsin y) \quad \text{(A 4.2)}
\]

Explicit Formulae.

\[
\mathcal{D}_p^{n}(n) = \delta_{0,p} - \sum_{m=0}^{2p} \frac{(-4)^p}{n^m} \left( \frac{1 + 2p}{1 + m} \right) \\
\quad \times \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{n + 2kn}{m + 2p + n + 2kn} \left( p + kn + \frac{m + n}{2} \right) \quad \text{(A 4.3a)}
\]

\[
\mathcal{D}_p^{n}(n) \quad \text{(A 4.3b)}
\]

Examples.

\[
\mathcal{D}_2^{n}(n) = \frac{(n + 1)(1 - n)}{6}
\]

\[
\mathcal{D}_4^{n}(n) = \frac{(n + 1)(1 - n)}{360} (17 + 7n^2)
\]

\[
\mathcal{D}_6^{n}(n) = \frac{(n + 1)(1 - n)}{15120} \{367 + 178n^2 + 31n^4\}
\]

\[
\mathcal{D}_8^{n}(n) = \frac{(n + 1)(1 - n)}{1814400} \{13 + 3n^2\} (2143 + 610n^2 + 127n^4)
\]

\[
\mathcal{D}_{10}^{n}(n) = \frac{(n + 1)(1 - n)}{119750400} \times \{1295803 + 689428n^2 + 192162n^4 + 31892n^6 + 2555n^8\}.
\]

Remark. It is trivial to check, by symmetry, that

\[
\sum_{k=1}^{n-1} (-1)^k \csc^{1+2p} k \frac{\pi}{n} = 0.
\]
Sketch of Proof. Reformulate the trigonometric identity (A.0.3a) as

\[ 1 - \frac{n \sin \theta}{\sin n\theta} = \sum_{k=1}^{n-1} \left( -1 \right)^k \frac{\sin^2 \theta \cos \left( \frac{k\pi}{n} \right)}{\sin^2 \left( \frac{k\pi}{n} \right) - \sin^2 \theta} \]

\[ = \sum_{p=1}^{\infty} \sin^{2p} \theta \sum_{k=1}^{n-1} \left( -1 \right)^k \frac{\cos \left( \frac{k\pi}{n} \right)}{\sin^{2p} \left( \frac{k\pi}{n} \right)} . \]

Then the substitution \( \sin \theta \rightarrow y \) leads us to the generating function (A.4.2).

For \( y = \sin \theta \), we have the power series expansions similar to those in A.3

\[ \frac{e^{ni\theta}}{1 - e^{2ni\theta}} = \frac{e^{ni\theta}}{-2niy} \left( 1 - \left( \frac{1 + e^{2ni\theta}}{2niy} \right) \right) \]

\[ = \frac{e^{ni\theta}}{-2niy} \sum_{\ell \geq 0} \left( \frac{1 + e^{2ni\theta}}{2niy} \right)^\ell \]

\[ = \frac{e^{ni\theta}}{-2niy} \sum_{\ell \geq 0} \sum_{m=0}^\ell \left( \frac{\ell}{m} \right) \frac{(1 - e^{2ni\theta})^m}{(2niy)^m} \]

\[ = -\frac{1}{2niy} \sum_{\ell \geq 0} \sum_{m=0}^\ell \left( \frac{\ell}{m} \right) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{e^{(1+2k)n\theta}}{(2niy)^m} \]

which enable us to restate, through (0.3b) and (0.3c), the generating function (A.4.2) as a multiple sum

\[ 1 - ny \csc \left( n \arcsin \frac{y}{n} \right) \]

\[ = 1 + \frac{2niye^{ni\theta}}{1 - e^{2ni\theta}} \]

\[ = 1 - \sum_{\ell \geq 0} \sum_{m=0}^\ell \left( \frac{\ell}{m} \right) \sum_{k=0}^{m} (-1)^k \]

\[ \times \sum_{\ell = m \text{ (mod 2)}} \left( -1 \right)^{(m-k)/2} \frac{(m-k)(n+2kn)}{(2ny)^m} \left( \sqrt{1 - \frac{y^2}{n^2}} \right)^{n+2kn-\ell} \]
where the last line can be expressed, by means of the binomial expansion (0.5a) as a formal Laurent series in $y$

$$\sum_{p} \frac{(-1)^p y^{2p}}{(2n)^m} = \sum_{\ell=m \text{(mod 2)}} \left( n + 2kn \right) \left( p + (m - \ell)/2 \right).$$

Evaluating the sum with respect to $\ell$ by (0.8b), we obtain an explicit formula

$$\mathcal{D}_{2\ell}^a(n) = \delta_{0,\ell} - \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \frac{(-4)^p}{n^m}
\times \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{nk + n/2}{p + nk + (m + n)/2} \left( p + kn + \frac{m + n}{2} \right).$$

According to (0.9b), the double sum with respect to $m$ and $k$ will vanish if $\ell > 2p$. Therefore we may replace the upper limit for the summation variable $\ell$ by $2p$ and simplify the summation through (0.7a), which lead us to (A 4.3).

A5. Sums of $\tan^{2p}(k\pi/n)$

**Definition.**

$$\mathcal{D}_{2\ell}^a(n) = \sum_{k=0}^{n-1} \tan^{2p}\left( \frac{k\pi}{n} \right)$$

**Generating Function.**

$$\sum_{p=0}^{\infty} \mathcal{D}_{2\ell}^a(n) y^{2p} = \frac{n}{1 + y^n \left( 1 + y \tan(n \arctan y) \right)}$$

**Explicit Formulae.**

$$\mathcal{D}_{2\ell}^a(n) = n(-1)^p + (-1)^p \sum_{m=0}^{2p} \sum_{k=0}^{m} \binom{m}{k}$$
$$\times (-1)^k \sum_{\ell=1}^{p} \binom{2nk}{2k-1} \binom{nk + p - \ell}{p - \ell}$$

(A 5.3a)

(A 5.3b)
Examples.

\[ \mathcal{E}^n_2(n) = n(n-1) \]
\[ \mathcal{E}^n_4(n) = \frac{n(n-1)}{3} \left( n^2 + n - 3 \right) \]
\[ \mathcal{E}^n_6(n) = \frac{n(n-1)}{15} \left( 2n^4 + 2n^3 - 8n^2 - 8n + 15 \right) \]
\[ \mathcal{E}^n_8(n) = \frac{n(n-1)}{315} \left( 17n^6 + 17n^5 - 95n^4 + 95n^3 + 213n^2 + 213n - 315 \right) \]
\[ \mathcal{E}^n_{10}(n) = \frac{n(n-1)}{2835} \left( 62n^8 + 62n^7 - 448n^6 - 448n^5 + 1358n^4 + 1358n^3 - 2232n^2 - 2232n + 2835 \right). \]

Remark. Among the results displayed above, \( \mathcal{E}^n_2(n) \) yields an elegant summation formula

\[ \sum_{k=1}^{n-1} \tan^2 \left( \frac{k\pi \theta}{n} \right) = n(n-1) \]

due to Heinz-Jürgen Seiffert [8], which is a primary motivation for the present work.

Sketch of Proof. According to the definition of \( \mathcal{E}^n_{2p}(n) \), we may formally compute its generating function by series rearrangement

\[
\sum_{p=0}^{\infty} \mathcal{E}^n_{2p}(n) y^{2p} = \sum_{k=0}^{n-1} \frac{1}{1 - y^2 \tan^2 (k\pi/n)} = \sum_{k=0}^{n-1} \frac{\cos^2 (k\pi/n)}{(1 + y^2) \cos^2 (k\pi/n) - y^2} = \frac{n}{1 + y^2} + \frac{1}{1 + y^2} \sum_{k=0}^{n-1} \frac{y^2/(1 + y^2)}{\cos^2 (k\pi/n) - y^2/(1 + y^2)}
\]

which reduces to (A.5.2) by means of (A.0.3b) with the replacements \( \cot \theta \to y \) and \( \cot(n\theta) \to \tan(n \arctan y) \).
For \( y = \tan \theta \), make the power series expansion

\[
\frac{1 - e^{2ni\theta}}{1 + e^{2ni\theta}} = \frac{(1 - e^{2ni\theta})}{2} \frac{1}{1 - (1 - e^{2ni\theta})/2} = \sum_{m=0}^\infty 2^{-m} (1 - e^{2ni\theta})^m = \sum_{m=0}^\infty 2^{-m} \sum_{k=0}^m (-1)^k \binom{m}{k} e^{2ni \theta}.
\]

By means of (0.4c), we may rewrite the generating function (A5.2) as

\[
\frac{n}{1 + y^2} \{ 1 + y \tan (n \arctan y) \} = \frac{n}{1 + y^2} + \frac{ny}{1 + y^2} \frac{1 - e^{2ni\theta}}{1 + e^{2ni\theta}} = \frac{n}{1 + y^2} + \sum_{m=0}^\infty n \sum_{k=0}^m \binom{m}{k} (-1)^k \times \sum_{j=0}^\infty (-1)^j \left( \frac{2kn}{2j - 1} \right) y^{2j} / (1 + y^2)^{1+k+n}.
\]

Noting that the last line can be restated, with the help of the binomial expansion (0.5b), as a formal power series in \( y \)

\[
\sum_{p \geq 0} (-1)^p y^{2p} \sum_{j=1}^p \binom{2kn}{2j - 1} \binom{kn + p - j}{p - j}
\]

we obtain an explicit formula for \( \mathcal{E}_a^{2p}(n) \)

\[
\mathcal{E}_a^{2p}(n) = n(-1)^p + \sum_{m=0}^\infty \frac{n}{2^m} \sum_{k=0}^m \binom{m}{k} (-1)^{k+p} \binom{2kn}{2j - 1} \binom{kn + p - j}{p - j}.
\]

The last sum with respect to \( k \) is the \( m \)th difference operation on a polynomial with degree \( 2p - 1 \), which vanishes for \( m \geq 2p \). Therefore we may replace the summation limit for \( m \) by \( 2p \) and the desired formula (A5.3) follows immediately.
The summation formula (A5.3) may also be reformulated as

\[ F_{2p}^n = n(-1)^p + \frac{n}{4^p} \sum_{m=0}^{2p} \sum_{k=0}^{m} \left( \frac{1 + 2p}{1 + m} \right) (-1)^{k+p} \]  
\[ \times \sum_{j=1}^{p} \binom{2kn}{2j-1} \binom{kn+p-j}{p-j} \]  
\[ A5.5a \]  
\[ A5.5b \]

by means of the binomial transform (0.7b).

A6. Alternating sums of $\sec(k\pi/n)\tan^2(\pi/n)$

**Definition.**

\[ F_{2p}^n(n) = \sum_{k=0}^{n-1} (-1)^k \sec\left( \frac{k\pi}{n} \right) \tan^2\left( \frac{k\pi}{n} \right) \]  
\[ A6.1 \]

**Generating Function.**

\[ \sum_{p=0}^{\infty} F_{2p}^n(n) y^{2p} = \frac{n \sin(n\pi/2)}{\sqrt{1 + y^2}} \sec(n \arctan y) \]  
\[ A6.2 \]

**Explicit Formulae.**

\[ F_{2p}^n(n) = \sin\left( \frac{n\pi}{2} \right) \sum_{m=0}^{2p} \frac{n}{2^m} \sum_{k=0}^{m} (-1)^{k+p} \binom{m}{k} \]  
\[ \times \sum_{\ell=0}^{p} \binom{n + 2nk}{2\ell} \binom{nk + \frac{n-1}{2} + p - \ell}{p - \ell} \]  
\[ A6.3a \]  
\[ A6.3b \]

**Examples.** Let $\pm$ denote the alternating signs corresponding to $n = \pm 1 \pmod{4}$ respectively. Then we have

\[ F_0^2(n) = \pm n \]
\[ F_2^2(n) = \pm \frac{n(n+1)(n-1)}{2} \]
\[ F_4^2(n) = \pm \frac{n(n+1)(n-1)}{24} (5n^2 - 9) \]
\[ F_6^2(n) = \pm \frac{n(n+1)(n-1)}{720} (61n^4 - 214n^2 - 225) \]
\[ F^u_6(n) = \pm \frac{n(n + 1)(n - 1)}{8064} \{277n^6 - 1431n^4 + 2783n^2 - 2205\} \]
\[ F^u_{10}(n) = \pm \frac{n(n + 1)(n - 1)}{3628800} \times \{50521n^8 - 344204n^6 + 959854n^4 - 1357596n^2 + 893025\}. \]

**Remark.** It is easy to see, by symmetry, that
\[
\sum_{k=1}^{n-1} \tan^{1+2p} \frac{k\pi}{n} = 0.
\]

**Sketch of Proof.** According to the definition of \( F^u_p(n) \), we may formally compute its generating function by series rearrangement
\[
\sum_{p=0}^{\infty} F^u_p(n) y^{2p} = \sum_{k=0}^{n-1} \frac{(-1)^k \sec(k\pi/n)}{1 - y^2 \tan^2(k\pi/n)}
\]
\[
= \sum_{k=0}^{n-1} \frac{(-1)^k \cos(k\pi/n)}{(1 + y^2)\cos^2(k\pi/n) - y^2}
\]
\[
= \frac{1}{1 + y^2} \sum_{k=0}^{n-1} \frac{(-1)^k \cos(k\pi/n)}{\cos^2(k\pi/n) - y^2/(1 + y^2)}
\]
which reduces to (A6.2) by means of (A0.3a) with the replacements \( \cot \theta \to y \) and \( \sin(n\pi) = \sin(n\pi/2) \cos(n \arctan y) \).

