ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 332 (2007) 1468–1476
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Cone metric spaces and fixed point theorems of contractive mappings

Huang Long-Guang, Zhang Xian *

School of Sciences, Jimei University, Xiamen 361021, People's Republic of China

Received 15 July 2004

Available online 22 December 2006

Submitted by G. Jungck

Abstract

In this paper we introduce cone metric spaces, prove some fixed point theorems of contractive mappings on cone metric spaces.

Keywords: Cone metric space; Fixed point; Contractive mapping; Ordered Banach space

In this paper, we replace the real numbers by ordering Banach space and define cone metric spaces (X, d). In Section 1, we discuss some properties of convergence of sequences. In Section 2, we prove some fixed point theorems for contractive mappings. Our results generalized some fixed point theorems in metric spaces [1,2].

1. Cone metric spaces

In this section we shall define cone metric spaces and prove some properties.

Let E always be a real Banach space and P a subset of E. P is called a cone if and only if:

* Corresponding author.
E-mail address: zhangxianjmj@sina.com (X. Zhang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.03.087

(i) P is closed, nonempty, and $P \neq \{0\}$;
(ii) $a, b \in R$, $a, b \geqslant 0$, $x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leqslant y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, $\operatorname{int} P$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leqslant x \leqslant y \quad \text{implies} \quad \|x\| \leqslant K\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots \leqslant y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ $(n \to \infty)$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose E is a Banach space, P is a cone in E with $\operatorname{int} P \neq \emptyset$ and $\leqslant$ is partial ordering with respect to P.

Definition 1. Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leqslant d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

It is obvious that cone metric spaces generalize metric spaces.

Example 1. Let $E = R^2$, $P = \{(x, y) \in E \mid x, y \geqslant 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geqslant 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

Lemma 1. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.*

Proof. Suppose that $\{x_n\}$ converges to x. For every real $\varepsilon > 0$, choose $c \in E$ with $0 \ll c$ and $K\|c\| < \varepsilon$. Then there is N, for all $n > N$, $d(x_n, x) \ll c$. So that when $n > N$, $\|d(x_n, x)\| \leqslant K\|c\| < \varepsilon$. This means $d(x_n, x) \to 0$ $(n \to \infty)$.

Conversely, suppose that $d(x_n, x) \to 0$ $(n \to \infty)$. For $c \in E$ with $0 \ll c$, there is $\delta > 0$, such that $\|x\| < \delta$ implies $c - x \in \operatorname{int} P$. For this δ there is N, such that for all $n > N$, $\|d(x_n, x)\| < \delta$. So $c - d(x_n, x) \in \operatorname{int} P$. This means $d(x_n, x) \ll c$. Therefore $\{x_n\}$ converges to x. $\square$

Lemma 2. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique.*

Proof. For any $c \in E$ with $0 \ll c$, there is N such that for all $n > N$, $d(x_n, x) \ll c$ and $d(x_n, y) \ll c$. We have

$$d(x, y) \leqslant d(x_n, x) + d(x_n, y) \leqslant 2c.$$

Hence $\|d(x, y)\| \leqslant 2K\|c\|$. Since c is arbitrary $d(x, y) = 0$; therefore $x = y$. $\square$

Definition 3. Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 4. Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 3. *Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Proof. For any $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x) \ll c/2$ and $d(x_m, x) \ll c/2$. Hence $d(x_n, x_m) \leqslant d(x_n, x) + d(x_m, x) \ll c$. Therefore $\{x_n\}$ is a Cauchy sequence. $\square$

Lemma 4. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.*

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence. For every $\varepsilon > 0$, choose $c \in E$ with $0 \ll c$ and $K\|c\| < \varepsilon$. Then there is N, for all $n, m > N$, $d(x_n, x_m) \ll c$. So that when $n, m > N$, $\|d(x_n, x_m)\| \leqslant K\|c\| < \varepsilon$. This means $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

Conversely, suppose that $d(x_n, x_m) \to 0$ $(n, m \to \infty)$. For $c \in E$ with $0 \ll c$, there is $\delta > 0$, such that $\|x\| < \delta$ implies $c - x \in \operatorname{int} P$. For this δ there is N, such that for all $n, m > N$, $\|d(x_n, x_m)\| < \delta$. So $c - d(x_n, x_m) \in \operatorname{int} P$. This means $d(x_n, x_m) \ll c$. Therefore $\{x_n\}$ is a Cauchy sequence. $\square$

