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The square of a block graph

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ABSTRACT

The square H^2 of a graph H is obtained from H by adding new edges between every two vertices having distance two in H. A block graph is one in which every block is a clique. For the first time, good characterizations and a linear time recognition of squares of block graphs are given in this paper. Our results generalize several previous known results on squares of trees.

1. Introduction

A graph H is a *square root* of a graph G and G is the *square* of H, written $G = H^2$, if two distinct vertices are joined by an edge in G if and only if they are of distance at most two in H. Graph square is a basic operation with a number of results about its properties in the literature. Ross and Harary [20] characterized squares of trees and showed that tree square roots, when they exist, are unique up to isomorphism. Mukhopadhyay [18] provided a characterization of graphs which have a square root, but this is not a good characterization in the sense that it does not give a short certificate when a graph does have a square root. In fact, such a good characterization may not exist as Motwani and Sudan [17] proved that it is NP-complete to determine if a given graph has a square root. On the other hand, there are polynomial time algorithms to recognize squares of trees [15,11,13,3,5], squares of bipartite graphs [13], and, very recently, squares of graphs having girth at least six [8] (the girth of a graph is the smallest length of a cycle in the graph). Note that bipartite graphs, as well as graphs having girth at least six generalize trees in such a way that these do not have cliques of size larger than two. It should be remarked that known polynomial time recognitions for squares of trees, of bipartite graphs, and of graphs having girth at least six depend partly on this fact; chordal graphs also generalize trees but deciding if a graph is the square of a chordal graph is NP-complete; see [14].

Another natural generalization of trees are *block graphs*; these are exactly the connected graphs in which every block (i.e., every maximal 2-connected subgraph) is a clique. Powers of block graphs have been considered in [6] in context of interval number, and in [2] in context of leaf powers and simplicial powers. To the best of our knowledge, the complexity of recognizing powers of block graphs, as well as the characterization problem are not yet discussed in the literature.

In this paper we consider the characterization and recognition problems of graphs that are squares of block graphs, i.e., for a given graph G, to determine if $G = H^2$ for some block graph H. We first give relevant properties of squares of block graphs in Section 2. Then, based on these properties, we will provide in Section 3 good characterizations for graphs that are squares of block graphs and in Section 4 a simple linear time algorithm to compute a square root that is a block graph (if any). In Section 5 we will derive known results for squares of trees from our discussions.

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Definitions and notation. All graphs considered are finite, undirected and simple. Let $G = (V_G, E_G)$ be a graph. We often write $xy \in E_G$ for $\{x, y\} \in E_G$. The *neighborhood* $N_G(v)$ in G of a vertex v is the set all vertices in G adjacent to v and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. Set $\deg_G(v) = |N_G(v)|$, the *degree* of v in G. A *universal vertex* of G is one that is adjacent to all other vertices.

Let $d_G(x, y)$ be the length, i.e., number of edges, of a shortest path in G between x and y. Let $G^k = (V_{G^k}, E_{G^k})$ with $V_{G^k} = V_G$ and $xy \in E_{G^k}$ if and only if $1 \le d_G(x, y) \le k$ denote the *kth power of* G. If $G = H^k$ then G is the kth power of the graph H and H is a *k-th root* of G. Since the power of a graph H is the union of the powers of the connected components of H, we may assume that all graphs considered are connected.

A set of vertices $Q \subseteq V_G$ is called a *clique* in G if every two distinct vertices in Q are adjacent; a *maximal clique* is a clique that is not properly contained in another clique. A *stable set* is a set of pairwise non-adjacent vertices. Given a set of vertices $X \subseteq V_G$, the subgraph induced by X is written $G[X]$ and $G - X$ stands for $G[V \setminus X]$. If $X = \{a, b, c, \ldots\}$, we write $G[a, b, c, \ldots]$ for $G[X]$. Also, we often identify a subset of vertices with the subgraph induced by that subset, and vice versa.

For $\ell \ge 1$, let P_ℓ denote a chordless path with ℓ vertices and $\ell - 1$ edges, and for $\ell \ge 3$, let C_ℓ denote a chordless cycle with ℓ vertices and ℓ edges. A complete graph is one in which every two distinct vertices are adjacent; a complete graph on ℓ vertices is also denoted by K_ℓ. A graph is *chordal* if it contains no induced C_ℓ, $\ell \ge 4$. It is well known (see, e.g, [9]) that a graph is chordal if and only if each of its induced subgraph admits a *simplicial* vertex; here, a vertex is simplicial if its neighborhood is a clique (equivalently, if it belongs to exactly one maximal clique).

For a positive integer k, a *k-connected component* in a graph G is a maximal (induced) k-connected subgraph of G; the 1-connected components of G are the usual connected components, and the 2-connected components of G are also called *blocks* of G. A *k-cut* in a graph is a cutset with k vertices; a 1-cut is also called a *cut-vertex*. An *endblock* in a graph is a block that contains at most one cut-vertex of the graph.

A connected graph is a *block graph* if its blocks are cliques. The following theorem collects several known characterizations of block graphs.

