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Expressive equivalence of least and inflationary fixed-point logic

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Abstract

We study the relationship between least and inflationary fixed-point logic. In 1986, Gurevich and Shelah proved that in the restriction to finite structures, the two logics have the same expressive power. On infinite structures however, the question whether there is a formula in IFP not equivalent to any LFP-formula was left open.

In this paper, we answer the question negatively, i.e. we show that the two logics are equally expressive on arbitrary structures. We give a syntactic translation of IFP-formulae to LFP-formulae such that the two formulae are equivalent on all structures.

As a consequence of the proof we establish a close correspondence between the LFP-alternation hierarchy and the IFP-nesting depth hierarchy. We also show that the alternation hierarchy for IFP collapses to the first level, i.e. the complement of any inflationary fixed point is itself an inflationary fixed point.

1. Introduction

Formal logics have played a crucial role in the development of theoretical computer science. Features that are pervasive to many diverse areas such as database theory, computer-aided verification, or computational and descriptive complexity theory are definitions by *recursion* or *iteration*.

Formalising recursive definitions in a logical language usually involves some kind of fixed-point construction. This can be incorporated into the logic in various ways. In second-order logic, recursion is modelled by quantifying over the individual stages of the iteration process or by defining the intersection of all fixed points, whereas in infinitary logics, the

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same is simulated by infinitary disjunctions defining arbitrary recursion depths. Another way of modelling recursive definitions is to incorporate an explicit operator for forming fixed points. Logics following this approach are called *fixed-point logics*. In the various areas of computer science where fixed-point logics have been considered, a huge variety of such logics has evolved. Regardless of how great the differences are elsewhere, the fixed-point part of most logics is formed according to the same common principle.

Consider a first-order formula $\varphi(R, \overline{x})$ with a free second-order variable R of arity k, and k free first-order variables $\overline{x}$. On any structure $\mathfrak{A}$, such a formula induces an operator F_φ taking a set $P \subseteq A^k$ to the set $\{\overline{a} : (\mathfrak{A}, P) \models \varphi[\overline{a}]\}$. Recursive definitions are now modelled by considering the various kinds of fixed points such an operator may possess. Among these, *least fixed points* play a fundamental role.

Least fixed points are usually incorporated into a logic as follows. If φ is positive in R, the operator F_φ is monotone, i.e. $X \subseteq Y$ implies $F_\varphi(X) \subseteq F_\varphi(Y)$. Monotone operators always have a least fixed point $\mathbf{lfp}(F_\varphi) := \bigcap\{X : F_\varphi(X) = X\}$ and therefore, on any structure $\mathfrak{A}$, a first-order formula $\varphi(R, \overline{x})$ positive in R naturally induces a set $\mathbf{lfp}(F_\varphi)$. This forms the basis of *least fixed-point logic* (LFP), an extension of first-order logic (FO) equipped with an explicit construct $[\mathbf{lfp}_{R,\overline{x}}\, \varphi(R, \overline{x})](\overline{x})$, for φ positive in R, defining the least fixed point of F_φ.

A different type of fixed point can be obtained by an explicit induction process. Here, we associate with each formula $\varphi(R, \overline{x})$ the *inflationary operator* I_φ taking a set $P \subseteq A^k$ to the set $P \cup F_\varphi(P) = P \cup \{\overline{a} \in A^k : (\mathfrak{A}, P) \models \varphi[\overline{a}]\}$. The operator I_φ is used to build up the following sequence $(R^\alpha)_{\alpha\in\mathrm{Ord}}$ of sets, indexed by ordinals α.

$$R^\alpha := I_\varphi(R^{<\alpha}) = R^{<\alpha} \cup \{\overline{a} : (\mathfrak{A}, R^{<\alpha}) \models \varphi[\overline{a}]\},$$

where $R^{<\alpha} := \bigcup_{\xi<\alpha} R^\xi$ for every $\alpha \in \mathrm{Ord}$. As this sequence is increasing, it leads to a fixed point R^∞ of I_φ defined as $R^\infty := R^\alpha$ for the least ordinal α such that $R^\alpha = R^{\alpha+1}$. R^∞ is called the *inflationary fixed point* of φ and is used to form the *inflationary fixed-point logic* (IFP) as the extension of FO by an operator $[\mathbf{ifp}_{R,\overline{x}}\varphi(R, \overline{x})](\overline{x})$ defining the inflationary fixed point of φ. The existence of this fixed point is independent of φ being positive in R. However, due to a theorem by Knaster and Tarski (see Theorem 2.2), if φ is positive in R, the inflationary and the least fixed point coincide. Thus, every LFP-formula is equivalent to a formula in IFP.

