Discrete Mathematics 78 (1989) 45–53
North-Holland 

# AN INDUCTIVE DEFINITION OF THE CLASS OF 3-CONNECTED QUADRANGULATIONS OF THE PLANE

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Received 11 January 1988
Revised 17 June 1988

An inductive definition of the class of all 3-connected quadrangulations of the plane is given. The dual inductive definition determines the class of all 3-connected 4-regular planar graphs.

The *inductive class* $\mathfrak{I} = \mathrm{Cn}(\mathfrak{B}; \mathfrak{R})$ is defined by giving the class $\mathfrak{B}$ of initial objects—the *basis* of $\mathfrak{I}$, and the class $\mathfrak{R}$ of *generating rules*. Any such rule applied to an appropriate sequence of objects, already in $\mathfrak{I}$, produces an object of $\mathfrak{I}$. The inductive class $\mathfrak{I}$ consists exactly of the objects which can be constructed from the basis by a finite number of applications of generating rules.

In this paper an inductive definition of the class of all 3-connected quadrangulations of the plane is given.

A simple graph $G$ is a quadrangulation of the plane iff it can be embedded in the plane in such a way that all faces are quadrangles. By $\mathfrak{Q}$ we shall denote the class of all connected quadrangulations of the plane. Some graphs from $\mathfrak{Q}$ are presented in *Fig. 1.* From the last of these examples we can see that the quadrangulations of the plane are not necessarily 3-connected; even if there is no vertex of degree 2.

*BQ* and *PQ* are given in *Fig. 2.* The base graph *BQ* and rule *PQ* should be understood as embedded in the plane (sphere). The small triangles attached to the vertices in the description of the rule denote any number (zero or more) of edges. It is easy to see that:

**Theorem 1.** $\mathfrak{Q} = \mathrm{Cn}(BQ; PQ)$.

In the following we shall limit our attention to the class $\mathfrak{Q}_3$ of all 3-connected quadrangulations of the plane, which have (essentially) unique embedding in the plane (sphere).

**Lemma 2.** *Every* 3-*connected quadrangulation of the plane contains at least* 8 *vertices of degree* 3.

Detailed version of the talk presented at the Eleventh British Combinatorial Conference, London, July 13–17, 1987.

This work was supported in part by the Research Council of Slovenia, Yugoslavia.

0012-365X/89/$3.50 

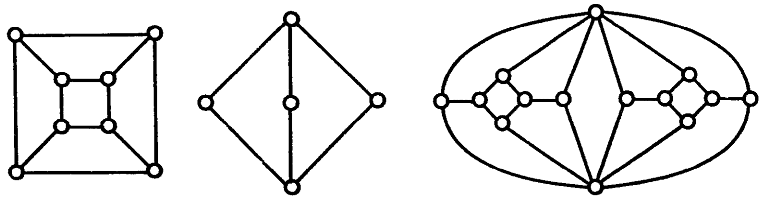


Fig. 1.

**Proof.** Let $G$ be a 3-connected quadrangulation of the plane with $p$ vertices, $q$ edges and $f$ faces. Also, let $F(G)$ denote the set of all faces, $F(e)$ the set of the two faces which have the edge $e$ in common; and $E(F)$ the set of edges on the boundary of the face $F$. Then it holds

$$\sum_{e\in E(G)} \operatorname{card} F(e) = \sum_{F\in F(G)} \operatorname{card} E(F).$$

But, $\operatorname{card} F(e) = 2$ for each $e \in E(G)$ and $\operatorname{card} E(F) = 4$ for each $F \in F(G)$. Therefore we obtain the equality

$$q = 2f$$

and from the Euler's formula $q + 2 = p + f$ finally

$$f + 2 = p.$$

Because $G$ is 3-connected it has no vertex of degree 1 or 2. Let $t$ be the number of vertices of degree 3 in $G$. Then we have:

$$\sum_{v\in V(G)} d(v) = 2q = 4(p-2)$$

and

$$\sum_{v\in V(G)} d(v) = \sum_{v:d(v)\geqslant 4} d(v) + \sum_{v:d(v)=3} d(v) \geqslant 4(p-t) + 3t$$