Theorem 1.1. *For all graphs G, the following statements are equivalent*

(i) *G is a block graph;*
(ii) *G is the intersection graph of blocks of some connected graph;*
(iii) *G is a connected diamond-free chordal graph;*
(iv) *Between every two vertices in G there is exactly one chordless path.*

Where the *diamond* is a K_4 minus an edge. The equivalence (i) ⇔ (ii) is Theorem 3.5 in [10], and the equivalence (i) ⇔ (iii) can be easily seen, e.g., by [3, Observation 3]. The equivalence (i) ⇔ (iv) can easily be seen as follows: The direction (i) ⇒ (iv) is obvious; (iv) implies that every 2-connected component of G must be a clique, hence (i).

Finally, we remark that block graphs can be recognized in linear time: By an algorithm in [22], the blocks of a given graph $G = (V_G, E_G)$ can be detected in linear time. Then, testing if all blocks of G are cliques can be done in an obvious way in $O(|V_G| + |E_G|)$ time.

2. Basic facts

In this section we give basic properties of squares of block graphs which form a starting point for our characterizations of such graphs in Section 3.

Let x, y be two non-adjacent vertices in a graph $G = (V_G, E_G)$. A subset $S \subseteq V_G$ is an *x, y-separator* if x and y belong to different connected components of $G - S$. A *separator* is an x, y-separator for some non-adjacent vertices x, y. A *minimal* separator is an x, y-separator that is not properly contained in other x, y-separator.

Observation 2.1. *Let $G = H^2$ for some block graph H. Let B be a non-endblock of H and let $u \ne v$ be two cut-vertices of H in B. Let X and Y be two connected components of $H - B$ such that $N_H(u) \cap X \ne \varnothing$ and $N_H(v) \cap Y \ne \varnothing$. Then B is a minimal x, y-separator in G for any pair of vertices $x \in X$, $y \in Y$.*

Proof. Clearly, B is a separator in G disconnecting any pair of vertices $x \in X$, $y \in Y$. Moreover, in G, every vertex $w \in B$ is adjacent to a vertex in X and to a vertex in Y, implying $B - w$ does not separate X and Y. Thus, B is a minimal x, y-separator in G for any pair of vertices $x \in X$, $y \in Y$. □

The following fact is the key observation for further discussions.

Proposition 2.2. *Let G be a connected, non-complete graph such that $G = H^2$ for some block graph H. Then the maximal cliques in G are exactly the closed neighborhoods $N_H[v]$ for cut-vertices v in H.*

Proof. (i) Let v be a cut-vertex in H. Clearly, $Q = N_H[v]$ is a clique in G. Consider an arbitrary vertex $x \in V_H \setminus Q$ (note that such a vertex exists as G is not complete), and let B be a block of H containing v such that x does not belong to the connected component of $H - v$ containing $B - v$. Then $d_H(x, y) \ge 3$ for all vertices $y \in B - v$, hence x cannot be adjacent, in G, to all vertices in Q. Therefore, Q is a maximal clique in G.

(ii) Let Q be a maximal clique in G. Among all vertices in Q, let $v \in Q$ be a vertex with inclusion-maximal $Q \cap N_H[v]$. We will see that v is a cut-vertex of H and $Q = N_H[v]$.

Assume first, by way of contradiction, that v is a simplicial vertex in H and let B be the unique block of H containing v. Then, as G is not complete, B contains at least one cut-vertex. Clearly, for all cut-vertices u of H in B and for all vertices $x \in V_H$, if $d_H(v, x) \le 2$, then $d_H(u, x) \le 2$. In particular $d_H(u, q) \le 2$ for all $q \in Q$. Hence, by the maximality of Q, $u \in Q$. Moreover, $Q \cap N_H[v] = Q \cap B \subseteq Q \cap N_H[u]$, and therefore, by the choice of v, $Q \cap B = Q \cap N_H[u] \subseteq B$ for all cut-vertices u of H in B. This implies that $Q \subseteq B$, contradicting the maximality of Q.

Hence v must be a cut-vertex in H. Next, we claim that $Q \setminus N_H[v] = \varnothing$. If not, consider a vertex $w \in Q \setminus N_H[v]$. As $d_H(w, v) = 2$, there exists a cut-vertex u such that $vu, uw \in E_H$. Note that, in H, u separates v and w. Hence $Q \subseteq N_H[u]$ because $d_H(q, v) \le 2$ and $d_H(q, w) \le 2$ for all $q \in Q$. By the maximality of Q, $Q = N_H[u]$. But then $Q \cap N_H[v]$ is the block in H containing vu which is properly contained in $N_H[u] = Q \cap N_H[u]$, contradicting the choice of v. Hence $Q \setminus N_H[v] = \varnothing$, as claimed.

Thus $Q \subseteq N_H[v]$, and by the maximality of Q, $Q = N_H[v]$. □

Minimal separators in squares of block graphs can be characterized as follows.