Following work in recursion theory on inductive definitions in arithmetic, the first systematic study of inductive definitions on abstract structures occurred in the 1970s. At that time, no explicit construct to form fixed points was considered and therefore fixed points could not be nested. Nevertheless, many fundamental methods in the theory of fixed-point logics date back to the investigations done then. See [1,9] for surveys of the results and methods established by then. We will briefly recall some results related to the present paper in Section 4.2.

Since the 1980s, fixed-point logics in the modern form are studied in various areas of computer science like database theory or finite model theory. The main evolution over the cases studied in the 1970s was the introduction of explicit fixed-point operators such as $[\mathbf{lfp}_{R,\overline{x}}\, \varphi](\overline{x})$ and $[\mathbf{ifp}_{R,\overline{x}}\, \varphi](\overline{x})$. In particular, the formulae φ can again contain fixed-point operators and thus fixed points can be nested and negated. Although different in scope and focus, finite model theory and database theory both concentrate on finite structures. One effect of this is that today a lot more is known about these logics on finite than on infinite structures.

An important question concerning the logics LFP and IFP is whether IFP is strictly more expressive than LFP. As noted by Dawar and Gurevich [2] it comes in two forms:

Question. Is there a formula φ of IFP and a structure $\mathfrak{A}$ such that for every formula ψ of LFP, $\mathfrak{A} \not\models (\varphi \leftrightarrow \psi)$?

Is there a formula φ of IFP such that for every formula ψ of LFP, there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models (\varphi \leftrightarrow \psi)$?

Using the stage comparison method, Gurevich and Shelah showed in 1986 that in the restriction to finite structures, the two logics are equivalent. It is clear that the proof does not extend to infinite structures as it crucially relies on the fact that on finite structures every fixed-point induction is finite itself and therefore only successor stages occur.

The main contribution of this paper is to show that the two logics are equivalent on arbitrary structures, rather than just on finite ones. In particular, we show that for every formula in IFP there is a formula in LFP equivalent to it on all structures. Thus, we give a negative answer to both questions above.

As a simple consequence of the method used to show this, we establish a close correspondence between the LFP alternation hierarchy and the IFP nesting-depth hierarchy. To be precise, the IFP nesting-depth hierarchy is infinite on a structure $\mathfrak{A}$, if, and only if, the alternation hierarchy for LFP is infinite on $\mathfrak{A}$.

We also show that there is a negation normal form for IFP, i.e. every formula of IFP is equivalent to a formula where negation occurs only in front of atoms. Thus, the alternation hierarchy for IFP collapses to the first level. This contrasts with least fixed-point logic, for which the strictness of the alternation hierarchy follows from results due to Moschovakis [9, Chapter 5D].

An extended abstract of the present paper was published in [8].

Organisation. In the next section, we give precise definitions of the fixed-point logics considered in this paper. The stage comparison relations and theorems are presented in Section 3. In Section 4 we establish our main result, the equivalence of LFP and IFP. We first give a brief review of the equivalence result for the logics on finite structures and recall a related theorem by Harrington and Kechris. Finally, Section 5 contains results about the nesting and alternation hierarchies for IFP and LFP.

2. Fixed-point logics

In this section we present the basic definitions for the following explorations. See [2–4] for details on fixed-point logics. We first present some notation used throughout the paper.

Let $\mathfrak{A} := (A, \tau)$ be a structure and let R be a k-ary relation symbol not occurring in τ.

- If $\bar{t}$ is a tuple of terms, we write $\bar{t}^{\mathfrak{A}}$ for the interpretation of $\bar{t}$ in $\mathfrak{A}$.
- Let $\bar{x}$ be a k-tuple of terms and let $\psi_1(\bar{x})$, $\psi_2(\bar{x})$ be formulae, which may or may not contain R. We write $\varphi(\bar{x}, R\bar{u}/\psi_1(\bar{u}))$ for the formula obtained from φ by replacing

every occurrence of an atom $R\overline{u}$ by $\psi_1(\overline{u})$, where $\overline{u}$ is a tuple of terms. Here, $\psi_1(\overline{u})$ means that the variables $\overline{x}$ in $\psi_1(\overline{x})$ are replaced by $\overline{u}$, where bound variables in ψ_1 are suitably renamed to avoid conflicts.