For \( y = \tan \theta \), applying (0.4b) to the power series expansion
\[
\frac{2e^{ni\theta}}{1 + e^{2ni\theta}} = \frac{e^{ni\theta}}{1 - (1 - e^{2ni\theta})/2}
\]
\[
= e^{ni\theta} \sum_{m \geq 0} 2^{-m} (1 - e^{2ni\theta})^m
\]
\[
= \sum_{m \geq 0} 2^{-m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} e^{i(2k-n)\theta}
\]
we may express the generating function (A.6.2) as
\[
\frac{n \sin \frac{n\pi}{2}}{\sqrt{1 + y^2}} \sec(n \arctan y)
\]
\[
= \frac{n \sin \frac{n\pi}{2}}{\sqrt{1 + y^2}} \frac{2e^{n\theta}}{1 + e^{2n\theta}}
\]
\[
= \sin \frac{n\pi}{2} \sum_{m \geq 0} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k
\]
\[
\times \sum_{j \geq 0} (-1)^j \left( \frac{n + 2kn}{2j} \right) \left( 1 + y^2 \right)^{kn + (1 + n)/2}.
\]
Rewriting the last line, by means of (0.5b), as a formal power series in \( y \)
\[
\sum_{p \geq 0} \left( -1 \right)^p y^{2p} \sum_{j=0}^{p} \left( \frac{n + 2kn}{2j} \right) \left( kn + \frac{n - 1}{2} + p - j \right)
\]
we get an explicit formula for \( \mathcal{F}_p(n) \)
\[
\mathcal{F}_p(n) = \sin \frac{n\pi}{2} \sum_{m \geq 0} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^{k+p}
\]
\[
\times \sum_{j=0}^{p} \left( \frac{n + 2kn}{2j} \right) \left( kn + \frac{n - 1}{2} + p - j \right). \tag{A.6.4a}
\]
\[
\text{The last sum with respect to } k \text{ is the } m \text{th difference operation on a}
\]
\[
\text{polynomial with degree } 2p, \text{ which vanishes for } m > 2p. \text{ Therefore we may}
\]
\[
\text{replace the summation limit for } m \text{ by } 2p \text{ which leads us to (A.6.3)}
\]
\[
\text{promptly.}
\]
An alternative form of (A.6.3) reads as
\[
\mathcal{F}_p(n) = \frac{n}{4^p} \sin \frac{n\pi}{2} \sum_{m \geq 0} \sum_{k=0}^{2p} \binom{1 + 2p}{1 + m} (-1)^{k+p}
\]
\[
\times \sum_{j=0}^{p} \left( \frac{n + 2kn}{2j} \right) \left( kn + \frac{n - 1}{2} + p - j \right). \tag{A.6.5a}
\]
on account of the binomial transform (0.7b).
A7. Sums of $\cot^2 p (k \pi/n)$

**Definition.**

$$G^p_{2p}(n) = \sum_{k=1}^{n-1} \cot^2 \left( \frac{k \pi}{n} \right)$$  \hspace{1cm} (A 7.1)

**Generating Function.**

$$\sum_{\rho=0}^{\infty} G^p_{2p}(n) y^{2\rho} = \frac{n}{1 + y \frac{1 - y \cot(n \arctan y)}{2 \pi}}$$  \hspace{1cm} (A 7.2)

**Explicit Formulae.**

$$G^p_{2p}(n) = n(-1)^p - \frac{1}{2} \sum_{m=0}^{2p} \frac{(-1)^p}{(2n)^m} \left( 1 + 2p \right)^{m+1} \sum_{k=0}^{m} \left( \frac{1 + m}{k} \right) \times (-1)^k \frac{1 + m - 2k}{1 + m} \sum_{\ell \geq 0} \left( \frac{nk + \ell}{\ell} \right) \left( m + 2p - 2\ell \right)$$  \hspace{1cm} (A 7.3a)

$$G^p_{2p}(n) = n(-1)^p - \frac{1}{2} \sum_{m=0}^{2p} \frac{(-1)^p}{(2n)^m} \left( 1 + 2p \right)^{m+1} \sum_{k=0}^{m} \left( \frac{1 + m}{k} \right) \times (-1)^k \frac{1 + m - 2k}{1 + m} \sum_{\ell \geq 0} \left( \frac{nk + \ell}{\ell} \right) \left( m + 2p - 2\ell \right)$$  \hspace{1cm} (A 7.3b)

**Examples.**

\[
\begin{align*}
G^2_{2}(n) &= \frac{(n - 1)(n - 2)}{3} \\
G^4_{2}(n) &= \frac{(n - 1)(n - 2)}{45} \{n^2 + 3n - 13\} \\
G^6_{2}(n) &= \frac{(n - 1)(n - 2)}{945} \{2n^4 + 6n^3 - 28n^2 - 96n + 251\} \\
G^8_{2}(n) &= \frac{(n - 1)(n - 2)}{14175} \\
&\quad \times \{3n^6 + 9n^5 - 59n^4 + 195n^3 + 457n^2 + 1761n - 3551\} \\
G^{10}_{2}(n) &= \frac{(n - 1)(n - 2)}{93555} \{2n^8 + 6n^7 - 52n^6 - 168n^5 + 546n^4 \\
&\quad + 1974n^3 - 3068n^2 - 13152n + 22417\}.
\end{align*}
\]
Remark. It is trivial to see that
\[ \sum_{k=1}^{n-1} \cot^{1+2p} \frac{k\pi}{n} = 0. \]

Sketch of Proof. According to the definition of \( G_{2p}^a(n) \), we may formally compute its generating function by series rearrangement
\[
\sum_{p=0}^{\infty} G_{2p}^a(n) y^{2p} = \sum_{k=1}^{n-1} \frac{1}{1 - y^2 \cot^2(k\pi/n)} = \sum_{k=1}^{n-1} \frac{1 - \cos^2(k\pi/n)}{1 - (1 + y^2)\cos^2(k\pi/n)} = \frac{n}{1 + y^2} - \frac{y^2}{1 + y^2} \sum_{k=0}^{n-1} \frac{1/(1 + y^2)}{\cos^2(k\pi/n) - 1/(1 + y^2)}
\]
which reduces to (A 7.2) by means of (A 0.3b) with the replacement \( \tan \theta \rightarrow y \).

Following the process in A 3, notice that \( 1 - e^{2ni\theta} = -2niy \) for \( y = \tan \theta \). We have the power series expansion in succession
\[
\frac{1 + e^{2ni\theta}}{1 - e^{2ni\theta}} = \frac{1 + e^{2ni\theta}}{-2niy} \left( 1 - \left( 1 + \frac{1 - e^{2ni\theta}}{2niy} \right) \right) = \frac{1 + e^{2ni\theta}}{-2niy} \sum_{\ell \geq 0} \left( 1 + \frac{1 - e^{2ni\theta}}{2niy} \right)^\ell = \frac{1 + e^{2ni\theta}}{-2niy} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \left( 1 - e^{2ni\theta} \right)^m \frac{e^{2ni\theta}}{(2niy)^m} = \frac{1 + e^{2ni\theta}}{-2niy} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} e^{2nki\theta} \frac{e^{2ni\theta}}{(2niy)^m} \times \sum_{k=0}^{1+m} (-1)^k \frac{1 + m - 2k}{1 + m} \frac{1 + m}{k} \frac{e^{2nki\theta}}{(2niy)^m}.\]
Then we may reformulate, through (0.4b) and (0.4c), the generating function \( A_{7.2} \) as a multiple sum

\[
\frac{n}{1 + y^2} \{ 1 - y \cot(n \arctan y) \}
\]

\[
= \frac{n}{1 + y^2} + \frac{n i y}{1 + y^2} \frac{1 + e^{2ni\theta}}{1 - e^{2ni\theta}}
\]

\[
= \frac{n}{1 + y^2} - \frac{1}{2} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \sum_{k=0}^{1+m} (-1)^k \binom{1 + m}{k} \frac{1 + m - 2k}{1 + m}
\]

\[
\times \sum_{i = m \pmod{2}} (-1)^{(m - i)/2} \frac{(2kn_i)}{(2ny)^m} y^i / (1 + y^2)^{n+1}.
\]

Rewriting the last line, by means of the binomial expansion (0.5b), as a formal Laurent series in \( y \)

\[
\sum_p \frac{(-1)^p y^{2p}}{(2n)^m} \sum_{i = m \pmod{2}} \binom{2kn_i}{p + (m - i)/2}(kn + p + (m - i)/2)
\]

we derive an explicit formula for \( G_{2p}^n(n) \)

\[
G_{2p}^n(n) = n(-1)^p - \frac{1}{2} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \frac{(-1)^p 1^m}{(2n)^m} \sum_{k=0}^{1+m} \binom{1 + m}{k} (-1)^k
\]

\[
\times \frac{1 + m - 2k}{1 + m} \sum_{i = m \pmod{2}} \binom{2kn_i}{p + (m - i)/2}(kn + p + (m - i)/2).
\]

According to (0.9c), the double sum with respect to \( m \) and \( k \) will vanish if \( \ell > 2p \). Therefore we may replace the upper limit for the summation variable \( \ell \) by \( 2p \) and simplify the summation through (0.7a), which result in (A7.3).

A8. Alternating sums of \( \cos(k\pi/n) \cot^{2p}(k\pi/n) \)

**Definition.**

\[
G_{2p}^n(n) = \sum_{k=1}^{n-1} (-1)^k \cos \left( \frac{k\pi}{n} \right) \cot^{2p} \left( \frac{k\pi}{n} \right) \quad (A8.1)
\]
Generating Function.

\[
\sum_{p=0}^{\infty} \mathcal{H}_{2p}(n) y^{2p} = \delta_{1,n} - \frac{ny}{1 + y^2} \csc(n \arctan y) \quad (A.8.2)
\]

Explicit Formulae.

\[
\mathcal{H}_{2p}(n) = \delta_{1,n} (-1)^p - \sum_{m=0}^{2p} \frac{(-1)^p}{(2n)^m} \left( \frac{1 + 2p}{1 + m} \right) \sum_{k=0}^{m} \binom{m}{k} \times (-1)^k \sum_{\ell \geq 0} \left( nk + \ell + \frac{n + 1}{2} \right) \left( \frac{n + 2nk}{m + 2p - 2\ell} \right) \quad (A.8.3a)
\]

\[
\mathcal{H}_{2p}(n) = \delta_{1,n} (-1)^p - \sum_{m=0}^{2p} \frac{(-1)^p}{(2n)^m} \left( \frac{1 + 2p}{1 + m} \right) \sum_{k=0}^{m} \binom{m}{k} \times (-1)^k \sum_{\ell \geq 0} \left( nk + \ell + \frac{n + 1}{2} \right) \left( \frac{n + 2nk}{m + 2p - 2\ell} \right) \quad (A.8.3b)
\]

Examples. For \( n > 1 \), we have the summation formulas

\[
\mathcal{H}_2(n) = -\frac{n^2 - 7}{6}
\]

\[
\mathcal{H}_4(n) = -\frac{7n^4 - 110n^2 + 463}{360}
\]

\[
\mathcal{H}_6(n) = -\frac{31n^6 - 735n^4 + 6489n^2 - 20905}{15120}
\]

\[
\mathcal{H}_8(n) = -\frac{381n^8 - 11780n^6 + 151998n^4 - 984420n^2 + 2658221}{1814400}
\]

\[
\mathcal{H}_{10}(n) = \frac{-1}{119750400} \{ 2555n^{10} - 96393n^8 + 1592470n^6 - 14734874n^4 + 77648175n^2 - 184162333 \}.
\]

Remark. It is not hard to check that

\[
\sum_{k=0}^{n-1} (-1)^k \cos \frac{k \pi}{n} = \delta_{1,n}.
\]
Sketch of Proof. According to the definition of $H_p^m(n)$, we may formally compute its generating function by series rearrangement

$$
\sum_{p=0}^{\infty} H_p^m(n) y^{2p} = \sum_{k=1}^{n-1} \frac{(-1)^k \cos(k\pi/n)}{1 - y^2 \cot^2(k\pi/n)}
$$

which reduces to (A.8.2) by means of (A.03a) with the replacement $\cot \theta \rightarrow y$.