Proposition 2.3. *Let G be a connected, non-complete graph such that $G = H^2$ for some block graph H. Then the following conditions are equivalent:*

(i) *S is a minimal separator of G;*
(ii) *S is a non-endblock of H;*
(iii) *$|S| \ge 2$ and S is the intersection of two maximal cliques of G.*

Proof. (i) ⇒ (ii) Let S be a minimal x, y-separator of G. Let $xv_1 \dots v_\ell y$ be the shortest path in H connecting x and y. Since $d_H(x, y) \ge 3$, $\ell \ge 2$. Note that all v_i are cut-vertices of H. For each $1 \le i < \ell$, let B_i be the block of H containing $v_i v_{i+1}$. By Observation 2.1, each B_i is a minimal x, y-separator of G. If $S \ne B_i$ for all i, then, by the minimality of the x, y-separators S and B_i, $B_i - S \ne \varnothing$ for all i. Let $b_i \in B_i - S$, $1 \le i < \ell$ (possibly $b_i = b_j$ for some $i \ne j$). Now by noting that x and b_1, $b_{\ell-1}$ and y are adjacent in G, as well as $G[\{b_i \mid 1 \le i < \ell\}]$ contains a path with endpoints $b_1, b_{\ell-1}$, we get the contradiction that S does not separate x and y. Thus, we conclude that $S = B_i$ for some i, hence (ii).

(ii) ⇒ (iii) Let S be a non-endblock of H. Then $|S| \ge 2$, and S contains at least two cut-vertices $u \ne v$ of H. By Proposition 2.2, $Q = N_H[u]$ and $Q' = N_H[v]$ are maximal cliques in G. Clearly, $S = Q \cap Q'$.

(iii) ⇒ (i) Let Q, Q' be two maximal cliques in G such that $S = Q \cap Q'$ has at least two vertices. By Proposition 2.2, $Q = N_H[u]$ and $Q' = N_H[v]$ for some cut-vertices $u \ne v$ in H. Since $|N_H[u] \cap N_H[v]| = |S| \ge 2$, u and v must be adjacent in H. Hence S is the non-endblock in H containing uv, and (i) follows from Observation 2.1. □

As a corollary of Proposition 2.3, all minimal separators of the square of a block graph are cliques with at least two vertices, hence squares of block graphs are chordal (indeed, it is well known that a graph is chordal if and only if each of its minimal separators is a clique; see, e.g., [9]) and 2-connected.

Recall that a chordal graph is *strongly chordal* if it does not contain any ℓ-sun as an induced subgraph; here an *ℓ-sun*, $\ell \ge 3$, consists of a clique $\{u_1, u_2, \dots, u_\ell\}$ and a stable set $\{v_1, v_2, \dots, v_\ell\}$ such that for $i \in \{1, \dots, \ell\}$, v_i is adjacent to exactly u_i and u_{i+1} (index arithmetic modulo ℓ). Clearly, block graphs are strongly chordal. It was shown in [7,16,19] that all powers of strongly chordal graphs are strongly chordal. In particular, squares of block graphs (hence of trees) are strongly chordal; later, unknowing this fact, [15,1] proved that the square of a tree is chordal. As another consequence of Proposition 2.3, we give a new and short proof for this fact:

Corollary 2.4 (*[7,16,19]*). *Squares of block graphs are strongly chordal.*

Proof. Let G be a non-complete graph that is the square of a block graph H. As pointed out, G is chordal. Suppose G contains an induced ℓ-sun with clique $\{u_1, u_2, \dots, u_\ell\}$ and stable set $\{v_1, v_2, \dots, v_\ell\}$. Let Q be a maximal clique in G containing $\{u_1, u_2, \dots, u_\ell\}$, and for each $i \in \{1, \dots, \ell\}$, let Q_i be a maximal clique of G containing v_i, u_i and u_{i+1}. Now, $Q \cap Q_i$, $1 \le i \le \ell$, contains u_i and u_{i+1} but none of $\{u_1, u_2, \dots, u_\ell\} \setminus \{u_i, u_{i+1}\}$, hence they are pairwise distinct blocks in H. But then the cycle in H with edges $u_1u_2, u_2u_3, \dots, u_{\ell-1}u_\ell, u_\ell u_1$ belongs to distinct blocks of H, a contradiction. Thus, G is a strongly chordal graph. □

The structure of the minimal separators in squares of block graphs is now described in the following proposition. Given a block graph H, a simplicial vertex of H belonging to an endblock of H is called a *leaf*.

Proposition 2.5. *Let G be a connected, non-complete graph such that $G = H^2$ for some block graph H.Let F be the subgraph of G formed by all minimal separators of G. Then*

(i) *F is obtained from H by deleting all leaves of H. In particular, F is a block graph whose blocks are exactly the minimal separators of G;*
(ii) *For all maximal cliques Q and Q' of G with $|Q \cap Q'| \ge 2$, $Q \cap Q'$ is a block of F;*
(iii) *Every block S of F belongs to at least two and at most $|S|$ maximal cliques of G;*
(iv) *Every two non-disjoint blocks of F belong to a common maximal clique of G;*

(v) *For all maximal cliques Q of G, $Q \cap V_F = N_F[v]$ for some vertex v of F;*
(vi) *$V_G \setminus V_F$ is the set of all simplicial vertices of G. Moreover, for every vertex $v \in V_G \setminus V_F$, $|N_H(v) \cap V_F| = 1$, and if $N_H(v) \cap V_F = \{u\}$, then $N_G(u) \setminus V_F = N_H(u) \setminus V_F$.*

Proof. (i) and (ii) follow from Proposition 2.3.