Lemma 5. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x$, $y_n \to y$ $(n \to \infty)$. Then*

$$d(x_n, y_n) \to d(x, y) \quad (n \to \infty).$$

Proof. For every $\varepsilon > 0$, choose $c \in E$ with $0 \ll c$ and $\|c\| < \frac{\varepsilon}{4K+2}$. From $x_n \to x$ and $y_n \to y$, there is N such that for all $n > N$, $d(x_n, x) \ll c$ and $d(y_n, y) \ll c$. We have

$$d(x_n, y_n) \leqslant d(x_n, x) + d(x, y) + d(y_n, y) \leqslant d(x, y) + 2c,$$
$$d(x, y) \leqslant d(x_n, x) + d(x_n, y_n) + d(y_n, y) \leqslant d(x_n, y_n) + 2c.$$

Hence

$$0 \leqslant d(x, y) + 2c - d(x_n, y_n) \leqslant 4c$$

and

$$\bigl\|d(x_n, y_n) - d(x, y)\bigr\| \leqslant \bigl\|d(x, y) + 2c - d(x_n, y_n)\bigr\| + \|2c\| \leqslant (4K + 2)\|c\| < \varepsilon.$$

Therefore $d(x_n, y_n) \to d(x, y)$ $(n \to \infty)$. □

Definition 5. Let (X, d) be a cone metric space. If for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X. Then X is called a sequentially compact cone metric space.

2. Fixed point theorems

In this section we shall prove some fixed point theorems of contractive mappings.

Theorem 1. *Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leqslant kd(x, y), \quad \text{for all } x, y \in X,$$

where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0, \ldots.$ We have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \leqslant kd(x_n, x_{n-1}) \\ &\leqslant k^2 d(x_{n-1}, x_{n-2}) \leqslant \cdots \leqslant k^n d(x_1, x_0). \end{aligned}$$

So for $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leqslant (k^{n-1} + k^{n-2} + \cdots + k^m) d(x_1, x_0) \leqslant \frac{k^m}{1-k} d(x_1, x_0). \end{aligned}$$

We get $\|d(x_n, x_m)\| \leqslant \frac{k^m}{1-k} K \|d(x_1, x_0)\|$. This implies $d(x_n, x_m) \to 0$ $(n, m \to \infty)$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Since

$$d(Tx^*, x^*) \leqslant d(Tx_n, Tx^*) + d(Tx_n, x^*) \leqslant kd(x_n, x^*) + d(x_{n+1}, x^*),$$
$$\big\|d(Tx^*, x^*)\big\| \leqslant K\big(k\big\|d(x_n, x^*)\big\| + \big\|d(x_{n+1}, x^*)\big\|\big) \to 0.$$

Hence $\|d(Tx^*, x^*)\| = 0$. This implies $Tx^* = x^*$. So x^* is a fixed point of T.

Now if y^* is another fixed point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant kd(x^*, y^*).$$

Hence $\|d(x^*, y^*)\| = 0$ and $x^* = y^*$. Therefore the fixed point of T is unique. □

Corollary 1. *Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c$ and $x_0 \in X$, set $B(x_0, c) = \{x \in X \mid d(x_0, x) \leqslant c\}$. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leqslant kd(x, y), \quad \text{for all } x, y \in B(x_0, c),$$

where $k \in [0, 1)$ is a constant and $d(Tx_0, x_0) \leqslant (1 - k)c$. Then T has a unique fixed point in $B(x_0, c)$.

Proof. We only need to prove that $B(x_0, c)$ is complete and $Tx \in B(x_0, c)$ for all $x \in B(x_0, c)$.

Suppose $\{x_n\}$ is a Cauchy sequence in $B(x_0, c)$. Then $\{x_n\}$ is also a Cauchy sequence in X. By the completeness of X, there is $x \in X$ such that $x_n \to x$ $(n \to \infty)$. We have

$$d(x_0, x) \leqslant d(x_n, x_0) + d(x_n, x) \leqslant d(x_n, x) + c.$$

Since $x_n \to x$, $d(x_n, x) \to 0$. Hence $d(x_0, x) \leqslant c$, and $x \in B(x_0, c)$. Therefore $B(x_0, c)$ is complete.