For $y = \tan \theta$, we have the power series expansions similar to those in A.7:

$$
e^{ni\theta} = \sum_{i \geq 0} \frac{1 - e^{2ni\theta}}{2niy} \left(1 - \left(1 + \frac{1 - e^{2ni\theta}}{2niy}\right)^{i}\right)
$$

which may be restated as

$$
\frac{\delta_{1,n}}{1 + y^2} - \frac{ny}{\sqrt{1 + y^2}} \csc(n \arctan y)
$$

By means of (0.4b) and (0.4c), the generating function (A.8.2) may be restated as

$$
\frac{\delta_{1,n}}{1 + y^2} + \frac{2niy}{\sqrt{1 + y^2}} \frac{e^{ni\theta}}{1 - e^{2ni\theta}}
$$
\[
\frac{\delta_{1,n}}{1 + y^2} = \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \sum_{\iota = m \mod 2} \frac{(-1)^{(m-\iota)/2}}{(2ny)^m} (n + 2kn)^{\ell/2} (1 + y^2)^{(n + \iota)/2}
\]

where the last line can be expressed, with the help of the binomial expansion (0.5b) as a formal Laurent series in \( y \)

\[
\sum_{p} \frac{(-1)^p y^{2p}}{(2n)^m} \sum_{\iota = m \mod 2} \frac{(-1)^{(m-\iota)/2}}{(2ny)^m} (n + 2kn)^{\ell/2} (1 + y^2)^{(n + \iota)/2}. 
\]

Therefore we get an explicit formula for \( \mathcal{H}_{2p}(n) \)

\[
\mathcal{H}_{2p}(n) = \delta_{1,n} (-1)^p - \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \binom{\ell}{m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \sum_{\iota = m \mod 2} \frac{(-1)^{(m-\iota)/2}}{(2ny)^m} (n + 2kn)^{\ell/2} (1 + y^2)^{(n + \iota)/2}. 
\]

According to (0.9c), the double sum with respect to \( m \) and \( k \) will vanish if \( \ell > 2p \). Hence we may replace the upper limit for the summation variable \( \ell \) by \( 2p \) and simplify the summation through (0.7a), which lead us to (A 8.3).

Recalling that

\[
1 + \tan^2 \theta = \sec^2 \theta, \quad 1 + \cot^2 \theta = \csc^2 \theta
\]

we can establish the following inverse relations for the trigonometric sums defined in this section:

\[
\mathcal{E}_{1,2p}(n) = \sum_{p \ell \in 0} \left( \binom{p}{\ell} \right) \mathcal{E}_{2p}(n) \quad \mathcal{E}_{2p}(n) = \sum_{p \ell \in 0} (-1)^{p-\ell} \left( \binom{p}{\ell} \right) \mathcal{E}_{2p}(n)
\]

\[
\mathcal{B}_{1+2p}(n) = \sum_{p \ell \in 0} \left( \binom{p}{\ell} \right) \mathcal{B}_{2p}(n) \quad \mathcal{B}_{2p}(n) = \sum_{p \ell \in 0} (-1)^{p-\ell} \left( \binom{p}{\ell} \right) \mathcal{B}_{1+2p}(n)
\]

\[
\mathcal{C}_{1,2p}(n) = \sum_{p \ell \in 0} \left( \binom{p}{\ell} \right) \mathcal{C}_{2p}(n) \quad \mathcal{C}_{2p}(n) = \sum_{p \ell \in 0} (-1)^{p-\ell} \left( \binom{p}{\ell} \right) \mathcal{C}_{1,2p}(n)
\]

\[
\mathcal{D}_{1,2p}(n) = \sum_{p \ell \in 0} \left( \binom{p}{\ell} \right) \mathcal{D}_{2p}(n) \quad \mathcal{D}_{2p}(n) = \sum_{p \ell \in 0} (-1)^{p-\ell} \left( \binom{p}{\ell} \right) \mathcal{D}_{1,2p}(n)
\]
Throughout this section, \( n \) will denote an even natural number. Then the trigonometric function \( \sin \theta \sin n\theta \) may be considered as a polynomial of degree \( n + 1 \) in \( \cos \theta \), whose \( n + 1 \) distinct zeros are \( \{\beta_k\}_{k=0}^{n} \) with \( \beta_k = k\pi/n \). If \( P(\theta) \) is a polynomial of degree \( \leq n \) in \( \cos \theta \), we have two expansions in partial fractions

\[
\frac{nP(\theta)}{\sin \theta \sin n\theta} = \sum_{k=0}^{n} \frac{\mu_k}{\cos \theta - \cos \beta_k} \tag{B0.1a}
\]

\[
\frac{nP(\pi - \theta)}{\sin \theta \sin n\theta} = \sum_{k=0}^{n} \frac{\mu_k}{\cos \theta + \cos \beta_k} \tag{B0.1b}
\]

where the coefficients \( \{\mu_k\} \) are determined by

\[
\mu_k = nP(\beta_k) \lim_{\theta \to \beta_k} \frac{\cos \theta - \cos \beta_k}{\sin \theta \sin n\theta}
\]

\[
= \begin{cases} 
(-1)^{k+1}P(k\pi/n), & k = 1, 2, \ldots, n - 1 \\
(-1)^{k+1}P(k\pi/n)/2, & k = 0, n.
\end{cases}
\]

From (B0.1a) ± (B0.1b), we obtain

\[
\frac{-n[P(\theta) + P(\pi - \theta)]}{\sin(2\theta)\sin n\theta} = \frac{P(0) - P(\pi)}{2\sin^2 \theta}
\]

\[
+ \sum_{k=0}^{n-1} (-1)^{k} \frac{P(k\pi/n)}{\cos^2 \theta - \cos^2(k\pi/n)} \tag{B0.2a}
\]

\[
\frac{-n[P(\theta) - P(\pi - \theta)]}{\sin \theta \sin n\theta} = \frac{P(0) + P(\pi)}{2\sin^2 \theta}
\]

\[
+ \sum_{k=0}^{n-1} (-1)^{k} \frac{P(k\pi/n)\cos(k\pi/n)}{\cos^2 \theta - \cos^2(k\pi/n)} . \tag{B0.2b}
\]

For \( P(\theta) = 1 \) and \( \cos n\theta \), we may write (B0.2a), respectively, as

\[
\frac{-n}{\sin n\theta} = \sum_{k=0}^{n-1} (-1)^{k} \frac{\sin \theta \cos \theta}{\cos^2 \theta - \cos^2(k\pi/n)} \tag{B0.3a}
\]

\[
\frac{-n \cos n\theta}{\sin n\theta} = \sum_{k=0}^{n-1} \frac{\sin \theta \cos \theta}{\cos^2 \theta - \cos^2(k\pi/n)} \tag{B0.3b}
\]

which will be used to demonstrate trigonometric identities in this section.
B1. Sums of $\sec^2p(k\pi/n)$

**Definition.**

\[
A_{2p}^b(n) = \sum_{k=0 \atop k+n/2}^{n-1} \sec^2p\left(\frac{k\pi}{n}\right) \tag{B1.1}
\]

**Generating Function.**

\[
\sum_{p=1}^{\infty} A_{2p}^b(n)y^{2p} = 1 - \frac{ny}{\sqrt{1-y^2}}\cot(n\arcsin y) \tag{B1.2}
\]

**Explicit Formulae.**

\[
A_{2p}^b(n) = \frac{1}{2} \sum_{m=0}^{2p} \frac{(-4)^p}{n^m}\left(\frac{1+2p}{1+m}\right)
\times \sum_{k=0}^{1+m} (-1)^k \left(\frac{1+m-2k}{k}\right) \left(\frac{1}{2p+m}\right) \tag{B1.3a}
\]

\[
\times \frac{1}{2} \left(1+\frac{m}{m+1}\right) \frac{1+2p}{1+m} \left(\frac{p+kn+m-1}{2p+m}\right) \tag{B1.3b}
\]

**Examples.**

\[
A_2^b(n) = \frac{(n+1)(n-1)}{3}
\]

\[
A_4^b(n) = \frac{(n+1)(n-1)}{45}(11+n^2)
\]

\[
A_6^b(n) = \frac{(n+1)(n-1)}{945}\{191+23n^2+2n^4\}
\]

\[
A_8^b(n) = \frac{(n+1)(n-1)}{14175}\{227+10n^2+3n^4\}
\]

\[
A_{10}^b(n) = \frac{(n+1)(n-1)}{93555}\{14797+2125n^2+321n^4+35n^6+2n^8\}.
\]

**Remark.** Following the same process as in A3, we can establish the explicit formula (B1.3) from the generating function (B1.2).
Sketch of Proof. Reformulate (B0.3b) as the formal power series

\[ 1 + \frac{n \cos \theta \cos n \theta}{\sin \theta \sin n \theta} = \sum_{k=0}^{n-1} \frac{\cos^2 \theta}{\cos^2(k\pi/n) - \cos^2 \theta} \]

\[ = \sum_{p=1}^{\infty} \sum_{k=0}^{n-1} \frac{\cos^{2p} \theta}{\cos^{2p}(k\pi/n)}. \]

Then the generating function (B1.2) for \( \{B_p^b(n)\} \) follows immediately from this expansion under the replacements \( \cos \theta \to y \) and \( \cot n \theta = -\cot(n \arcsin y). \]

B2. Alternating sums of \( \sec^{2p}(k\pi/n) \)

Definition.

\[ B_p^b(n) = \sum_{k=0}^{n-1} (-1)^k \sec^{2p}\left(\frac{k\pi}{n}\right) \quad (B2.1) \]

Generating Function.

\[ \sum_{p=1}^{\infty} B_p^b(n) y^{2p} = (-1)^{n/2} \left(1 - \frac{ny}{\sqrt{1 - y^2}} \csc(n \arcsin y)\right) \quad (B2.2) \]

Explicit Formulae.

\[ B_p^b(n) = (-1)^{1+n/p/2} \sum_{m=0}^{2p} \left(\frac{1 + 2p}{1 + m}\right) \frac{4^p}{n^m} \]

\[ \times \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(\frac{p + nk + \frac{m + n - 1}{2}}{m + 2p}\right) \quad (B2.3a) \]

\[ \times \left(\frac{1}{2}\right)^{1+n/2} \left(\frac{1 + 2p}{1 + m}\right) \frac{4^p}{n^m} \]

\[ \times \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(\frac{p + nk + \frac{m + n - 1}{2}}{m + 2p}\right) \quad (B2.3b) \]

Examples. Let \( \mp \) denote the alternating signs corresponding to \( n = 0, 2 \mod 4 \) respectively. Then we have

\[ B_2^b(n) = \mp \frac{2 + n^2}{6} \]

\[ B_4^b(n) = \mp \frac{88 + 40n^2 + 7n^4}{360} \]
\[ \mathcal{B}_6(n) = \pm \frac{3056 + 1344n^2 + 294n^4 + 31n^6}{15120} \]

\[ \mathcal{B}_8(n) = \pm \frac{319616 + 138240n^2 + 32928n^4 + 4960n^6 + 318n^8}{1814400} \]

\[ \mathcal{B}_{10}(n) = \frac{1}{23950080} \left( 3788032 + 1622016n^2 + 404096n^4 + 70928n^6 + 8382n^8 + 511n^{10} \right). \]

**Sketch of Proof.** Notice that \((B.0.3a)\) may be restated as a formal power series

\[ (-1)^{n/2} + \frac{n \cos \theta}{\sin \theta \sin n \theta} = \sum_{k=0}^{n-1} \frac{(-1)^k \cos^2 \theta}{\cos^2(k \pi/n) - \cos^2 \theta} \]

\[ = \sum_{p=1}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^k \cos^2 \theta}{\cos^2(k \pi/n)}. \]

We get, from this expansion, the generating function \((B.2.2)\) for \(\mathcal{B}_{2p}(n)\) under the replacements \(\cos \theta \to y\) and \(\sin n \theta = -\cos (n \pi/2) \sin(n \arcsin y)\).