(iii) follows from Propositions 2.2 and 2.3, by noting that any block S in H clearly contains at most $|S|$ cut-vertices, and if S is a non-endblock, S contains at least two cut-vertices.

(iv) Two non-disjoint minimal separators S and S' of G are two non-endblocks of H (by Proposition 2.3) having a common cut-vertex, say v. Hence $S \cup S' \subseteq N_H[v]$, and (iv) follows from Proposition 2.2.

(v) Let Q be a maximal clique in G. By Proposition 2.2, $Q = N_H[v]$ for some cut-vertex v of H. Since G is not complete, some block of H in $N_H[v]$ must be a non-endblock, hence $v \in V_F$, and by (i), $N_F[v] = Q \cap V_F$.

(vi) If v has two non-adjacent neighbors $x \neq y$, then any minimal x, y-separator in G must contain v, hence $v \in V_F$. Thus, every vertex in $V_G \setminus V_F$ must be simplicial in G. On the other side, by (iii), every vertex in F belongs to at least two maximal cliques. Thus, $V_G \setminus V_F$ consists of exactly the simplicial vertices of G. The second part follows directly from the following observation: By (i), any vertex $v \in V_G \setminus V_F$ is a leaf of H and belongs to an endblock B_v of H. As G is not complete, the cut-vertex u of H in B_v must belong to a non-endblock of H, hence $N_H(v) \cap V_F = (B_v \setminus \{v\}) \cap V_F = \{u\}$, and $N_G(u) \setminus V_F$ consists of exactly the leaves of H that belong to an endblock containing u. □

Unlike tree roots, block graph roots are not unique in general. Indeed, if H is a block graph and H' is the block graph obtained from H by deleting all edges joining two simplicial vertices in an endblock of H (thus, every endblock in H' is an edge), then clearly H^2 and $(H')^2$ coincide; see also Fig. 1.

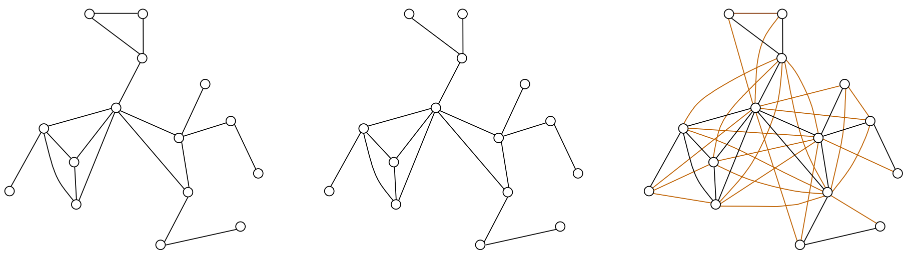

Fig. 1. Two block graphs (left and middle) with the same square (right).

Theorem 2.6. *Block graph square roots in which every endblock is an edge are unique up to isomorphism.*

Proof. Let H_1, H_2 be two block graphs in which every endblock is an edge, and assume that $f : H_1^2 \to H_2^2$ is an isomorphism. We will show that H_1 and H_2 are isomorphic by pointing out that the restriction $f : H_1 \to H_2$ of f is an isomorphism.

Write $G_i = H_i^2$, $i = 1, 2$. If G_1 or G_2 is a clique, then H_i must be stars (as every endblock in H_i is an edge) with the same vertex number, hence they are isomorphic. So, assume that G_i are non-complete, and let F_i be the subgraph of G_i formed by the minimal separators of G_i. By Proposition 2.5(i), F_i is a block graph and each block of F_i is a non-endblock of H_i.

Claim 1: The restrictions $f : V_{F_1} \to V_{F_2}$ and $f : V_{H_1} \setminus V_{F_1} \to V_{H_2} \setminus V_{F_2}$ of f are bijections, and $V_{H_i} \setminus V_{F_i}$ is a stable set in H_i, $i = 1, 2$.

Proof of Claim 1: The first part follows from Proposition 2.5(vi), the second part follows from our assumption on the block graphs H_i.

Claim 2: For all $v, v' \in V_{F_1}$: $vv' \in E_{F_1}$ if and only if $f(v)f(v') \in E_{F_2}$.

Proof of Claim 2: Note that, by Claim 1, $f(v), f(v') \in V_{F_2}$. Let $vv' \in E_{F_1}$. Then $f(v)f(v')$ is an edge in G_2. If $f(v)f(v') \notin E_{F_2}$, then $f(v)$ and $f(v')$ must belong to different blocks $B_2 \neq B_2'$ in F_2 with $B_2 \cap B_2' \neq \emptyset$. Consider the block B in F_1 containing vv'. As B is a non-endblock of H_1, there are different blocks B_1, B_1' of H_1 with $\emptyset \neq B \cap B_1 \neq B \cap B_1' \neq \emptyset$. Let $x \in B_1 \setminus B$, $x' \in B_1' \setminus B$. Then $xx' \notin E_{G_1}$ but $f(x)f(x') \in E_{G_2}$ because $f(x)$ and $f(x')$ are adjacent in G_2 to both $f(v)$ and $f(v')$, hence $f(x)$ and $f(x')$ must belong to some blocks in H_2 containing the cut-vertex $B_2 \cap B_2'$. This contradiction shows that $f(v)f(v') \in E_{F_2}$. Along the same line, it can be seen that $f(v)f(v') \in E_{F_2}$ implies $vv' \in E_{F_1}$.