For every $x \in B(x_0, c)$,

$$d(x_0, Tx) \leqslant d(Tx_0, x_0) + d(Tx_0, Tx) \leqslant (1 - k)c + kd(x_0, x) \leqslant (1 - k)c + kc = c.$$

Hence $Tx \in B(x_0, c)$. □

Corollary 2. *Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose a mapping $T : X \to X$ satisfies for some positive integer n,*

$$d(T^n x, T^n y) \leqslant kd(x, y), \quad \text{for all } x, y \in X,$$

where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof. From Theorem 1, T^n has a unique fixed point x^*. But $T^n(Tx^*) = T(T^n x^*) = Tx^*$, so Tx^* is also a fixed point of T^n. Hence $Tx^* = x^*$, x^* is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique. □

Theorem 2. *Let (X, d) be a sequentially compact cone metric space, P be a regular cone. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) < d(x, y), \quad \text{for all } x, y \in X,\ x \neq y.$$

Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots,\ x_{n+1} = Tx_n = T^{n+1}x_0, \ldots$. If for some n, $x_{n+1} = x_n$, then x_n is a fixed point of T, the proof is complete. So we assume that for all n, $x_{n+1} \neq x_n$. Set $d_n = d(x_n, x_{n+1})$, then

$$d_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) < d(x_n, x_{n+1}) = d_n.$$

Therefore d_n is a decreasing sequence bounded below by 0. Since P is regular, there is $d^* \in E$ such that $d_n \to d^*$ $(n \to \infty)$. From the sequence compactness of X, there are subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x^* \in X$ such that $x_{n_i} \to x^*$ $(i \to \infty)$. We have

$$d(Tx_{n_i}, Tx^*) \leqslant d(x_{n_i}, x^*), \quad i = 1, 2, \ldots.$$

So

$$\left\| d(Tx_{n_i}, Tx^*) \right\| \leqslant K \left\| d(x_{n_i}, x^*) \right\| \to 0 \quad (i \to \infty),$$

where K is the normal constant of E. Hence $Tx_{n_i} \to Tx^*$ $(i \to \infty)$. Similarly $T^2x_{n_i} \to T^2x^*$ $(i \to \infty)$. By using Lemma 5, we have $d(Tx_{n_i}, x_{n_i}) \to d(Tx^*, x^*)$ $(i \to \infty)$ and $d(T^2x_{n_i}, Tx_{n_i}) \to d(T^2x^*, Tx^*)$ $(i \to \infty)$. It is obvious that $d(Tx_{n_i}, x_{n_i}) = d_{n_i} \to d^* = d(Tx^*, x^*)$ $(i \to \infty)$. Now we shall prove that $Tx^* = x^*$. If $Tx^* \neq x^*$, then $d^* \neq 0$. We have

$$d^* = d(Tx^*, x^*) > d(T^2x^*, Tx^*) = \lim_{i \to \infty} d(T^2x_{n_i}, Tx_{n_i}) = \lim_{i \to \infty} d_{n_i+1} = d^*.$$

We have a contradiction, so $Tx^* = x^*$. That is x^* is a fixed point of T. The uniqueness of fixed point is obvious. □

Theorem 3. *Let (X, d) be a complete cone metric space, P a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leqslant k\big(d(Tx, x) + d(Ty, y)\big), \quad \text{for all } x, y \in X,$$

where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0, \ldots$. We have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \leqslant k\big(d(Tx_n, x_n) + d(Tx_{n-1}, x_{n-1})\big) \\ &= k\big(d(x_{n+1}, x_n) + d(x_n, x_{n-1})\big). \end{aligned}$$

So

$$d(x_{n+1}, x_n) \leqslant \frac{k}{1-k} d(x_n, x_{n-1}) = h d(x_n, x_{n-1}),$$

where $h = \frac{k}{1-k}$. For $n > m$,

$$d(x_n, x_m) \leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$
$$\leqslant (h^{n-1} + h^{n-2} + \cdots + h^m) d(x_1, x_0) \leqslant \frac{h^m}{1-h} d(x_1, x_0).$$

We get $\|d(x_n, x_m)\| \leqslant \frac{h^m}{1-h} K \|d(x_1, x_0)\|$. This implies $d(x_n, x_m) \to 0$ $(n, m \to \infty)$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Since

$$d(Tx^*, x^*) \leqslant d(Tx_n, Tx^*) + d(Tx_n, x^*)$$
$$\leqslant k\big(d(Tx_n, x_n) + d(Tx^*, x^*)\big) + d(x_{n+1}, x^*),$$
$$d(Tx^*, x^*) \leqslant \frac{1}{1-k}\big(kd(Tx_n, x_n) + d(x_{n+1}, x^*)\big),$$
$$\big\|d(Tx^*, x^*)\big\| \leqslant K \frac{1}{1-k}\big(k\big\|d(x_{n+1}, x_n)\big\| + \big\|d(x_{n+1}, x^*)\big\|\big) \to 0.$$

Hence $\|d(Tx^*, x^*)\| = 0$. This implies $Tx^* = x^*$. So x^* is a fixed point of T.