Similar to this procedure in A4, we have the power series expansions for \(y = \sin \theta\)

\[ \frac{e^{ni\theta}}{1 - e^{2ni\theta}} = \frac{e^{ni\theta}}{-2n\theta} \left( 1 - \left(1 + \frac{1 - e^{2ni\theta}}{2n\theta} \right) \right) \]

\[ = \frac{e^{ni\theta}}{-2n\theta} \sum_{\ell \geq 0} \left(1 + \frac{1 - e^{2ni\theta}}{2n\theta} \right) \]

\[ = \frac{e^{ni\theta}}{-2n\theta} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \left(\frac{\ell}{m}\right)(1 - e^{2ni\theta})^m (2n\theta)^m \]

\[ = -\frac{1}{2n\theta} \sum_{\ell \geq 0} \sum_{m=0}^{\ell} \left(\frac{\ell}{m}\right) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{e^{(1+2k)n\theta}}{(2n\theta)^m}. \]
which enable us to restate through (0.3b) and (0.3c) the generating function (B2.2) as a multiple sum

\[
(-1)^{n/2}\left(1 - \frac{ny}{\sqrt{1-y^2}} \csc(n \arcsin y)\right)
\]

\[
= (-1)^{n/2}\left(1 + \frac{2niy}{\sqrt{1-y^2}} \frac{e^{ni\theta}}{1 - e^{2ni\theta}}\right)
\]

\[
= (-1)^{n/2} - \sum_{\ell \geq 0} \sum_{m=0}^\ell \binom{\ell}{m} \sum_{k=0}^m \binom{m}{k} (-1)^{k+n/2}
\]

\[
\times \sum_{i=m(m \mod 2)} \frac{(-1)^{(m-i)/2}}{(2ny)^m} \left( n + 2kn \right) y^i \left( \sqrt{1-y^2} \right)^{n+2kn-i-1}
\]

where the last line can be expressed, by means of the binomial expansion (0.5a) as a formal Laurent series in \( y \)

\[
\sum_p \frac{(-1)^p y^{2p}}{(2n)^m} \sum_{i=m(m \mod 2)} \left( n + 2kn \right) \binom{k + \left( n-i-1 \right)/2}{p + \left( m-i \right)/2}.
\]

Evaluating the sum with respect to \( i \) by (0.8a), we derive an explicit formula

\[
\mathcal{B}_p(n) = (-1)^{1+p+n/2} \sum_{\ell \geq 0} \sum_{m=0}^\ell \binom{\ell}{m} \frac{4^p}{n^m}
\]

\[
\times \sum_{k=0}^m (-1)^k \binom{m}{k} \left( p + kn + \frac{m+n-1}{m+2p} \right),
\]

(According to (0.9a), the double sum with respect to \( m \) and \( k \) will vanish if \( \ell > 2p \). Therefore we may replace the upper limit for the summation variable \( \ell \) by \( 2p \) and simplify the summation through (0.7a), which lead us to (B2.3).)

B3. Sums of \( \csc^{2p}(k\pi/n) \)

Definition.

\[
\mathcal{B}_p(n) = \sum_{k=1}^{n-1} \csc^{2p}\left(\frac{k\pi}{n}\right),
\]

(B3.1)
Generating Function.

\[
\sum_{p=1}^{\infty} \mathcal{E}_p^b(n) y^{2p} = 1 - \frac{ny}{\sqrt{1 - y^2}} \cot(n \arcsin y)
\]  

(B3.2)

Explicit Formulae.

\[
\mathcal{E}_p^b(n) = \delta_{0,p} - \frac{1}{2} \sum_{m=0}^{2p} \frac{(-4)^p}{n^m} \left( \frac{1 + 2p}{1 + m} \right)
\times \sum_{k=0}^{1+m} (-1)^k \binom{1+m}{k} \frac{1 + m - 2k}{1 + m} \left( \frac{p + kn + \frac{m-1}{2}}{2p + m} \right)
\]  

(B3.3b)

EXAMPLES.

\[
\mathcal{E}_2^b(n) = \frac{(n+1)(n-1)}{3}
\]

\[
\mathcal{E}_4^b(n) = \frac{(n+1)(n-1)}{45} (11 + n^2)
\]

\[
\mathcal{E}_6^b(n) = \frac{(n+1)(n-1)}{945} \{191 + 23n^2 + 2n^4\}
\]

\[
\mathcal{E}_8^b(n) = \frac{(n+1)(n-1)}{14175} (11 + n^2) \{227 + 10n^2 + 3n^4\}
\]

\[
\mathcal{E}_{10}^b(n) = \frac{(n+1)(n-1)}{93555} \{14797 + 2125n^2 + 321n^4 + 35n^6 + 2n^8\}.
\]

Remark. The explicit formula (B3.3) can be established from the generating function (B3.2) in the same way as in A3.

Sketch of Proof. Restate (B0.3b) as the formal power series

\[
1 - \frac{n \sin \theta \cos n \theta}{\cos \theta \sin n \theta} = \sum_{k=1}^{n-1} \frac{\sin^2 \left( k \frac{\pi}{n} \right)}{\sin^2 (k \frac{\pi}{n}) - \sin^2 \theta} = \sum_{p=1}^{n-1} \sum_{k=1}^{n-1} \frac{\sin^{2p} \theta}{\sin^{2p} (k \frac{\pi}{n})}.
\]
It becomes the generating function (B3.2) for \( (\mathcal{D}_k^b(n)) \) under the replacement \( \sin \theta \to y \) and some trivial modification.

**B4. Alternating sums of \( \csc^2 p(k \pi/n) \)**

**Definition.**

\[
\mathcal{D}_k^b(n) = \sum_{k=1}^{n-1} (-1)^k \csc^2 p \left( \frac{k \pi}{n} \right) \tag{B4.1}
\]

**Generating Function.**

\[
\sum_{p=1}^{\infty} \mathcal{D}_k^b(n) y^{2p} = 1 - \frac{ny}{\sqrt{1 - y^2}} \csc(n \arcsin y) \tag{B4.2}
\]

**Explicit Formulae.**

\[
\mathcal{D}_k^b(n) = (-1)^{1+p} \sum_{m=0}^{2p} \frac{(1+2p)}{1+m} \frac{4^p}{n^m} \times \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left( \frac{p + nk + \frac{m + n - 1}{2}}{m + 2p} \right) \tag{B4.3a}
\]

**Examples.**

\[
\mathcal{D}_2^b(n) = -\frac{2 + n^2}{6}
\]
\[
\mathcal{D}_3^b(n) = -\frac{88 + 40n^2 + 7n^4}{360}
\]
\[
\mathcal{D}_4^b(n) = -\frac{3056 + 1344n^2 + 294n^4 + 31n^6}{15120}
\]
\[
\mathcal{D}_5^b(n) = -\frac{319616 + 138240n^2 + 32928n^4 + 4960n^6 + 381n^8}{1814400}
\]
\[
\mathcal{D}_6^b(n) = -\frac{3788032 + 1622016n^2 + 404096n^4}{23950090}
\]
\[
\mathcal{D}_10^b(n) = \frac{1}{23950090} \left( 3788032 + 1622016n^2 + 404096n^4 \right.
\]
\[
\left. + 70928n^6 + 8382n^8 + 511n^{10} \right).
\]
Remark. Notice that the generating function (B4.2) differs from (B2.2) only in the factor \((-1)^{n/2}\). Therefore we may write down the explicit formula (B4.3) from (B2.3) immediately.

Sketch of Proof. The formal power series expansion of (B0.3a) reads as

\[
1 - \frac{n \sin \theta}{\cos \theta \sin n\theta} = \frac{(-1)^k \sin^2 \theta}{\sin^2(k\pi/n) - \sin^2 \theta}
\]

\[
= \sum_{p=1}^{\infty} \sum_{k=1}^{n-1} (-1)^k \frac{\sin^{2p} \theta}{\sin^{2p}(k\pi/n)}
\]

which reduces to the generating function (B4.2) for \(\mathcal{G}^b_{2p}(n)\) under the replacement \(\sin \theta \to y\) and some trivial modification.

B5. Sums of \(\tan^{2p}(k\pi/n)\)

**Definition.**

\[
\mathcal{G}^b_{2p}(n) = \sum_{k=0, k \neq n/2}^{n-1} \tan^{2p}\left(\frac{k\pi}{n}\right) \quad (B5.1)
\]

**Generating Function.**

\[
\sum_{p=0}^{\infty} \mathcal{G}^b_{2p}(n)y^{2p} = \frac{n}{1 + y^2} \left(1 - y \cot(n \arctan y)\right) \quad (B5.2)
\]

**Explicit Formulae.**

\[
\mathcal{G}^b_{2p}(n) = n(-1)^p - \frac{1}{2} \sum_{m=0}^{2p} \frac{(-1)^p}{(2n)^m} \left(1 + 2p\right) \sum_{k=0}^{m+1} \left(\frac{1 + m}{k}\right) \times (-1)^k \frac{1 + m - 2k}{1 + m} \sum_{\ell \geq 0} \left(\frac{nk + \ell}{\ell}\right)\left(2nk\right) \left(m + 2p - 2\ell\right) \quad (B5.3a)
\]

\[
\times (-1)^k \frac{1 + m - 2k}{1 + m} \sum_{\ell \geq 0} \left(\frac{nk + \ell}{\ell}\right)\left(2nk\right) \left(m + 2p - 2\ell\right) \quad (B5.3b)
\]

**Examples.**

\[
\mathcal{G}^b_2(n) = \frac{(n-1)(n-2)}{3}
\]

\[
\mathcal{G}^b_4(n) = \frac{(n-1)(n-2)}{45} \{n^2 + 3n - 13\}
\]
Remark. The explicit formula (B5.3) follows from (A7.3) directly because the corresponding generating functions (A7.2) and (B5.2) coincide with each other.

Sketch of Proof. According to the definition of $E_b^p(n)$, we may formally compute its generating function by series rearrangement

$$
\sum_{p=0}^{\infty} E_b^p(n) y^{2p} = \sum_{k=0}^{n-1} \frac{1}{1 - y^2 \tan^2(k \pi/n)}
$$

$$
= \sum_{k=0}^{n-1} \frac{\cos^2(k \pi/n)}{(1 + y^2) \cos^2(k \pi/n) - y^2}
$$

$$
= \frac{n}{1 + y^2} + \frac{y}{1 + y^2} \sum_{k=0}^{n-1} \frac{\cos^2(k \pi/n) - y^2/(1 + y^2)}
$$

which reduces to (B5.2) by means of (B0.3b) with the replacements $\cot \theta \to y$ and $\cot(n \theta) = -\cot(n \arctan y)$.

B6. Alternating sums of $\tan^{2p}(k \pi/n)$

Definition.

$$
\mathcal{F}_b^p(n) = \sum_{k=0}^{n-1} (-1)^k \tan^{2p}(k \pi/n) \quad (B6.1)
$$

Generating Function.

$$
\sum_{p=0}^{\infty} \mathcal{F}_b^p(n) y^{2p} = (-1)^n / (1 + y^2) \csc(n \arctan y) \quad (B6.2)
$$
**Explicit Formulae.**

\[
\mathcal{F}_p^2(n) = \sum_{m=0}^{2p} \left( \frac{1 + 2p}{1 + m} \right) \left( -1 \right)^{1+p+n/2} \sum_{k=0}^{m} \frac{(-1)^k}{(2n)^m} \sum_{l=0}^{m} \left( \frac{m}{k} \right) (-1)^l (B6.3a)
\]

\[
\times \sum_{l \geq 0} \left( \frac{n + 2nk}{m + 2p - 2l} \right) \left( \frac{nk + l + n/2}{l} \right) (B6.3b)
\]

**Examples.** Let \( \pm \) denote the alternating signs corresponding to \( n = 0, 2 \) (mod 4) respectively. Then we have

\[
\mathcal{F}_2^2(n) = \pm \frac{(n + 2)(n - 2)}{6}
\]

\[
\mathcal{F}_4^2(n) = \pm \frac{(n + 2)(n - 2)}{360} \{7n^2 - 52 \}
\]

\[
\mathcal{F}_6^2(n) = \pm \frac{(n + 2)(n - 2)}{15120} \{31n^4 - 464n^2 + 2008 \}
\]

\[
\mathcal{F}_8^2(n) = \pm \frac{(n + 2)(n - 2)}{1814400} \{381n^6 - 8396n^4 + 69904n^2 - 227264 \}
\]

\[
\mathcal{F}_{10}^2(n) = \pm \frac{(n + 2)(n - 2)}{23950080} \times \{511n^8 - 14720n^6 + 175728n^4 - 1066240n^2 + 2869376 \}
\]

**Sketch of Proof.** According to the definition of \( \mathcal{F}_p^2(n) \), we may formally compute its generating function by series rearrangement

\[
\sum_{p = 0}^{\infty} \mathcal{F}_p^2(n) y^{2p} = \sum_{k=0}^{n-1} \frac{(-1)^k}{1 - y^2 \tan^2(k \pi/n)} = \frac{1}{1 + y^2} \sum_{k=0}^{n-1} \frac{(-1)^k \cos^2(k \pi/n)}{(1 + y^2)\cos^2(k \pi/n) - y^2} = \frac{y}{1 + y^2} \sum_{k=0}^{n-1} \frac{(-1)^k y/(1 + y^2)}{\cos^2(k \pi/n) - y^2/(1 + y^2)}
\]
which reduces to (B6.2) by means of (B0.3a) with the replacements \( \cot \theta \to y \) and \( \sin(n \theta) = -\cos n\pi/2 \sin(n \arctan y) \).