Sometimes we need to replace positive and negative occurrences of atoms $R\overline{u}$ by separate formulae. In this case we write $\varphi(\overline{x}, R\overline{u}/\psi_1(\overline{u}), \neg R\overline{u}/\psi_2(\overline{u}))$ to denote the formula obtained from φ by replacing each positive occurrence of atoms of the form $R\overline{u}$ by $\psi_1(\overline{u})$ and each negative occurrence of atoms of the form $R\overline{u}$ by $\neg\psi_2(\overline{u})$. For instance, if $\varphi(R, \overline{x})$ is the formula $R\overline{x} \vee \neg R\overline{x}$, then $\varphi(\overline{x}, R\overline{u}/\psi_1(\overline{u}), \neg R\overline{u}/\psi_2(\overline{u}))$ would just be $\psi_1(\overline{x}) \vee \neg(\neg\psi_2(\overline{x}))$ which is equivalent to $\psi_1(\overline{x}) \vee \psi_2(\overline{x})$.

Clearly, any such formula $\varphi(\overline{x}, R\overline{u}/\psi_1(\overline{u}), \neg R\overline{u}/\psi_2(\overline{u}))$ is positive in both ψ_1 and ψ_2 and thus positive in R if ψ_1 and ψ_2 are.

- Finally, if $\mathcal{L}$ and $\mathcal{L}'$ are logics, we write $\mathcal{L} \leq \mathcal{L}'$ if the logic $\mathcal{L}$ is no more expressive than $\mathcal{L}'$, i.e. for every formula $\varphi \in \mathcal{L}$ there is an equivalent formula $\varphi' \in \mathcal{L}'$.

Let τ be a signature and $\mathfrak{A} := (A, \tau)$ a τ-structure. Let $\varphi(R, \overline{x})$ be a first-order formula with k free variables $\overline{x}$ and a free relation symbol R not occurring in τ. The formula φ defines an operator

$$\begin{aligned} F_\varphi\colon\ \mathcal{P}(A^k) &\longrightarrow \mathcal{P}(A^k) \\ R &\longmapsto \{\overline{a} : (\mathfrak{A}, R) \models \varphi[\overline{a}]\}. \end{aligned}$$

A fixed point of the operator F_φ is any set R such that $F_\varphi(R) = R$. Clearly, as φ is arbitrary, the corresponding operator F_φ need not have any fixed points. For instance, the formula $\varphi(R, \overline{x}) := \neg\forall\overline{y}\, R\overline{y}$ defines the operator F_φ mapping any set $R \subsetneq A^k$ to A^k and the set A^k itself to the empty set.

However, if the class of admissible formulae φ is suitably restricted, then the existence of fixed points can be guaranteed. A formula $\varphi(R, \overline{x})$ is *monotone in* R, if for all τ-structures $\mathfrak{A} := (A, \tau)$ and all sets $R, R' \subseteq A^k$, $R \subseteq R'$ implies $F_\varphi(R) \subseteq F_\varphi(R')$. It is easily seen that for monotone operators F_φ fixed points always exist and in fact even a least fixed point exists, defined as

$$\mathbf{lfp}(F_\varphi) := \bigcap\{R : F_\varphi(R) = R\}.$$

A different kind of fixed point is obtained by an explicit induction process. Here we associate with a formula $\varphi(R, \overline{x})$ the *inflationary operator*

$$\begin{aligned} I_\varphi\colon\ \mathcal{P}(A^k) &\longrightarrow \mathcal{P}(A^k) \\ R &\longmapsto R \cup F_\varphi(R) = R \cup \{\overline{a} : (\mathfrak{A}, R) \models \varphi[\overline{a}]\}. \end{aligned}$$

The operator I_φ is used to build up the following sequence $(R^\alpha)_{\alpha\in\mathrm{Ord}}$ of sets, indexed by ordinals α:

$$R^\alpha := I_\varphi(R^{<\alpha}) = R^{<\alpha} \cup \{\overline{a} : (\mathfrak{A}, R^{<\alpha}) \models \varphi[\overline{a}]\}, \tag{1}$$

where $R^{<\alpha} := \bigcup_{\xi<\alpha} R^\xi$ for every $\alpha \in \mathrm{Ord}$. Clearly this sequence of sets is increasing and thus leads to a limit $R^\infty := R^\alpha$ for the least ordinal α such that $R^\alpha = R^{\alpha+1}$. The set R^α is called the *inflationary fixed point* of I_φ, in terms $\mathbf{ifp}(I_\varphi)$. With abuse of notation we also refer to R^∞ as the inflationary fixed point of the formula φ. Least and inflationary fixed points are the basis for the fixed-point logics studied in this paper. Since the three fixed-point logics considered here are syntactically all rather similar, we present the three logics at once.