Now if y^* is another fixed point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant k\big(d(Tx^*, x^*) + d(Ty^*, y^*)\big) = 0.$$

Hence $x^* = y^*$. Therefore the fixed point of T is unique. □

Theorem 4. *Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leqslant k\big(d(Tx, y) + d(Ty, x)\big), \quad \textit{for all } x, y \in X,$$

where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0, \ldots.$ We have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leqslant k\big(d(Tx_n, x_{n-1}) + d(Tx_{n-1}, x_n)\big)$$
$$\leqslant k\big(d(x_{n+1}, x_n) + d(x_n, x_{n-1})\big).$$

So

$$d(x_{n+1}, x_n) \leqslant \frac{k}{1-k} d(x_n, x_{n-1}) = hd(x_n, x_{n-1}),$$

where $h = \frac{k}{1-k}$. For $n > m$,

$$d(x_n, x_m) \leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$
$$\leqslant (h^{n-1} + h^{n-2} + \cdots + h^m) d(x_1, x_0) \leqslant \frac{h^m}{1-h} d(x_1, x_0).$$

We get $\|d(x_n, x_m)\| \leqslant \frac{h^m}{1-h} K \|d(x_1, x_0)\|$. This implies $d(x_n, x_m) \to 0$ $(n, m \to \infty)$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Since

$$\begin{aligned} d(Tx^*, x^*) &\leqslant d(Tx_n, Tx^*) + d(Tx_n, x^*) \\ &\leqslant k\big(d(Tx^*, x_n) + d(Tx_n, x^*)\big) + d(x_{n+1}, x^*) \\ &\leqslant k\big(d(Tx^*, x^*) + d(x_n, x^*) + d(x_{n+1}, x^*)\big) + d(x_{n+1}, x^*), \\ d(Tx^*, x^*) &\leqslant \frac{1}{1-k}\Big(k\big(d(x_n, x^*) + d(x_{n+1}, x^*)\big) + d(x_{n+1}, x^*)\Big), \\ \big\|d(Tx^*, x^*)\big\| &\leqslant K\frac{1}{1-k}\Big(k\big(\big\|d(x_n, x^*)\big\| + \big\|d(x_{n+1}, x^*)\big\|\big) + \big\|d(x_{n+1}, x^*)\big\|\Big) \to 0. \end{aligned}$$

Hence $\|d(Tx^*, x^*)\| = 0$. This implies $Tx^* = x^*$. So x^* is a fixed point of T.

Now if y^* is another fixed point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant k\big(d(Tx^*, y^*) + d(Ty^*, x^*)\big) = 2kd(x^*, y^*).$$

Hence $d(x^*, y^*) = 0$, $x^* = y^*$. Therefore the fixed point of T is unique. □

Remark 1. Theorems 1–4 generalize the fixed point theorems of contractive mappings in metric spaces to cone metric spaces.

We conclude with an example.

Let $E = R^2$, the Euclidean plane, and $P = \{(x, y) \in R^2 \mid x, y \geqslant 0\}$ a normal cone in P. Let $X = \{(x, 0) \in R^2 \mid 0 \leqslant x \leqslant 1\} \cup \{(0, x) \in R^2 \mid 0 \leqslant x \leqslant 1\}$. The mapping $d : X \times X \to E$ is defined by

$$\begin{aligned} d\big((x, 0), (y, 0)\big) &= \left(\frac{4}{3}|x-y|, |x-y|\right), \\ d\big((0, x), (0, y)\big) &= \left(|x-y|, \frac{2}{3}|x-y|\right), \\ d\big((x, 0), (0, y)\big) &= d\big((0, y), (x, 0)\big) = \left(\frac{4}{3}x + y, x + \frac{2}{3}y\right). \end{aligned}$$

Then (X, d) is a complete cone metric space.

Let mapping $T : X \to X$ with

$$T((x, 0)) = (0, x) \quad \text{and} \quad T((0, x)) = \left(\frac{1}{2}x, 0\right).$$

Then T satisfies the contractive condition

$$d\big(T((x_1, x_2)), T((y_1, y_2))\big) \leqslant kd\big((x_1, x_2), (y_1, y_2)\big), \quad \text{for all } (x_1, x_2), (y_1, y_2) \in X,$$

with constant $k = \frac{3}{4} \in [0, 1)$. It is obvious that T has a unique fixed point $(0, 0) \in X$. On the other hand, we see that T is not a contractive mapping in the Euclidean metric on X.

References

[1] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, 1985.
[2] B.E. Rhoades, A comparison of various definition of contractive mappings, Trans. Amer. Math. Soc. 266 (1977) 257–290.