For \( y = \tan \theta \), we have the power series expansions similar to those in A8:

\[
\frac{e^{ni\theta}}{1 - e^{2ni\theta}} = \frac{e^{ni\theta}}{-2niy} \left( 1 - \left( 1 + \frac{1 - e^{2ni\theta}}{2niy} \right) \right)
\]

\[
= \frac{e^{ni\theta}}{-2niy} \sum_{\ell \geq 0} \left( 1 + \frac{1 - e^{2ni\theta}}{2niy} \right)^\ell
\]

\[
= \frac{e^{ni\theta}}{-2niy} \sum_{\ell \geq 0} \sum_{m=0}^\ell \binom{\ell}{m} \left( \frac{1 - e^{2ni\theta}}{2niy} \right)^m
\]

\[
= \frac{-1}{2niy} \sum_{\ell \geq 0} \sum_{m=0}^\ell \binom{\ell}{m} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{e^{(1+2k)ni\theta}}{(2niy)^m}
\]

which permit us to rewrite, through the power series expansions (0.4b) and (0.4c), the generating function (B6.2) as

\[
(-1)^{1+n/2} \frac{ny}{1+y^2} \csc(n \arctan y)
\]

\[
= (-1)^{n/2} \frac{2niy}{1+y^2} \frac{e^{ni\theta}}{1 - e^{2ni\theta}}
\]

\[
= (-1)^{1+n/2} \sum_{\ell \geq 0} \sum_{m=0}^\ell \binom{\ell}{m} \sum_{k=0}^m (-1)^k \frac{e^{(1-2k)ni\theta}}{(2niy)^m}
\]

\[
\times \sum_{\ell = m \text{(mod 2)}} (-1)^{(m-1)/2} \binom{n + 2kn}{\ell} y^{\ell} \left( 1 + y^2 \right)^{1+nk+n/2}
\]

where the last line can be expressed, by means of the binomial expansion (0.5b), as a formal Laurent series in \( y \):

\[
\sum_{p} \frac{(-1)^p y^{2p}}{(2n)^m} \sum_{\ell = m \text{(mod 2)}} \binom{n + 2kn}{\ell} \left( \frac{p + kn + (m + n - \ell)/2}{p + (m - \ell)/2} \right).
\]
Therefore we get an explicit formula for $\mathcal{S}_{2p}^b(n)$

$$
\mathcal{S}_{2p}^b(n) = \sum_{\ell \geq 0} \sum_{m=0}^n \left( \left( \frac{1}{m} \right) \frac{(-1)^{1+p+n/2}}{(2n)^m} \sum_{k=0}^m \binom{m}{k}(-1)^k \right) \times \sum_{\ell \equiv m \mod 2} \left( n + 2kn \right) \left( \frac{p + kn + (m + n - \ell)/2}{p + (m - \ell)/2} \right).
$$

According to (0.9c), the double sum with respect to $m$ and $k$ will vanish if $\ell > 2p$. Hence we may replace the upper limit for the summation variable $\ell$ by $2p$ and simplify the summation through (0.7a), which lead us to (B6.3).

**B7. Sums of cot$^2p(k\pi/n)$**

**Definition.**

$$
\mathcal{S}_{2p}^b(n) = \sum_{k=1}^{n-1} \cot^2p\left( \frac{k\pi}{n} \right)
$$

**Generating Function.**

$$
\sum_{p=0}^{\infty} \mathcal{S}_{2p}^b(n) y^{2p} = \frac{n}{1 + y^2} \left( 1 - y \cot(n \arctan y) \right)
$$

**Explicit Formulae.**

$$
\mathcal{S}_{2p}^b(n) = n(-1)^p - \frac{1}{2} \sum_{m=0}^{2p} \left( \frac{(-1)^p}{(2n)^m} \left( \frac{1 + 2p}{1 + m} \right) \sum_{k=0}^{m+1} \binom{m+1}{k} \right) \times (-1)^k \frac{1 + m - 2k}{1 + m} \sum_{\ell \geq 0} \frac{2nk}{2m + 2p - 2\ell}.
$$

**Examples.**

$$
\mathcal{S}_2^b(n) = \frac{(n - 1)(n - 2)}{3}
$$
$$
\mathcal{S}_4^b(n) = \frac{(n - 1)(n - 2)}{45} \{n^2 + 3n - 13\}
$$
$$
\mathcal{S}_6^b(n) = \frac{(n - 1)(n - 2)}{945} \{2n^4 + 6n^2 - 28n - 96n + 251\}
\[ \mathcal{F}_b(n) = \frac{(n-1)(n-2)}{14175} \times \{3n^6 + 9n^5 - 59n^4 - 195n^3 + 457n^2 + 1761n - 3551\} \]
\[ \mathcal{F}_{20}(n) = \frac{(n-1)(n-2)}{93555} \{2n^8 + 6n^7 - 52n^6 - 168n^5 + 546n^4 \]
\[ + 1974n^3 - 3068n^2 - 13152n + 22417\} \]

**Remark.** The explicit formula (B.7.3) reads from (A.7.3) immediately for the generating function (B.7.2) is the same as (A.7.2).

**Sketch of Proof.** According to the definition of \( \mathcal{F}_b(n) \), we may formally compute its generating function by series rearrangement

\[ \sum_{p=0}^{\infty} \mathcal{F}_b^c(n) y^{2p} = \sum_{k=1}^{n-1} \frac{1}{1 - y^2 \cot^2(k\pi/n)} \]
\[ = \sum_{k=1}^{n-1} \frac{1 - \cos^2(k\pi/n)}{1 - (1 + y^2)\cos^2(k\pi/n)} \]
\[ = \frac{n}{1 + y^2} - \frac{y}{1 + y^2} \sum_{k=0}^{n-1} \frac{y/(1 + y^2)}{\cos^2(k\pi/n) - 1/(1 + y^2)} \]

which reduces to (B.7.2) by means of (B.0.3b) with the replacement of \( \tan \theta \to y \) and some trivial modification.

**B8. Alternating sums of \( \cot^{2p}(k\pi/n) \)**

**Definition.**

\[ \mathcal{F}_b^h(n) = \sum_{k=1}^{n-1} (-1)^k \cot^{2p}(k\pi/n) \] (B8.1)

**Generating Function.**

\[ \sum_{p=0}^{\infty} \mathcal{F}_b^h(n) y^{2p} = \frac{-ny}{1 + y^2} \csc(n \arctan y) \] (B8.2)

**Explicit Formulae.**

\[ \mathcal{F}_b^h(n) = \sum_{m=0}^{2p} \frac{1 + 2p}{1 + m} \frac{(-1)^{1+p}}{(2n)^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \] (B8.3a)
\[ \times \sum_{\ell \geq 0} \binom{n + 2nk}{m + 2p - 2\ell} \binom{nk + \ell + n/2}{\ell} \] (B8.3b)
Examples.

\[ \mathcal{H}^b_0(n) = -\frac{(n + 2)(n - 2)}{6} \]
\[ \mathcal{H}^b_2(n) = -\frac{(n + 2)(n - 2)}{360} \{7n^2 - 52\} \]
\[ \mathcal{H}^b_4(n) = -\frac{(n + 2)(n - 2)}{15120} \{31n^4 - 464n^2 + 2008\} \]
\[ \mathcal{H}^b_6(n) = -\frac{(n + 2)(n - 2)}{1814400} \{381n^6 - 8396n^4 + 69904n^2 - 227264\} \]
\[ \mathcal{H}^b_{10}(n) = -\frac{(n + 2)(n - 2)}{23950080} \times \{511n^8 - 14720n^6 + 175728n^4 - 1066240n^2 + 2869376\}. \]

Remark. The generating function \( \mathcal{B}^b_{8.2} \) differs from \( \mathcal{B}^b_{6.2} \) only in the factor \((-1)^n/2\). Therefore the explicit formula \( \mathcal{B}^b_{8.2} \) follows from \( \mathcal{B}^b_{6.3} \) directly.

Sketch of Proof. According to the definition of \( \mathcal{H}^b_p(n) \), we may formally compute its generating function by series rearrangement

\[
\sum_{p=0}^{\infty} \mathcal{H}^b_p(n) y^p = \sum_{k=1}^{n-1} \frac{(-1)^k}{1 - y^2 \cot^2(k\pi/n)} = \sum_{k=1}^{n-1} \frac{(-1)^k}{1 - \cos^2(k\pi/n)} \frac{1 - \cos(k\pi/n)}{1 - (1 + y^2)\cos^2(k\pi/n)} \] 

\[ = \frac{y^{-1} \sum_{k=0}^{n-1} (-1)^{1+k} y/(1 + y^2)}{1 + y^2 \cos^2(k\pi/n) - 1/(1 + y^2)} \]

which reduces to \( \mathcal{B}^b_{8.2} \) by means of \( \mathcal{B}^b_{0.3} \) with the replacement \( \tan \theta \to y \) and some trivial modification.

Similar to Section A, the inverse relations for the trigonometric sums defined in this section may be displayed as follows:

\[ \mathcal{A}^b_{2p}(n) = \sum_{\ell=0}^{p} \binom{p}{\ell} \mathcal{A}^b_{2\ell}(n) \]
\[ \mathcal{B}^b_{2p}(n) = \sum_{\ell=0}^{p} (-1)^{p-\ell} \binom{p}{\ell} \mathcal{B}^b_{2\ell}(n) \]
\[ \mathcal{A}^b_{2p}(n) = \sum_{\ell=0}^{p} \binom{p}{\ell} \mathcal{A}^b_{2\ell}(n) \]
\[ \mathcal{B}^b_{2p}(n) = \sum_{\ell=0}^{p} (-1)^{p-\ell} \binom{p}{\ell} \mathcal{B}^b_{2\ell}(n) \]
\[ \mathcal{F}_{2p}^b(n) = \sum_{\ell=0}^{p} \left( \frac{p}{\ell} \right) \mathcal{F}_{2\ell}^b(n) \]
\[ \mathcal{U}_{2p}^b(n) = \sum_{\ell=0}^{p} (-1)^{\ell-\ell} \left( \frac{p}{\ell} \right) \mathcal{U}_{2\ell}^b(n) \]
\[ \mathcal{V}_{2p}^b(n) = \sum_{\ell=0}^{p} \left( \frac{p}{\ell} \right) \mathcal{V}_{2\ell}^b(n) \]
\[ \mathcal{W}_{2p}^b(n) = \sum_{\ell=0}^{p} (-1)^{\ell-\ell} \left( \frac{p}{\ell} \right) \mathcal{W}_{2\ell}^b(n). \]

In addition, the combination of A3 and B3 asserts that for two natural numbers \( n \) and \( p \), there holds the generating function due to Stanley [9] (see also [5])

\[ \sum_{p=1}^{\infty} \sum_{k=1}^{n-1} \frac{y^{2p}}{\sin^2(p(k\pi/n))} = 1 - \frac{ny}{\sqrt{1-y^2}} \cot(n \arcsin y) \]

and the explicit formula

\[ \sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n}\right) = \delta_{0,p} - \frac{1}{2} \sum_{m=0}^{2p} \frac{(-4)^p}{n^m} \left( \frac{1+2p}{1+m} \right) \left( \sum_{k=0}^{1+m} (-1)^k \left( \frac{1+m}{k} \right) \frac{1+m-2k}{1+m} \right) \left( p + kn + \frac{m-1}{2p+m} \right). \]