Claim 3: For all $v \in V_{F_1}$, $v' \in V_{H_1} \setminus V_{F_1}$: $vv' \in E_{H_1}$ if and only if $f(v)f(v') \in E_{H_2}$.

Proof of Claim 3: Note that, by Claim 1, $f(v) \in V_{F_2}$, $f(v') \in V_{H_2} \setminus V_{F_2}$. Let $vv' \in E_{H_1}$. Then $f(v)f(v') \in E_{G_2}$. Assume that $f(v)f(v') \notin E_{H_2}$. Then there exists vertex $u \in V_{H_1}$ such that $f(v)f(u)$ and $f(v')f(u)$ are edges of H_2. As $f(v)$ is a cut-vertex of H_2, there exists $w \in V_{H_1}$ such that $f(w)f(v) \in E_{H_2}$ and $f(w), f(u)$ belong to different blocks of H_2. Hence $f(w)f(v') \notin E_{G_2}$, and by Proposition 2.5(vi) (second part), $f(w) \in V_{F_2}$. Therefore, $w \in V_{F_1}$ (by Claim 1) and $wv \in E_{H_1}$ (by Claim 2), implying $wv' \in E_{G_1}$. This contradicts $f(w)f(v') \notin E_{G_2}$. Thus, $f(v)f(v') \in E_{H_2}$, as claimed. Similarly, it can be seen that $f(v)f(v') \in E_{H_2}$ implies $vv' \in E_{H_1}$.

It follows from Claim 1, Claim 2 and Claim 3 that the restriction $f : H_1 \to H_2$ of f is an isomorphism. □

3. Good characterizations of squares of block graphs

We are now ready to characterize graphs that are squares of a block graph. Our characterizations are good in the sense that they lead to polynomial time recognition algorithms for such graphs.

Theorem 3.1. *Let G be a connected non-complete graph and let F be the subgraph of G formed by all minimal separators of G. Then G is the square of a block graph if and only if the following conditions hold.*

(i) *F is a block graph whose blocks are exactly the minimal separators of G;*
(ii) *For all maximal cliques Q and Q' of G with $|Q \cap Q'| \geq 2$, $Q \cap Q'$ is a block of F;*
(iii) *Every block S of F belongs to at least two and at most $|S|$ maximal cliques of G;*
(iv) *Every two non-disjoint blocks of F belong to a common maximal clique of G;*
(v) *For all maximal cliques Q of G, $Q \cap V_F = N_F[s]$ for some vertex s of F.*

Proof. The only if-part follows directly from Proposition 2.5.

For the if-part, let G be a connected, non-complete graph satisfying (i)–(v). Then note that $V_G \setminus V_F$ is the set of all simplicial vertices of G (cf. also the proof of Proposition 2.5 (vi)): If v has two non-adjacent neighbors $x \neq y$, then any minimal x, y-separator in G must contain v, hence $v \in V_F$. On the other side, by (iii), every vertex in F belongs to at least two maximal cliques.

Now, we will construct a spanning subgraph H of G such that H is a block graph and $G = H^2$ by attaching the simplicial vertices in $V_G \setminus V_F$ to F in a suitable way (see also Fig. 2): For each $v \in V_G \setminus V_F$, $Q = N_G[v]$ is a maximal clique of G (as v is simplicial in G). By (v) we have two cases. If $Q \cap V_F = N_F[s]$ for some cut-vertex s of F, then $Q \cap V_F$ consists of all blocks of F at s. Since H should be a square root of G, $d_H(v, s) \leq 2$ for all $v \in Q$. Hence we must attach all $v \in Q \setminus V_F$ to F at the vertex s. In other case, $S = Q \cap V_F$ is a block of F. Then we take a simplicial vertex $s \in S$ of F and attach all $v \in Q \setminus V_F$ to F at the vertex s. A simplicial vertex of F in S always exists: If $s \in S$ is a cut-vertex of F, i.e., there is another block S' of F at s, then by (iv), $N_F[s] = Q' \cap V_F$ for another maximal clique $Q' \neq Q$ (hence we cannot join $v \in Q \setminus V_F$ to s). Thus, letting q_1 be number of the maximal cliques C of G with $C \cap V_F = S$ and q_2 be the number of cut-vertices of F in S, we have $q_1 \leq |S| - q_2$ because of (iii) at most $|S|$ maximal cliques may contain S.