from where the inequality $t \geqslant 8$ follows. □

**Theorem 3.** $\mathfrak{Q}_3 = \mathrm{Cn}(B; P1, P2, P3)$.

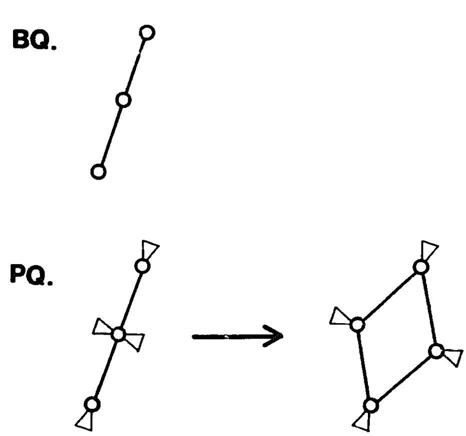



Fig. 2.

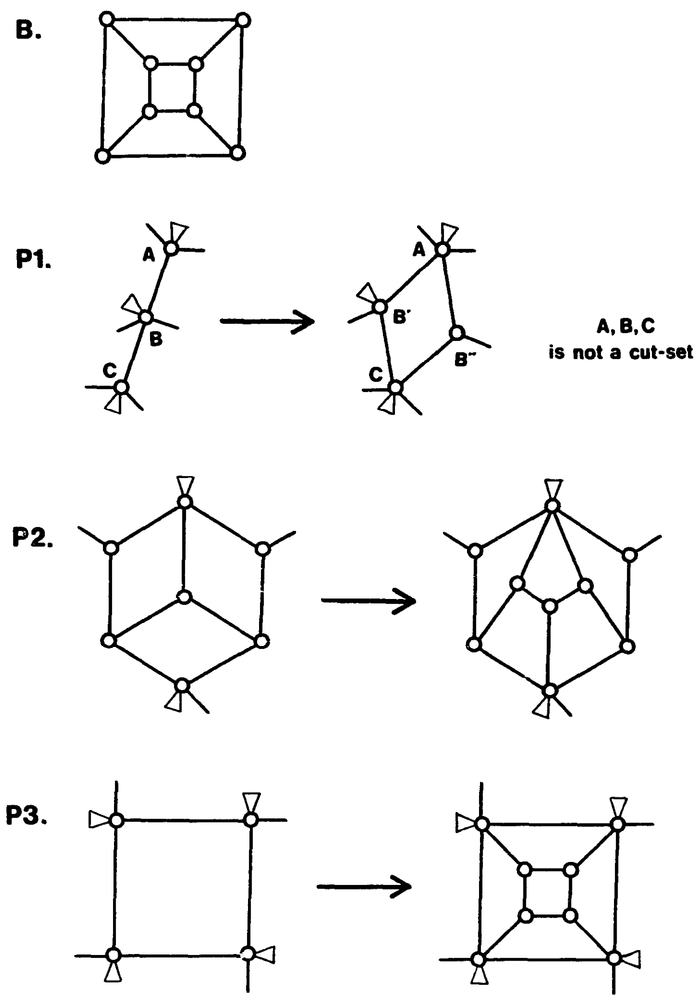



Fig. 3.

$B$, $P1$, $P2$ and $P3$ are given in *Fig. 3.* The halfedges in the figure indicate that there must be an edge.

**Proof.** The proof of the theorem follows the standard pattern [2]. By inductive generalization we prove:

($A$) $\mathrm{Cn}(B; P1, P2, P3) \subseteq \mathfrak{Q}_3$.

By proving the existence of a construction in $\mathfrak{I}$ for each $G \in \mathfrak{Q}_3$ we show:

($B$) $\mathfrak{Q}_3 \subseteq \mathrm{Cn}(B; P1, P2, P3)$.

($A$) The base graph $B$ (1-skeleton of the 3-cube) is 3-connected quadrangulation. Obviously the rules $P1$, $P2$ and $P3$ transform a quadrangulation into a quadrangulation. Let us prove that they also preserve 3-connectedness:

The effect of an application of rule $P1$ to graph $G$ is represented in *Fig. 4.* Let us suppose that the extended graph $G^*$ is 2-connected and not 3-connected. Then

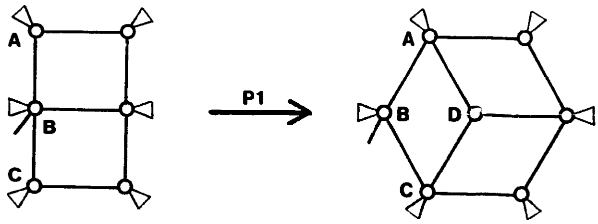



Fig. 4.