Similarly, from A7 and B7, we deduce that for two natural numbers \( n \) and \( p \), there are the generating function

\[ \sum_{p=1}^{\infty} \sum_{k=1}^{n-1} \frac{y^{2p}}{\tan^2(p(k\pi/n))} = \frac{n}{1+y^2} \left( 1 - y \cot(n \arctan y) \right) \]

and the summation formula

\[ \sum_{k=1}^{n-1} \cot^2\left(\frac{k\pi}{n}\right) = n(-1)^p - \frac{1}{2} \sum_{m=0}^{2p} \frac{(-1)^p}{(2n)^m} \left( \frac{1+2p}{1+m} \right) \sum_{k=0}^{m+1} \left( \frac{1+m}{k} \right) \left( -1 \right)^k \frac{1+m-2k}{1+m} \sum_{\ell=0}^{\infty} \left( \frac{nk+\ell}{m+2p-2\ell} \right). \]
C. SUMMATIONS OVER $\pi(1 + 2k)/n$ WITH $n = 0 \mod 4$

Throughout this section, $n$ will denote an even natural number. Then the trigonometric function $\cos n\theta$ may be considered as a polynomial of degree $n$ in $\cos \theta$, whose $n$ distinct zeros are $\gamma_k$ with $\gamma_k = \pi(1 + 2k)/(2n)$. If $P(\theta)$ is a polynomial of degree $< n$ in $\cos \theta$, we have two expansions in partial fractions

$$\frac{n P(\theta)}{\cos n\theta} = \sum_{k=0}^{n-1} \frac{\nu_k}{\cos \theta - \cos \gamma_k} \quad (C0.1a)$$

$$\frac{n P(\pi - \theta)}{\cos n\theta} = \sum_{k=0}^{n-1} \frac{-\nu_k}{\cos \theta + \cos \gamma_k} \quad (C0.1b)$$

where the coefficients $\nu_k$ are determined by

$$\nu_k =nP(\gamma_k) \lim_{\theta \to \gamma_k} \frac{\cos \theta - \cos \gamma_k}{\cos n\theta}$$

$$= (-1)^k P\left(\frac{1 + 2k}{2n}\pi\right) \sin\left(\frac{1 + 2k}{2n}\pi\right), \quad k = 0, 1, \ldots, n - 1.$$ 

From (C0.1a) ± (C0.1b), we find

$$\frac{n\{P(\theta) + P(\pi - \theta)\}}{\cos n\theta}$$

$$= \sum_{k=0}^{n-1} (-1)^k \frac{P(\pi(1 + 2k)/(2n)) \sin(\pi(1 + 2k)/(2n))}{\cos^2 \theta - \cos^2(\pi(1 + 2k)/(2n))} \quad (C0.2a)$$

$$\frac{n\{P(\theta) - P(\pi - \theta)\}}{2 \cos \theta \cos n\theta}$$

$$= \sum_{k=0}^{n-1} (-1)^k \frac{P(\pi(1 + 2k)/(2n)) \sin(\pi(1 + 2k)/(2n))}{\cos^2 \theta - \cos^2(\pi(1 + 2k)/(2n))} \quad (C0.2b)$$

For $P(\theta) = 1$ and $\sin n\theta/\sin \theta$, we may state (C0.2a) and (C0.2b), respectively, as

$$\frac{n}{\cos n\theta} = \sum_{k=0}^{n-1} (-1)^k \frac{\sin(\pi(1 + 2k)/(2n)) \cos(\pi(1 + 2k)/(2n))}{\cos^2 \theta - \cos^2(\pi(1 + 2k)/(2n))} \quad (C0.3a)$$

$$\frac{n}{\cos n\theta} = \sum_{k=0}^{n-1} \frac{\sin \theta \cos \theta}{\cos\theta - \cos(\pi(1 + 2k)/(2n))} \quad (C0.3b)$$
which will be used to establish trigonometric summation formulas in this section.

C1. Sums of $\sec^2(p(\pi(1+2k)/(2n))$

Definition.

\[
\mathcal{A}_p^2(n) = \sum_{k=0}^{n-1} \sec^2 p \left( \frac{1+2k}{2n} \right) \pi
\]  

(C1.1)

Generating Function.

\[
\sum_{p=1}^{\infty} \mathcal{A}_p^2(n) y^{2p} = \frac{ny}{\sqrt{1-y^2}} \tan(n \arcsin y)
\]  

(C1.2)

Explicit Formulae.

\[
\mathcal{A}_p^2(n) = n \sum_{k=1}^{2p-1} (-1)^{p+k} \left( \frac{p-1+kn}{2p-1} \right) \sum_{j=k}^{2p-1} \binom{2p}{j} \left( \frac{j+1}{2} \right)
\]  

(C1.3)

Examples.

\[
\mathcal{A}_2^2(n) = n^2
\]

\[
\mathcal{A}_3^2(n) = \frac{n^2}{3} \left[ 2 + n^2 \right]
\]

\[
\mathcal{A}_4^2(n) = \frac{n^2}{15} \left[ 8 + 5n^2 + 2n^4 \right]
\]

\[
\mathcal{A}_6^2(n) = \frac{n^2}{315} \left[ 144 + 98n^2 + 56n^4 + 17n^6 \right]
\]

\[
\mathcal{A}_{10}^2(n) = \frac{n^2}{2835} \left[ 1152 + 820n^2 + 546n^4 + 255n^6 + 62n^8 \right].
\]

Remark. Following the same process as in A1, we can establish, from the generating function (C1.2), the explicit formula (C1.3). It may also be expressed as

\[
\mathcal{A}_p^2(n) = n \sum_{k=1}^{2p-1} \left( \frac{p-1+kn}{2p-1} \right) (-1)^{p+k} \sum_{m=k}^{2p-1} \binom{m}{k} 2^{p-m-1}
\]  

(C1.4)

by means of the binomial transform (0.7b).
Sketch of Proof. The formal power series of (C0.3b) may be stated as
\[
\frac{n \cos \theta \sin n \theta}{\sin \theta \cos n \theta} = \sum_{k=0}^{n-1} \cos^2 \theta - \cos^2(\pi(1 + 2k)/(2n))
\]
\[
= - \sum_{p=1}^{\infty} \sum_{k=0}^{n-1} \cos^{2p} \frac{\theta}{(\pi(1 + 2k)/(2n))}
\]
where the summation order has been changed. Then the generating function (C1.2) for \(C_\theta(n)\) follows immediately from this expansion under the replacements \(\cos \theta \rightarrow y\) and \(n \theta = -\tan(n \arcsin y)\).

C2. Sums of \(\csc^2(\pi(1 + 2k)/(2n))\)

**Definition.**
\[
\mathcal{B}_p^c(n) = \sum_{k=0}^{n-1} \csc^{2p} \left( \frac{1 + 2k}{2n} \pi \right)
\]

**Generating Function.**
\[
\sum_{p=1}^{\infty} \mathcal{B}_p^c(n) y^{2p} = \frac{ny}{\sqrt{1 - y^2}} \tan(n \arcsin y)
\]

**Explicit Formulae.**
\[
\mathcal{B}_p^c(n) = n \sum_{k=1}^{2p-1} (-1)^{p+k} \left( \frac{p - 1 + kn}{2p - 1} \right)^{2p-1} \binom{2p}{j+1}
\]

**Examples.**
\[
\mathcal{B}_2^c(n) = n^2
\]
\[
\mathcal{B}_4^c(n) = \frac{n^2}{3} \{ 2 + n^2 \}
\]
\[
\mathcal{B}_6^c(n) = \frac{n^2}{15} \{ 8 + 5n^2 + 2n^4 \}
\]
\[
\mathcal{B}_8^c(n) = \frac{n^2}{315} \{ 144 + 98n^2 + 56n^4 + 17n^6 \}
\]
\[
\mathcal{B}_{10}^c(n) = \frac{n^2}{2835} \{ 1152 + 820n^2 + 546n^4 + 255n^6 + 62n^8 \}
\]
Remark. The same process displayed in A1 may be used to derive the explicit formula (C2.3) from the generating function (C2.2). Its reformulation via (0.7b) reads as

\[ \mathcal{A}^{c}_{2p}(n) = n \sum_{k=1}^{2p-1} \left( \frac{p - 1 + kn}{2p - 1} \right) (-1)^{p+k} \sum_{m=k}^{2p-1} \binom{m}{k} 2^{2p-m-1}. \] (C2.4)

Sketch of Proof. Reformulate (C0.3b) as the formal power series

\[
\frac{n \sin \theta \sin n \theta}{\cos \theta \cos n \theta} = \sum_{k=0}^{n-1} \sin^2 \left( \frac{\pi (1 + 2k)/(2n)}{2} \right) - \sin^2 \theta \\
= \sum_{p=1}^{\infty} \sum_{k=0}^{n-1} \sin^{2p} \left( \frac{\pi (1 + 2k)/(2n)}{2} \right)
\]

which reduces to the generating function (C2.2) for \( \mathcal{A}^{c}_{2p}(n) \) under the replacement \( \sin \theta \rightarrow y \) and some trivial modification.

C3. Alternating sums of \( \sin(\pi(1 + 2k)/(2n)) \sec^{1+2p}(\pi(1 + 2k)/(2n)) \)

Definition.

\[ \mathcal{G}^{c}_{1+2p}(n) = \sum_{k=0}^{n-1} (-1)^k \sin \left( \frac{1 + 2k}{2n} \pi \right) \sec^{1+2p} \left( \frac{1 + 2k}{2n} \pi \right) \] (C3.1)

Generating Function.

\[ \sum_{p=0}^{\infty} \mathcal{G}^{c}_{1+2p}(n) y^{1+2p} = (-1)^{1+n/2} ny \sec(\text{arcsin} \ y) \] (C3.2)

Explicit Formulae.

\[ \mathcal{G}^{c}_{1+2p}(n) = (-1)^{1+p+n/2} n \sum_{m=0}^{2p} \binom{1 + 2p}{1 + m} \times \sum_{k=0}^{m} (-1)^k \left( \frac{p + kn + n/2}{2p} \right) \frac{kn + n/2}{p + kn + n/2} \] (C3.3a)
Examples. Let \( \mp \) denote the alternating signs corresponding to \( n \equiv 0, 2 \pmod{4} \), respectively. Then we have

\[
\begin{align*}
\mathcal{A}_1(n) &= \mp n \\
\mathcal{A}_2(n) &= \mp \frac{n^3}{2} \\
\mathcal{A}_3(n) &= \mp \frac{n^3}{24} \{4 + 5n^2\} \\
\mathcal{A}_4(n) &= \mp \frac{n^3}{720} \{64 + 100n^2 + 61n^4\} \\
\mathcal{A}_5(n) &= \mp \frac{n^3}{40320} \{2304 + 3920n^2 + 3416n^4 + 1385n^6\}.
\end{align*}
\]

Sketch of Proof. Expand \( \mathcal{C}_0.3a \) in terms of the formal power series

\[
\frac{n \cos \theta}{\cos n \theta} = \sum_{k=0}^{n-1} (-1)^k \frac{\sin(\pi(1+2k)/(2n)) \cos(\pi(1+2k)/(2n))}{\cos^2 \theta - \cos^2 \pi(1+2k)/(2n)}
\]

\[
= \sum_{p=0}^{\infty} \sum_{k=0}^{n-1} (-1)^{1+k} \frac{\sin(\pi(1+2k)/(2n))}{\cos^{1+2p} \pi(1+2k)/(2n)} \cos^{1+2p} \theta
\]

which reduces to the generating function \( \mathcal{C}_3.2 \) for \( \{\mathcal{A}_1, 2p(n)\} \) under the replacement \( \cos \theta \rightarrow y \) and \( \cos n \theta = \cos(n \pi/2) \cos(n \arcsin y) \).