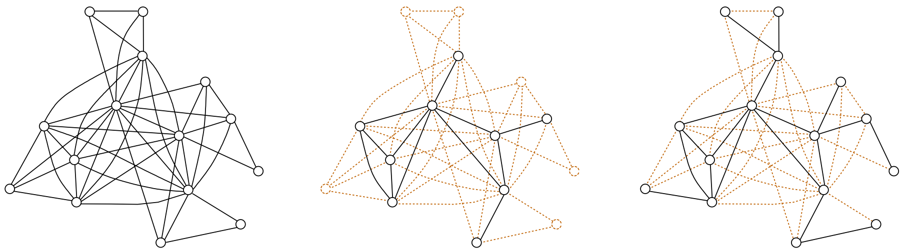

Fig. 2. An input graph G (left), the subgraph F of G (middle) and a square root H constructed by Algorithm `BlockGraphRoot`.

To sum up, a block graph H that will be a square root of G is constructed by the following Algorithm `BlockGraphRoot`:

Algorithm `BlockGraphRoot`

1. $H := F$
2. let X be the set of all cut-vertices of F
3. **for** each $v \in V_G \setminus V_H$ **do**
4. $\quad Q := N_G[v]$ //note: Q is a maximal clique in G
5. $\quad$ **if** $Q \cap V_F$ is a block of F **then**
6. $\quad\quad$ choose an arbitrary vertex $s_Q \in (Q \cap V_F) \setminus X$
7. $\quad\quad X := X \cup \{s_Q\}$
8. $\quad\quad$ **else** let s_Q be the universal vertex of $F[Q \cap V_F]$
9. $\quad$ //note: $Q \cap V_F = N_F[s_Q]$ by (v)
10. $\quad V_H := V_H \cup Q$
11. $\quad E_H := E_H \cup \{vs_Q \mid v \in Q \setminus V_H\}$

The output graph H of Algorithm `BlockGraphRoot` has the following properties:

Claim 1: The following facts hold:

(a) H is a spanning subgraph of G and contains F as an induced subgraph;

(b) Every vertex $v \in V_G \setminus V_F$ has exactly one neighbor in H, and if $N_H(v) = \{u\}$, then $u \in V_F$;
(c) If $v \in V_G \setminus V_F$ with $N_H(v) = \{u\}$, then $vw \in E_G$ for each $w \in N_F(u)$, and for all $v \neq v'$ in $V_G \setminus V_F$: $vv' \in E_G$ if and only if $N_H(v) = N_H(v')$;
(d) H is a block graph.

Proof of Claim 1: (a) As discussed before, by Conditions (iii)–(v), the vertex s_Q chosen in the for-loop at line 6, respectively, line 8 always exists, hence H is a spanning subgraph of G. Since the algorithm only attaches vertices outside F to F to obtain H, F is an induced subgraph of H.

(b) is obvious by construction; c.f. lines 6, 8 and 11 of the algorithm: Every $v \in V_G \setminus V_H$ is contained in a unique maximal clique Q of G and, by construction, $N_H(v) = \{s_Q\}$ and $s_Q \in V_F$.

(c) First, let $v \in V_G \setminus V_F$ with $N_H(v) = \{u\}$. Then, by (b), $u \in F$, and by construction, $Q \cap V_F = N_F[u]$ where Q is the unique maximal clique of G containing v. Hence the first assertion holds.

Next, let $v \neq v'$ in $V_G \setminus V_F$. If $vv' \in E_G$, then v and v' belong to a unique maximal clique Q of G, hence by construction $N_H(v) = N_H(v') = \{s_Q\}$. Conversely, if $vv' \notin E_G$, then the maximal cliques Q, Q' of G containing v, respectively v', are distinct. By construction, $s_Q \neq s_{Q'}$ whenever s_Q or $s_{Q'}$ is a simplicial vertex of F. So, let us consider the case $Q \cap V_F = N_F[s_Q]$, $Q' \cap V_F = N_F[s_{Q'}]$ with cut-vertices $s_Q, s_{Q'}$ in F. If $s_Q = s_{Q'}$, then $Q \cap Q' \cap V_F = N_F[s_Q] \subseteq Q \cap Q'$, contradicting (ii) because $N_F[s_Q]$ contains at least two blocks of F. Thus, $s_Q \neq s_{Q'}$, i.e., $N_H(v) \neq N_H(v')$.

(d) Since F is a block graph (by (i)), (c) directly follows from (a) and (b). (It should be remarked that every endblock in H is an edge.)

Claim 2: $E_{H^2} \subseteq E_G$.

Proof of Claim 2: Let $vv' \in E_{H^2} \setminus E_H$. Then there exists a vertex u such that $vu, v'u \in E_H$. We distinguish three cases. First, if $v, v' \in V_F$, then also, by Claim 1(b), $u \in V_F$, and hence by Claim 1(a), $vu, v'u \in E_F$. Now, as $vv' \notin E_H$, uv and uv' belong to different blocks of F, and by (iii), v and v' are contained in a common maximal clique of G, hence $vv' \in E_G$. Second, if $v, v' \in V_G \setminus V_F$, then by Claim 1(c), $vv' \in E_G$. Third, without loss of generality, we may assume that $v \in V_G \setminus V_F$ and $v' \in V_F$. Then by Claim 1(b), $u \in V_F$ is the unique neighbor of v in H, and by Claim 1(a), $v'u \in E_F$. Now, again by Claim 1(c), $vv' \in E_G$.