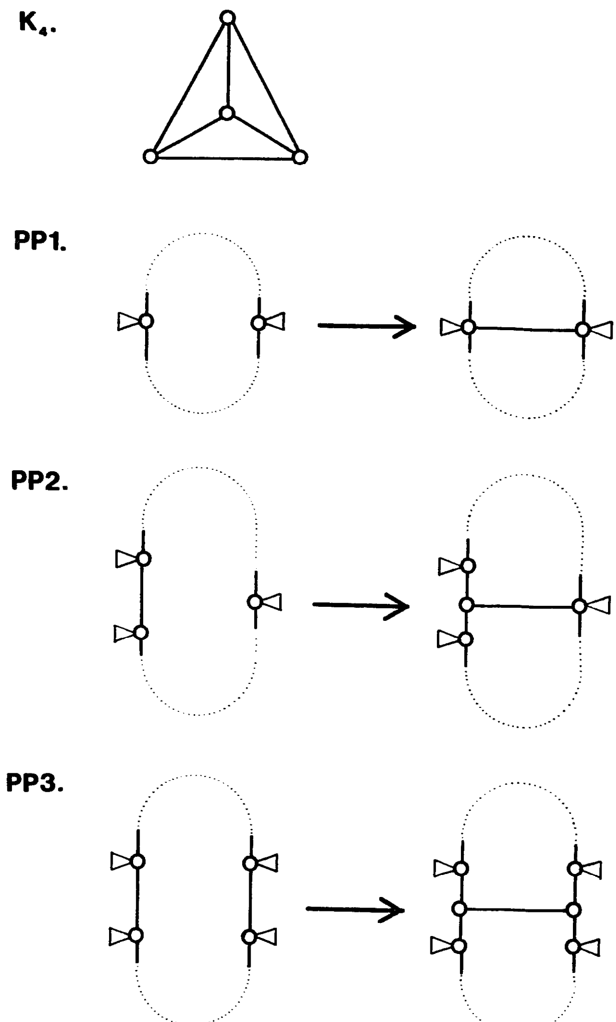



Fig. 5.

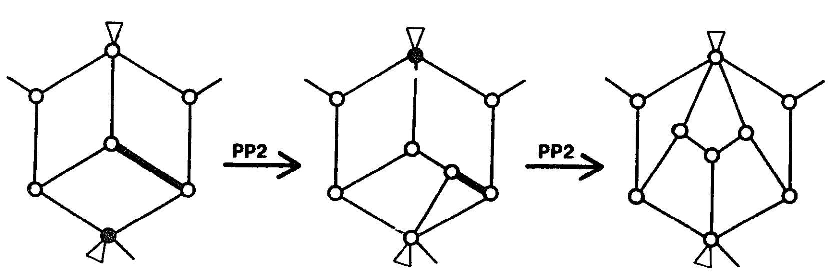



Fig. 6.

the graph $G^*$ contains a (vertex) cut-set $\{U, V\}$ which has a non-empty intersection with the set $\{A, B, C, D\}$; otherwise the set $\{U, V\}$ would be a cut-set also in the original graph $G$. There are two possibilities:

(a) The sets $\{U, V\}$ and $\{A, B, C, D\}$ have only one vertex in common. Because the vertices $A$, $B$, $C$, $D$ lie on the cycle, the remaining three vertices belong to the same component with respect to the cut-set $\{U, V\}$. Therefore the set $\{U, V\}$ is a cut-set also in $G$.

(b) The set $\{U, V\}$ is a subset of $\{A, B, C, D\}$. Obviously there are six cases. We can easily see that in five of these cases $G$ would also have to be 2-connected. The remaining case $\{U, V\} = \{A, C\}$ requires a special argument: If $\{A, C\}$ is a cut-set of $G^*$ then the set $\{A, B, C\}$ is a cut-set of $G$, but this violates the condition of the applicability of the rule $P1$.

In all cases we obtained a contradiction. The extended graph $G^*$ has to be 3-connected.