For \( y = \sin \theta \), we have the formal power series expansion similar to those in A2

\[
\frac{2e^{ni\theta}}{1 + e^{2ni\theta}} = \frac{e^{ni\theta}}{1 - (1 - e^{2ni\theta})/2}
\]

\[
= e^{ni\theta} \sum_{m \geq 0} \frac{2^{-m} \{1 - e^{2ni\theta}\}^m}{m!}
\]

\[
= \sum_{m \geq 0} \frac{2^{-m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} e^{(2k+1)ni\theta}}{m!}.
\]
Combining it with (0.3b), we may restate the generating function (C.3.2) as

\[( -1)^{1+n/2}ny \sec(n \arcsin y)\]

\[= ( -1)^{1+n/2} \frac{2nye^{n\theta}}{1 + e^{2n\theta}} \]

\[= ( -1)^{1+n/2} \sum_{m \geq 0} \sum_{k=0}^{m} \frac{n}{2^m} \binom{m}{k} (-1)^k \]

\[\times \sum_{j \geq 0} (-1)^j \left( \frac{n + 2kn}{2j} \right) y^{1+2j} \left( \sqrt{1 - y^2} \right)^{n+2kn-2j} \]

where the last line can be restated, with the help of the binomial expansion (0.5a) as a formal power series in \(y\)

\[\sum_{p \geq 0} (-1)^p y^{1+2p} \sum_{j=0}^{p} \binom{n + 2kn}{2j} \left( \frac{kn - j + n/2}{p - j} \right).\]

Evaluating the last sum through (0.8b), we get an explicit formula

\[G_{1+2p}^c(n) = ( -1)^{1+p+n/2}4^p \sum_{m \geq 0} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \] (C.3.4a)

\[\times \frac{kn + n/2}{p + kn + n/2} \left( p + kn + n/2 \right). \] (C.3.4b)

The last sum with respect to \(k\) is the \(m\)th difference operation on a polynomial with degree \(2p\), which vanishes for \(m > 2p\). Therefore we may replace the summation limit for \(m\) by \(2p\) and the resulting formula becomes

\[G_{1+2p}^c(n) = ( -1)^{1+p+n/2}4^p \sum_{m = 0}^{2p} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \] (C.3.5a)

\[\times \frac{kn + n/2}{p + kn + n/2} \left( p + kn + n/2 \right). \] (C.3.5b)

which is equivalent to (C.3.3) thanks to the binomial transform (0.7b).
C4. Alternating sums of $\cos(\pi(1 + 2k)/(2n))\csc^{1 + 2p}(\pi(1 + 2k)/(2n))$

**Definition.**

$$D_{1+2p}^c(n) = \sum_{k=0}^{n-1} (-1)^k \cos\left(\frac{1 + 2k}{2n}\right)\csc^{1 + 2p}\left(\frac{1 + 2k}{2n}\right) \quad (C4.1)$$

**Generating Function.**

$$\sum_{p=0}^{\infty} D_{1+2p}^c(n)y^{1+2p} = ny\sec(n\arcsin y) \quad (C4.2)$$

**Explicit Formulae.**

$$D_{1+2p}^c(n) = n \sum_{m=0}^{2p} \binom{1 + 2p}{1 + m}$$

$$\times \sum_{k=0}^{m} (-1)^{k+p}\left(\frac{p + kn + n/2}{2}p\right)\frac{kn + n/2}{p + kn + n/2} \quad (C4.3)$$

**Examples.**

$$D_1^c(n) = n$$

$$D_2^c(n) = \frac{n^3}{2}$$

$$D_3^c(n) = \frac{n^3}{24}(4 + 5n^2)$$

$$D_4^c(n) = \frac{n^3}{720}(64 + 100n^2 + 61n^4)$$

$$D_5^c(n) = \frac{n^3}{40320}(2304 + 3920n^2 + 3416n^4 + 1385n^6).$$
Remark. The generating function (C4.2) differs from (C3.2) only in the alternating factor \((-1)^{1+n/2}\). Therefore we may read the explicit formula (C4.3) directly from (C3.3), whose alternative form may be stated as

\[
\mathcal{G}_{1+2p}^e(n) = 4^p \sum_{m=0}^{2p} \frac{n}{2^m} \times \sum_{k=0}^{m} \frac{(-1)^{k+p}}{k!} \binom{m}{k} \left(\frac{p + kn + n/2}{2p}\right) \frac{kn + n/2}{p + kn + n/2}
\]

(C4.4)

in view of the binomial transform (0.7b).

Sketch of Proof. The power series expansion of (C0.3a) reads as

\[
\frac{n \sin \theta}{\cos n \theta} = \sum_{k=0}^{n-1} (-1)^k \frac{\sin (\pi (1 + 2k)/(2n)) \cos (\pi (1 + 2k)/(2n))}{\sin^2 (\pi (1 + 2k)/(2n)) - \sin^2 \theta} \sin \theta
\]

\[
= \sum_{p=0}^{\infty} \sum_{k=0}^{n-1} L(-1)^k \frac{\cos (\pi (1 + 2k)/(2n))}{\sin^{1+2p} (\pi (1 + 2k)/(2n))} \sin^{1+2p} \theta
\]

which reduces to the generating function (C4.2) for \(\mathcal{G}_{1+2p}^e(n)\) under the replacement \(\sin \theta \rightarrow y\) and some trivial modification.

\[\blacksquare\]

C5. Sums of \(\tan^2(\pi(1 + 2k)/(2n))\)

**Definition.**

\[
\mathcal{E}_{2p}^e(n) = \sum_{k=0}^{n-1} \tan^2 \left(\frac{1 + 2k}{2n} \pi\right)
\]

(C5.1)

**Generating Function.**

\[
\sum_{p=0}^{\infty} \mathcal{E}_{2p}^e(n) y^{2p} = \frac{n}{1 + y^2} \left(1 + y \tan (n \arctan y)\right)
\]

(C5.2)

**Explicit Formulae.**

\[
\mathcal{E}_{2p}^e(n) = n(-1)^p + (-1)^p \sum_{m=0}^{2p} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k}
\]

\[
\times (-1)^k \sum_{\ell=1}^{p} \binom{2nk}{2\ell - 1} \binom{nk + p - \ell}{p - \ell}
\]

(C5.3a)


**Examples.**

\[
E_2^c(n) = n(n-1)
\]

\[
E_4^c(n) = \frac{n(n-1)}{3} \{ n^2 + n - 3 \}
\]

\[
E_6^c(n) = \frac{n(n-1)}{15} \{ 2n^4 + 2n^3 - 8n^2 - 8n + 15 \}
\]

\[
E_8^c(n) = \frac{n(n-1)}{315} \times \{ 17n^6 + 17n^5 - 95n^4 - 95n^3 + 213n^2 + 213n - 315 \}
\]

\[
E_{10}^c(n) = \frac{n(n-1)}{2835} \{ 62n^8 + 62n^7 - 448n^6 - 448n^5 + 1358n^4 + 1358n^3 - 2232n^2 - 2232n + 2835 \}.
\]

**Remark.** The explicit formula (C.5.3) may be deduced from the generating function (C.5.2) in the same way as in A.5. It can be reformulated as

\[
E_{2p}^c(n) = n(-1)^p + \frac{n}{4^p} \sum_{m=0}^{2p} \sum_{k=0}^{m} \left( \frac{1 + 2p}{1 + m} \right)(-1)^{k+p} \quad (C.5.4a)
\]

\[
\times \sum_{j=1}^{p} \left( \frac{kn}{2j-1} \right) \left( \frac{kn + p - j}{p - j} \right) \quad (C.5.4b)
\]

by means of the binomial transform (0.7b).

**Sketch of Proof.** According to the definition of \( E_{2p}^c(n) \), we may formally compute its generating function by series rearrangement

\[
\sum_{p=0}^{\infty} E_{2p}^c(n) y^{2p} = \sum_{k=0}^{n-1} \frac{1}{1 - y^2 \tan^2(\pi (1 + 2k)/(2n))}
\]

\[
= \sum_{k=0}^{n-1} \frac{\cos^2(\pi (1 + 2k)/(2n))}{(1 + y^2) \cos^2(\pi (1 + 2k)/(2n)) - y^2}
\]

\[
= \frac{n}{1 + y^2} + \frac{y}{1 + y^2} \times \sum_{k=0}^{n-1} \frac{y/(1 + y^2)}{\cos^2(\pi (1 + 2k)/(2n)) - y^2/(1 + y^2)}
\]
which reduces to (C5.2) by means of (C0.3b) with the replacements
\[ \cos \theta \rightarrow y \text{ and } \tan(n \theta) = -\tan(n \arctan y). \]

C6. \textit{Sums of } \cot^2(p(1 + 2k)/(2n))

\textbf{Definition.}

\[ \mathcal{F}_p^c(n) = \sum_{k=0}^{n-1} \cot^2 \left( \frac{1 + 2k}{2n} \pi \right) \quad (C6.1) \]

\textbf{Generating Function.}

\[ \sum_{p=0}^{\infty} \mathcal{F}_p^c(n) y^{2p} = \frac{n}{1 + y^2} \left\{ 1 + y \tan(n \arctan y) \right\} \quad (C6.2) \]

\textbf{Explicit Formulae.}

\[ \mathcal{F}_p^c(n) = n(-1)^p + (-1)^p \sum_{m=0}^{2p} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k} \]
\[ \times (-1)^k \sum_{\ell=1}^{p} \left( \frac{2nk}{2\ell-1} \right) \binom{nk+p-\ell}{p-\ell} \quad (C6.3a) \]

\textbf{Examples.}

\[ \mathcal{F}_2(n) = n(n-1) \]
\[ \mathcal{F}_4(n) = \frac{n(n-1)}{3} \{n^2 + n - 3\} \]
\[ \mathcal{F}_6(n) = \frac{n(n-1)}{15} \{2n^4 + 2n^3 - 8n^2 - 8n + 15\} \]
\[ \mathcal{F}_8(n) = \frac{n(n-1)}{315} \]
\[ \times \{17n^6 + 17n^5 - 95n^4 - 95n^3 + 213n^2 + 213n - 315\} \]
\[ \mathcal{F}_{10}(n) = \frac{n(n-1)}{2835} \{62n^8 + 62n^7 - 448n^6 - 448n^5 + 1358n^4 \]
\[ + 1358n^3 - 2232n^2 - 2232n + 2835\}. \]
Remark. The explicit formula (C6.3) follows directly from (C5.3) for they have the same generating function. It has, in turn, an alternative expression

\[
\mathcal{F}_p^c(n) = n(-1)^p + \frac{n}{4^p} \sum_{m=0}^{2p} \sum_{k=0}^{m} \frac{1 + 2p}{1 + m} (-1)^{k+p} \times \sum_{j=1}^{p} \left( \frac{2kn}{2j-1} \right) \left( \frac{kn + p - j}{p - j} \right)
\]

(C6.4a)

(C6.4b)

on account of the binomial transform (0.7b).

Sketch of Proof. According to the definition of \(\mathcal{F}_p^c(n)\), we may formally compute its generating function by series rearrangement

\[
\sum_{p=0}^{\infty} \mathcal{F}_p^c(n)\, \frac{1}{y^{2p}} = \sum_{k=0}^{n-1} \frac{1}{1 - y^2 \cot^2(\pi(1 + 2k)/(2n))}
\]

\[
= \sum_{k=0}^{n-1} \frac{1 - \cos^2(\pi(1 + 2k)/(2n))}{1 - (1 + y^2)\cos^2(\pi(1 + 2k)/(2n))}
\]

\[
= \frac{n}{1 + y^2} \times \sum_{k=0}^{n-1} \frac{y/(1 + y^2)}{\cos^2(\pi(1 + 2k)/(2n)) - 1/(1 + y^2)}
\]

which reduces to (C6.2) by means of (C0.3b) with the replacement \(\tan \theta \rightarrow y\) and some trivial modification.