Claim 3: $E_G \subseteq E_{H^2}$.

Proof of Claim 3: Let $vv' \in E_G \setminus E_H$ and let Q be a maximal clique in G containing vv'. First assume that $Q \cap V_F = N_F[s]$ for some cut-vertex s of F. Then, as $vv' \notin E_H$, $s \notin \{v, v'\}$. Hence $sv, sv' \in E_F$ (if $v, v' \in V_F$), or by construction $sv, sv' \in E_H$ (if $v, v' \notin V_F$) or one of sv, sv' is in E_F and the other is in E_H (otherwise). Thus $vv' \in E_{H^2}$. Next, if $Q \cap V_F$ is a block of F, then $Q \setminus V_F \neq \emptyset$ (by (iii)), and hence by construction $sv, sv' \in E_H$ for some $s \in Q \cap V_F$, $s \neq v, v'$. Thus $vv' \in E_{H^2}$.

It follows by Claims 2 and 3 that $G = H^2$, and Theorem 3.1 is proved. □

Another formulation of Theorem 3.1 is:

Theorem 3.2. *Let G be a connected graph. Then G is the square of a block graph if and only if G is 2-connected, chordal, and satisfies the following conditions:*

(i) *Every two distinct minimal separators of G have at most one vertex in common;*
(ii) *For all maximal cliques Q and Q' of G with $|Q \cap Q'| \geq 2$, $Q \cap Q'$ is a minimal separator of G;*
(iii) *Every minimal separator S belongs to at least two and at most $|S|$ maximal cliques of G;*
(iv) *Every two non-disjoint minimal separators belong to a common maximal clique of G;*
(v) *All minimal separators belonging to the same maximal clique have exactly one vertex in common.*

Proof. For complete graphs is the theorem trivially true. So, let us assume that G is non-complete. The if-part then follows from Observation 2.1, Corollary 2.4, and Theorem 3.1.

For the only if-part, let G be a 2-connected, non-complete, chordal graph satisfying (i)–(v). Let F be the subgraph of G formed by all minimal separators. We will show that F is a block graph in which each block is a minimal separator, and thus G satisfies the conditions in Theorem 3.1 and we are done.

To this end, we first note that for every maximal clique Q of G there exists another maximal clique Q' with $|Q \cap Q'| \geq 2$ (this is because G is non-complete, chordal and 2-connected). This together with (ii) and (v) imply that F is connected.

Next we show that F is chordal. Suppose not. Then there exists an induced cycle $v_1v_2 \dots v_\ell v_1$ in F, $\ell \geq 4$. Since every minimal separator is a clique (by (iii)), all edges v_iv_{i+1} (indices taken modulo ℓ) belong to pairwise distinct minimal separators. Hence, by (iv),

$$v_iv_{i+2} \in E_G \setminus E_F \quad \text{for all } i. \tag{1}$$

Therefore, by (v),

$$v_iv_{i+3} \notin E_G \quad \text{for all } i. \tag{2}$$

In particular, $\ell \geq 6$. Consider the cycle C in G, $C = v_1v_3v_5 \dots v_\ell v_1$ if ℓ is odd and $C = v_1v_3v_5 \dots v_{\ell-1}v_1$ otherwise.

If $\ell = 6$, let Q be a maximal clique of G containing $v_1v_3v_5$, Q' be a maximal clique containing $v_1v_2v_3$. By (2), $Q \neq Q'$. Now $v_1v_3 \in Q \cap Q'$, hence by (ii), $v_1v_3 \in F$, contradicting (1).

If $\ell \geq 7$, C has length $\lceil \frac{\ell}{2} \rceil \geq 4$. Since G is chordal, C has a short chord, say $v_1v_5 \in E_G$. As before we conclude $v_1v_3 \in F$, a contradiction to (1).

Thus, F is chordal. Furthermore, (i) and (v) imply that F cannot contain an induced diamond, hence F is a block graph. Finally, (i) implies that the blocks of F are exactly the minimal separators of G. □

Note that all conditions in Theorem 3.2, respectively, Theorem 3.1, can be tested in polynomial time (in fact, it is straightforward to do this in $O(nm)$ steps). Hence these characterizations give polynomial time recognition algorithms for squares of block graphs.

4. A linear time recognition for squares of block graphs

In this section we will describe how to recognize squares of block graphs in linear time. Instead of testing the Conditions (i)–(v) given in the characterizations explicitly, we will need the following fact:

Lemma 4.1. *Given a graph G and a block graph H on the same vertex set, testing if $G = H^2$ can be done in $O(m)$ time.*

Proof. The argument is similar to the case of tree squares given in [13, Lemma 6.1]. For the sake of completeness, we give the proof here.