The proof that rules $P2$ and $P3$ preserve 3-connectedness is simpler—we deduce these two rules in the class of all 3-connected planar graphs [7, 1]:

$$\mathfrak{G}_{P3} = \mathrm{Cn}(K_4; PP1, PP2, PP3).$$

Graph $K_4$ and rules $PP1$, $PP2$, $PP3$ are presented in *Fig. 5.* The deductions of rules $P2$ and $P3$ are given in *Fig.* 6 and *Fig. 7.*

($B$) The basic graph $B$ is the only 3-connected quadrangulation on at most 8 vertices. To prove that also $\mathfrak{Q}_3 \subseteq \mathrm{Cn}(B; P1, P2, P3)$, we must show that every 3-connected quadrangulation $G \in \mathfrak{Q}_3$, different from $B$, can be reduced by the inverse rules $P1^-$, $P2^-$ and $P3^-$ to a quadrangulation of the same type.

In Lemma 2 we proved that every 3-connected quadrangulation of the plane contains at least 8 vertices of degree 3. In the following figures we shall represent the vertices of degree 3 by black circles. There are two cases to be considered:

(a) There exists a vertex $X$ of degree 3 which has at least two neighbours of degree at least 4. The situation is presented on the left half of *Fig. 8.*

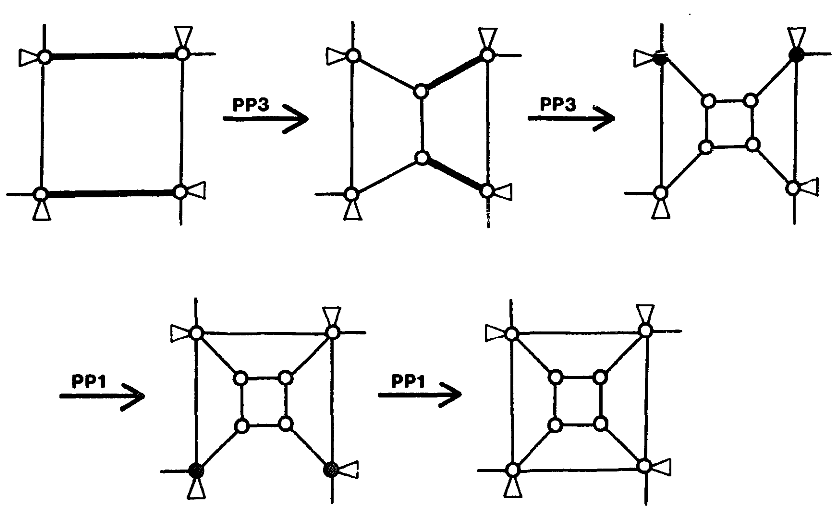



Fig. 7.

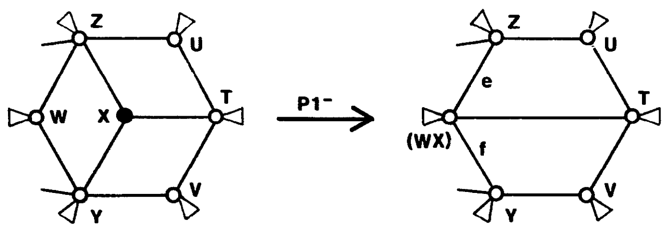



Fig. 8.

If we do not destroy the 3-connectedness we can apply the rule $P1^-$ to the quadrangle $YXZW$. Otherwise the reduced graph $G'$ is only 2-connected. The edges $e$ and $f$ are in different components of graph $G'$ with respect to the 2-cut-set; otherwise $G$ would also be only 2-connected. Therefore the vertex $(WX)$ belongs to the 2-cut-set of graph $G'$, and vertices $X$ and $W$ belong to the 3-cut-set of graph $G$. Since the vertices $W$, $Y$, $V$, $T$, $U$, $Z$ lie on the same cycle, the third vertex of the cut-set has to be $U$, $T$, or $V$. Therefore the graph $G$ has one of the two forms represented in *Fig. 9* or the graph $G$ has the form given on the right of *Fig. 9* with the roles of $U$ and $V$ interchanged. Other forms are impossible because all cycles in a quadrangulation are of even length [6]. In both cases we can apply the rule $P1^-$ to the quadrangle $YVTX$ without destroying 3-connectedness.