C7. Alternating sums of \(\tan^{1+2p}(\pi(1 + 2k)/(2n))\)

Definition.

\[
\mathcal{F}_{1+2p}^c(n) = \sum_{k=0}^{n-1} (-1)^k \tan^{1+2p} \left( \frac{1 + 2k}{2n} \right)
\]

(C7.1)
**Generating Function.**

\[
\sum_{p=0}^{\infty} \mathcal{G}^c_{1+2p}(n)y^{1+2p} = (-1)^{1+n/2} \frac{ny}{1+y^2} \sec(n \arctan y) \quad (C7.2)
\]

**Explicit Formulae.**

\[
\mathcal{G}^c_{1+2p}(n) = (-1)^{1+p+n/2} \sum_{m=0}^{2p} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \times \sum_{\ell} \left( \frac{n+2kn}{2\ell} \right) \left( \frac{p-\ell+kn+n/2}{p-\ell} \right) \quad (C7.3a)
\]

\[
\mathcal{G}^c_{1+2p}(n) = (-1)^{1+p+n/2} \sum_{m=0}^{2p} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \times \sum_{\ell} \left( \frac{n+2kn}{2\ell} \right) \left( \frac{p-\ell+kn+n/2}{p-\ell} \right) \quad (C7.3b)
\]

**Examples.** Let \( \mp \) denote the alternating signs corresponding to \( n \equiv 0, 2 \pmod{4} \) respectively. Then we have

\[
\mathcal{G}^c_1(n) = \mp n
\]

\[
\mathcal{G}^c_3(n) = \mp \frac{n}{2} (n^2 - 2)
\]

\[
\mathcal{G}^c_5(n) = \mp \frac{n}{24} (5n^4 - 20n^2 + 24)
\]

\[
\mathcal{G}^c_7(n) = \mp \frac{n}{720} (61n^6 - 350n^4 + 784n^2 - 720)
\]

\[
\mathcal{G}^c_9(n) = \mp \frac{n}{40320} (1385n^8 - 10248n^6 + 31920n^4 - 52352n^2 + 40320).
\]

**Sketch of Proof.** According to the definition of \( \mathcal{G}^c_{1+2p}(n) \), we may formally compute its generating function by series rearrangement

\[
\sum_{p=0}^{\infty} \mathcal{G}^c_{1+2p}(n)y^{1+2p}
\]

\[
= \sum_{k=0}^{n-1} (-1)^k \frac{y \tan(\pi(1+2k)/(2n))}{1-y^2 \tan^2(\pi(1+2k)/(2n))}
\]

\[
= \sum_{k=0}^{n-1} (-1)^k \frac{y \sin(\pi(1+2k)/(2n)) \cos(\pi(1+2k)/(2n))}{(1+y^2) \cos^2(\pi(1+2k)/(2n))}
\]

\[
= \frac{y}{1+y^2} \sum_{k=0}^{n-1} (-1)^k \frac{\sin(\pi(1+2k)/(2n)) \cos(\pi(1+2k)/(2n))}{\cos^2(\pi(1+2k)/(2n))} - y^2/(1+y^2)
\]
which reduces to (C7.2) by means of (C0.3a) with the replacements
\[ \cot \theta \to y \text{ and } \cos(n \theta) = (-1)^{n/2} \cos(n \arctan y). \]
For \( y = \tan \theta \), we have power series expansions similar to those in A6

\[
\frac{2 e^{ni \theta}}{1 + e^{2ni \theta}} = e^{ni \theta} \sum_{m \geq 0} \frac{1}{2m+1} \left( \frac{1}{1 - e^{2ni \theta}} \right)^m = \sum_{m \geq 0} 2^{-m} \sum_{k=0}^m (-1)^k \binom{m}{k} e^{(1+2k)ni \theta}.
\]

By means of (0.4b), we may express the generating function (C7.2) as

\[
(-1)^{1+n/2} \frac{ny}{1+y^2} \sec(n \arctan y)
= (-1)^{1+n/2} \frac{2ny}{1+y^2} \frac{e^{ni \theta}}{1 + e^{2ni \theta}}
= (-1)^{1+n/2} \sum_{m \geq 0} \frac{n}{2^m} \sum_{k=0}^m \binom{m}{k} (-1)^k
\times \sum_{j \geq 0} (-1)^j \binom{n+2kn}{2j} y^{1+2j} \left(1+y^2\right)^{1+kn+n/2}.
\]

Rewriting the last line, through the binomial expansion (0.5b), as a formal power series in \( y \)

\[
\sum_{p \geq 0} (-1)^p y^{1+2p} \sum_{j=0}^p \binom{n+2kn}{2j} \binom{kn+p-n+n/2}{p-j}
\]

we get an explicit formula for \( \mathcal{G}_{i+2p}^c(n) \)

\[
\mathcal{G}_{i+2p}^c(n) = (-1)^{1+p+n/2} \sum_{m \geq 0} \frac{n}{2^m} \sum_{k=0}^m \binom{m}{k} (-1)^k \times \sum_{j=0}^p \binom{n+2kn}{2j} \binom{kn+p-j+n/2}{p-j}.
\] (C7.4a)

\[
\mathcal{G}_{i+2p}^c(n) = (-1)^{1+p+n/2} \sum_{m \geq 0} \frac{n}{2^m} \sum_{k=0}^m \binom{m}{k} (-1)^k \times \sum_{j=0}^p \binom{n+2kn}{2j} \binom{kn+p-j+n/2}{p-j}.
\] (C7.4b)
The last sum with respect to \( k \) is the \( m \)th difference operation on a polynomial with degree \( 2p \), which vanishes for \( m > 2p \). Therefore we may replace the summation limit for \( m \) by \( 2p \) which leads us to (C7.3) promptly.

Alternatively, the summation formula (C7.3) may also be reformulated as

\[
\mathcal{F}_{1+2p}(n) = (-1)^{1+p+n/2} \frac{n}{2^p} \sum_{m=0}^{2p} \sum_{k=0}^{m} \binom{1+2p}{1+m} (-1)^k 
\times \sum_{j=0}^{p} \binom{n+2kn}{2j} \binom{kn+n/2+p-j}{p-j} \tag{C7.5a}
\]

by means of the binomial transform (0.7b).

C8. Alternating sums of \( \cot^{1+2p}(\pi(1+2k)/(2n)) \)

**Definition.**

\[
\mathcal{F}_{1+2p}(n) = \sum_{k=0}^{n-1} (-1)^k \cot^{1+2p} \left( \frac{1+2k}{2n} \pi \right) \tag{C8.1}
\]

**Generating Function.**

\[
\sum_{p=0}^{\infty} \mathcal{F}_{1+2p}(n) y^{1+2p} = \frac{ny}{1+y^2} \sec(n \arctan y) \tag{C8.2}
\]

**Explicit Formulae.**

\[
\mathcal{F}_{1+2p}(n) = (-1)^p \sum_{m=0}^{2p} \frac{n}{2^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k 
\times \sum_{j=0}^{p} \binom{n+2kn}{2j} \binom{p-j+kn+n/2}{p-j} \tag{C8.3a}
\]
Examples.

\[ C^2_1 (n) = n \]
\[ C^2_2 (n) = \frac{n}{2} \{ n^2 - 2 \} \]
\[ C^2_3 (n) = \frac{n}{24} \{ 5n^4 - 20n^2 + 24 \} \]
\[ C^2_4 (n) = \frac{n}{720} \{ 61n^6 - 350n^4 + 784n^2 - 720 \} \]
\[ C^2_5 (n) = \frac{n}{40320} \{ 1385n^8 - 10248n^6 + 31920n^4 - 52352n^2 + 40320 \}. \]

Remark. Notice that the generating function \( C^2_3 \) differs from \( C^2_2 \) only in the alternating factor \( (-1)^{1+n/2} \). Therefore we may write down the explicit formula \( C^2_4 \) directly from \( C^2_3 \). An alternative version of \( C^2_4 \) reads as

\[ C^2_{1+2a} (n) = \frac{n}{4^p} \sum_{m=0}^{2p} \sum_{k=0}^{m} \left( \frac{1+2p}{1+m} \right)(-1)^{k+p} \]
\[ \times \sum_{j=0}^{p} \left( n + 2kn \right) \left( \frac{kn + p - j + n/2}{p-j} \right) \]

in view of the binomial transform \( 0.7_b \).

Sketch of Proof. According to the definition of \( C^2_{1+2a} (n) \), we may formally compute its generating function by series rearrangements

\[ \sum_{p=0}^{\infty} C^2_{1+2p} (n) y^{1+2p} \]
\[ = \sum_{k=0}^{n-1} (-1)^k \frac{y \cot(\pi (1+2k)/(2n))}{1-y^2 \cot^2(\pi (1+2k)/(2n))} \]
\[ = \sum_{k=0}^{n-1} (-1)^k \frac{y \sin(\pi (1+2k)/(2n)) \cos(\pi (1+2k)/(2n))}{1-(1+y^2) \cos^2(\pi (1+2k)/(2n))} \]
\[ = \frac{-y}{1+y^2} \sum_{k=0}^{n-1} (-1)^k \frac{\sin(\pi (1+2k)/(2n)) \cos(\pi (1+2k)/(2n))}{\cos^2(\pi (1+2k)/(2n)) - 1/(1+y^2)} \]

which reduces to \( C^2_3 \) by means of \( C^2_{1+2a} \) with the replacement \( \tan \theta \rightarrow y \) and some trivial modification. \[\square\]
Similar to Sections A and B, the inverse relations for the trigonometric sums defined in this section may be displayed as follows:

\[
\begin{align*}
\mathcal{A}_2^c(n) &= \sum_{\ell=0}^{p} \left( \frac{p}{\ell} \right) \mathcal{A}_2(n) \\
\mathcal{B}_2^c(n) &= \sum_{\ell=0}^{p} (-1)^{p-\ell} \left( \frac{p}{\ell} \right) \mathcal{B}_2(n) \\
\mathcal{C}_2^c(n) &= \sum_{\ell=0}^{p} \left( \frac{p}{\ell} \right) \mathcal{C}_2(n) \\
\mathcal{D}_2^c(n) &= \sum_{\ell=0}^{p} (-1)^{p-\ell} \left( \frac{p}{\ell} \right) \mathcal{D}_2(n) \\
\mathcal{E}_2^c(n) &= \sum_{\ell=0}^{p} \left( \frac{p}{\ell} \right) \mathcal{E}_2(n) \\
\mathcal{F}_2^c(n) &= \sum_{\ell=0}^{p} (-1)^{p-\ell} \left( \frac{p}{\ell} \right) \mathcal{F}_2(n) \\
\mathcal{G}_2^c(n) &= \sum_{\ell=0}^{p} \left( \frac{p}{\ell} \right) \mathcal{G}_2(n) \\
\mathcal{H}_2^c(n) &= \sum_{\ell=0}^{p} (-1)^{p-\ell} \left( \frac{p}{\ell} \right) \mathcal{H}_2(n).
\end{align*}
\]

**APPENDIX: FURTHER PARTIAL FRACTIONS**

Consider the cyclotomic polynomial defined by

\[
1 - z^n = \prod_{k=0}^{n-1} (1 - z \omega_n^k)
\]

where \( \omega_n = \exp(2\pi i/n) \) is the \( n \)th primitive root of unity in the complex field. Its expansion in partial fractions reads as

\[
\frac{n}{1 - z^n} = \sum_{k=0}^{n-1} \frac{1}{1 - z \omega_n^k} \quad (1a)
\]
which may be restated, with the parameter replacement $z \to 1/z$, as

$$\frac{n z^n}{1 - z^n} = \sum_{k=0}^{n-1} \frac{z \omega_n^k}{1 - z \omega_n^k}. \quad (1b)$$

For $z = xe^{iy}$, both relations have the same imaginary part

$$\frac{nx^n \sin ny}{1 + x^{2n} - 2x^n \cos ny} = \sum_{k=0}^{n-1} \frac{x \sin(y + 2k \pi/n)}{1 + x^2 - 2x \cos(y + 2k \pi/n)}. \quad (2a)$$

While their real parts may be displayed, respectively, as

$$\frac{n(1 - x^n \cos ny)}{1 + x^{2n} - 2x^n \cos ny} = \sum_{k=0}^{n-1} \frac{1 - x \cos(y + 2k \pi/n)}{1 + x^2 - 2x \cos(y + 2k \pi/n)} \quad (2b)$$

$$\frac{nx^n(x^n - \cos ny)}{1 + x^{2n} - 2x^n \cos ny} = \sum_{k=0}^{n-1} \frac{x^2 - x \cos(y + 2k \pi/n)}{1 + x^2 - 2x \cos(y + 2k \pi/n)}. \quad (2c)$$

The linear combinations of the last two relations yield

$$\frac{n(1 - x^{2n})}{1 + x^{2n} - 2x^n \cos ny} = \sum_{k=0}^{n-1} \frac{1 - x^2}{1 + x^2 - 2x \cos(y + 2k \pi/n)} \quad (3a)$$

$$\frac{nx^n(x^2 - 2^n) + x^n(1 - x^2) \cos ny}{1 + x^{2n} - 2x^n \cos ny} = \sum_{k=0}^{n-1} \frac{x(1 - x^2) \cos(y + 2k \pi/n)}{1 + x^2 - 2x \cos(y + 2k \pi/n)}. \quad (3b)$$

which may be used to establish some more trigonometric summation formulas. The interested reader is encouraged to try further.

REFERENCES