Recall that leaves in a block graph H are simplicial vertices of H belonging to an endblock of H. Pick an arbitrary leaf v of H, and let B be the endblock of H containing v. Let u be the cut-vertex of H in B if $H \neq B$. Otherwise let u be an arbitrary vertex in $B - v$. Obviously, $N_{H^2}[v] = N_H[u]$. Therefore, if $G = H^2$, then $N_G[v] = N_H[u]$. Thus, if $N_G[v] \neq N_H[u]$, we return 'NO', meaning $G \neq H^2$. Otherwise, we replace G and H by $G - v$ and $H - v$, respectively, and repeat the process. If only one vertex is remained in H and G, it implies that $N_{H^2}[w] = N_G[w]$ for all vertices w. In this case $H^2 = G$ and we return 'YES'. The total time complexity is bounded by $\sum_{v \in V_G} O(\deg_G(v)) = O(\sum_{v \in V_G} \deg_G(v)) = O(m)$. □

Theorem 4.2. *Given a graph G, it can be recognized in $O(n + m)$ time if G is the square of a block graph, and if so, such a block square root can be computed in the same time.*

Proof. It is well known that 2-connectedness can be tested in linear time $O(n + m)$ (see [22]). It is also well known that testing chordality and listing all maximal cliques, as well as all minimal separators of a given chordal graph can be done in linear time (see, for example, [4,9,12,21]).

So, given $G = (V_G, E_G)$, we assume that G is chordal and 2-connected, otherwise we just return 'NO', meaning that G is not the square of a block graph. We may also assume that all maximal cliques and all minimal separators of G are available, and that there are at most $n = |V_G|$ maximal cliques (cf. Proposition 2.2) and at most $m = |E_G|$ minimal separators (cf. Proposition 2.3). In particular, we may assume further that the subgraph F of G formed by all minimal separators is a block graph, otherwise we return 'NO' (cf. Proposition 2.5(i)).

Next, construct the block graph H from F according to Algorithm `BlockGraphRoot` in the proof of Theorem 3.1; with small modifications: In line 5 instead of testing if $Q \cap V_F$ is a block of F we just test if $Q \cap V_F$ is a clique. Then, in line 6, respectively, line 8, if the vertex s_Q does not exists, we return 'NO' and stop. This takes $\sum_{v \in V_G \setminus V_F} O(\deg(v)) = O(m)$ steps.

Note that if G is indeed the square of a block graph, then all Conditions (i)–(v) are satisfied, hence the so constructed block graph H is indeed exactly the block graph obtained from Algorithm `BlockGraphRoot`, and therefore, H is a square root of G (cf. proof of Theorem 3.1). Thus, we have to check if H is really a square root of G. If not, we correctly return 'NO'. This last check takes $O(m)$ steps as pointed out by Lemma 4.1, and Theorem 4.2 follows. □

5. Squares of trees revisited

Given the fact that the squares of trees have been widely discussed in the literature, we will derive from our results some previous known results for tree squares.

First, tree squares are strongly chordal by Corollary 2.4. Second, as every endblock in a tree is an edge, Theorem 2.6 implies directly:

Theorem 5.1 (*[3,13,20]*). *The tree roots of squares of trees are unique up to isomorphism.*

Third, observe that Proposition 2.3 shows that each minimal separator in a tree square consists of exactly two vertices, and therefore, in tree squares minimal separators and 2-cuts coincide. Hence, in Theorem 3.2, applied for tree squares, (i) is trivially satisfied, (ii) means that every two maximal cliques have at most two vertices in common (this plus chordality and 2-connectedness implies that if $|Q \cap Q'| = 2$ then $Q \cap Q'$ is a 2-cut), and (iii) means that every 2-cut belongs to exactly two maximal cliques. Thus, we obtain:

Theorem 5.2 (*[3]*). *Let G be a connected graph. Then G is the square of a tree if and only if G is 2-connected, chordal and satisfies the following conditions:*

(i) *Every two distinct maximal cliques of have at most two vertices in common;*

(ii) *Every 2-cut belongs to exactly two maximal cliques;*
(iii) *Every pair of non-disjoint 2-cuts belongs to a common maximal clique;*
(iv) *All 2-cuts contained in the same maximal clique have a common vertex.*

Furthermore, in the proof of Theorem 4.2, if F is a tree, then H is also a tree. Hence we obtain:

Theorem 5.3 ([3,5,11,13,15]). *Given a graph $G = (V_G, E_G)$, it can be recognized in $O(|V_G| + |E_G|)$ time if G is the square of a tree. Moreover, a tree root of a square of a tree can be computed in the same time.*

6. Conclusion

Block graphs generalize trees in a very natural way, and in a sense, they are not too far from trees. Discussing powers of block graphs is motivated by a number of results on tree powers in the literature. In this paper we have found good characterizations for squares of block graphs and a linear time recognition algorithm for such squares. Our algorithm also constructs a square block graph root if one exists. Furthermore, our discussion on squares of block graphs generalizes some previous known results on squares of trees.

For $k \geq 3$, the complexity status of recognizing kth powers of block graphs is not yet determined. However, we strongly believe that k-TH POWER OF BLOCK GRAPH should be efficiently solvable for all fixed k:

k-TH POWER OF BLOCK GRAPH

Instance: A graph G.
Question: Does there exist a block graph H such that $G = H^k$?

Also, it would be interesting to see if there exists a good graph-theoretic characterization for kth powers of block graphs for all k.

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