(b) Each vertex of degree 3 has at least two neighbours of degree 3. Let $X$ be one among them. The neighbourhood of $X$ is represented in *Fig. 10.* There are

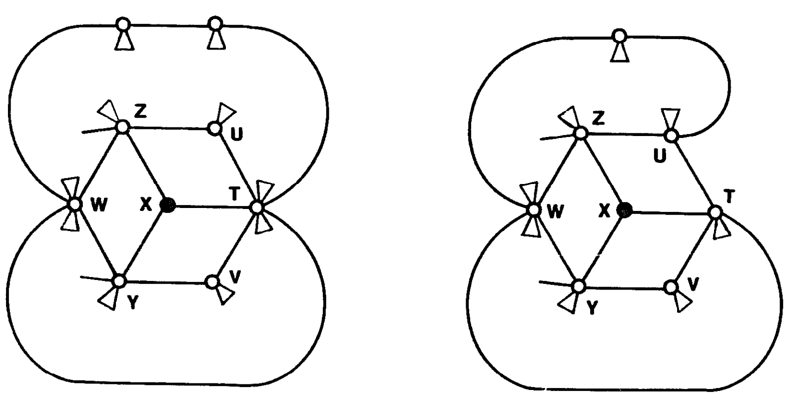



Fig. 9.

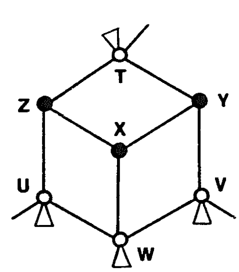



Fig. 10.

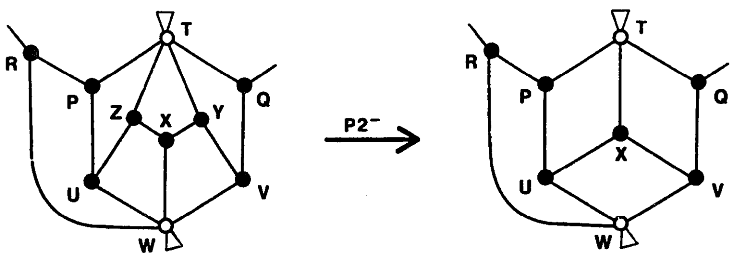



Fig. 11.

two possibilities:

(b1) There exists a vertex $X$ such that the vertex $T$ is of degree at least 4. In this case *Fig. 10* can be uniquely extended to the situation presented on the left part of *Fig. 11.* By the assumption of the case $b$ the vertices $U$ and $V$ are of degree 3. The vertices $P$ and $Q$ are not adjacent, neither is vertex $W$ their neighbour; otherwise we would get a cycle of length 3. Also $P \neq Q$ because otherwise $PTYV$ would bound a face, thus forcing $T$ to have degree 3. Vertex $W$ is of degree at least 4; otherwise the face with boundary $\ldots PUWV \ldots$ would not be a quadrangle. Adjacent to $P$ there exists a vertex $R$ (of degree 3), different from $Q$ and $W$. In this case we can apply the rule $P2^-$. The reduced graph $G'$ is also 3-connected because there is no pair of vertices, candidate for a 2-cut-set in the reduced graph, which is not a 2-cut-set already in $G$.

(b2) Each vertex of degree 3 belongs to a quadrangle which has all of its vertices of degree 3. Again we can uniquely extend it to the situation represented in *Fig. 12.* Let us show that all four vertices $S$, $U$, $V$ and $W$ are of degree at least 4. At least three of them are of degree at least 4 because otherwise there would exist a 1- or 2-cut-set in graph $G$. Suppose now that one vertex among $S$, $U$, $V$ and $W$ is of degree 3. But, this vertex would have two neighbours of degree at least 4, in contradiction with the assumption of the case $b$.

Let us show that among the quadrangles with all vertices of degree 3 we can always find one to which we can apply the rule $P3^-$ obtaining a 3-connected reduced graph $G'$.

Suppose the contrary – each application of the rule $P3^-$ produces a reduced graph which is not 3-connected. Let $XYTZ$ be such a quadrangle. The reduced graph $G'$ is (only) 2-connected. Then either $\{S, W\}$ or $\{U, V\}$ is a 2-cut-set of

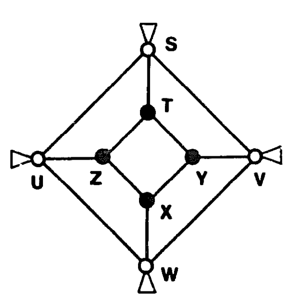



Fig. 12.

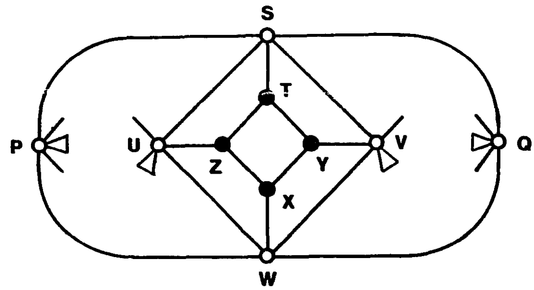


Fig. 13.

$G'$—two adjacent vertices of $S$, $W$, $U$, $V$ cannot form a 2-cut-set because this would be also a 2-cut-set in $G$. Let (because of symmetry) $\{S, W\}$ be a cut-set of $G'$. Because in a quadrangulation there is no odd cycle the graph $G$ has the form presented in *Fig. 13.* By the assumption of case $b$ the vertices $P$ and $Q$ are of degree at least 4.

Let us now concentrate our attention to the subgraph consisting of the quadrangle *PWUS* and its interior. It is also a quadrangulation and by Lemma 2 and other assumptions it contains in its interior at least one quadrangle with all vertices of degree 3. Because an application of the rule $P3^-$ to it destroys 3-connectedness, graph $G$ must have the form presented in *Fig. 14.* But now we can repeat the same argument on the subgraph determined by quadrangle *PWIS*, and so on, infinitely many times in contradiction with the finiteness of graph $G$.

This completes the proof of Theorem 3. □

In the dual form we can express Theorem 3 as follows:

**Theorem 3'.** *The inductive class* $\mathrm{Cn}(b; p1, p2, p3)$ (*see Fig. 15*) *is equal to the class of all* 3-*connected* 4-*regular planar graphs.*

This result complements the inductive definition of the class of all 4-regular graphs [3] and the inductive definition of the class of all 4-regular planar graphs [5, 4].

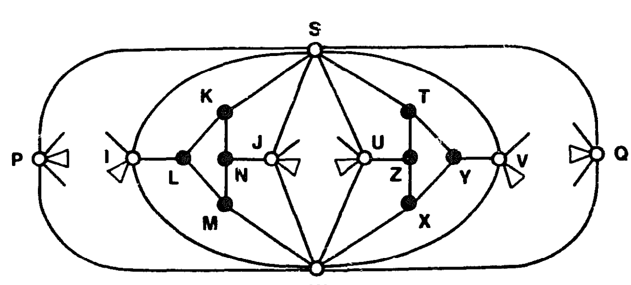
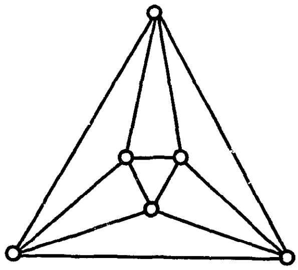
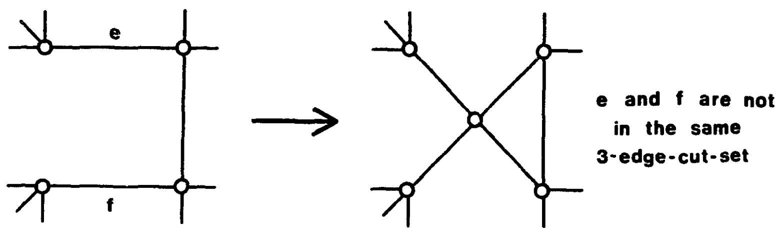


Fig. 14.

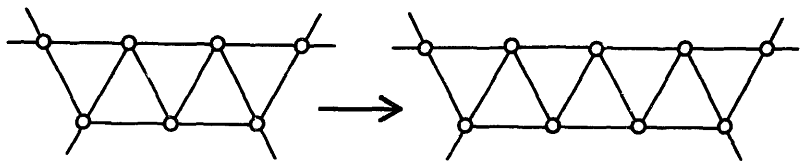
b.



p1.




p2.



p3.

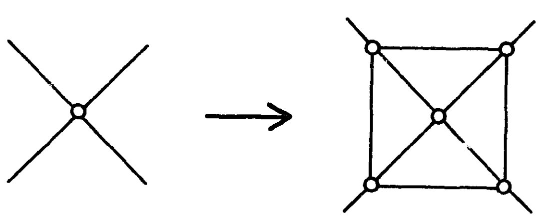


Fig. 15.

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