Definition 2.1. Let $\mathfrak{A}$ be a structure. The syntax of least, monotone and inflationary fixed-point logic is defined by the usual rules for first-order logic augmented with the following formula building rule: If $\varphi(R, \overline{x})$ is a formula with free first-order variables $\overline{x} := x_1, \ldots, x_k$ and a free second-order variable R of arity k then

(i) $\psi := [\mathbf{lfp}_{R,\overline{x}}\, \varphi](\overline{t})$ is a formula of *monotone fixed-point logic* (MFP) if $\varphi \in$ MFP defines on all structures a monotone operator,

(ii) $\psi := [\mathbf{lfp}_{R,\overline{x}}\, \varphi](\overline{t})$ is a formula of *least fixed-point logic* (LFP) provided that $\varphi \in$ LFP is positive in R and

(iii) $\psi := [\mathbf{ifp}_{R,\overline{x}}\, \varphi](\overline{t})$ is a formula of *inflationary fixed-point logic* (IFP) for arbitrary formulae $\varphi \in$ IFP.

In each case, the free variables of ψ are the variables occurring in $\overline{t}$ and the free variables of φ other than $\overline{x}$.

Let $\mathfrak{A}$ be a structure providing an interpretation of the free variables of φ except for $\overline{x}$. For formulae in MFP and LFP, $\mathfrak{A} \models [\mathbf{lfp}_{R,\overline{x}}\, \varphi](\overline{t})$ if, and only if, $\overline{t}^{\mathfrak{A}} \in \mathbf{lfp}(F_\varphi)$. For IFP, $\mathfrak{A} \models [\mathbf{ifp}_{R,\overline{x}}\, \varphi](\overline{t})$ if, and only if, $\overline{t}^{\mathfrak{A}} \in \mathbf{ifp}(I_\varphi)$.

As explained above, for any monotone operator F the least fixed point of F always exists. Therefore the semantics of the monotone fixed-point logic is well defined. However, the property of a formula to define an operator which is monotone on all structures is undecidable and therefore the monotone fixed-point logic has an undecidable syntax.

To avoid this, one considers syntactical restrictions of MFP which guarantee monotonicity of the corresponding operators. The most important of these is the least fixed-point logic, where the application of the fixed-point rule is restricted to formulae $\varphi(R, \overline{x})$ which are positive in the relation variable R. Clearly, if $\varphi(R, \overline{x})$ is positive in R, then the corresponding operator F_φ is monotone. Thus, LFP $\leq$ MFP.

As a corollary of the following theorem due to Knaster and Tarski we get that MFP is contained in inflationary fixed-point logic.

Theorem 2.2 (Knaster and Tarski). *Let M be a set. Every monotone operator F : $\mathrm{Pow}(M) \to \mathrm{Pow}(M)$ has a least fixed point*

$$\mathbf{lfp}(F) = \bigcap \{P : F(P) = P\}.$$

Further, this fixed point can also be obtained as the fixed point of the sequence of sets defined as

$$R^\alpha := F_\varphi(R^{<\alpha}). \tag{2}$$

As F is monotone, the sequence in the previous theorem is increasing and therefore the least fixed point reached in this way must also be the inflationary fixed point of F. It follows that

$$\text{LFP} \leq \text{MFP} \leq \text{IFP}.$$

When writing fixed-point formulae it is often convenient to use variants of the logics where the fixed points of several formulae can be built up simultaneously.

Let $R_1, \dots, R_k$ be relation symbols of arities r_i, respectively. *Simultaneous inflationary fixed-point formulae* are of the form $\psi(\overline{x}) := [\mathbf{ifp}\ R_i : S](\overline{x})$, where

$$S := \begin{cases} R_1\overline{x}_1 & \leftarrow \varphi_1(R_1, \dots, R_k, \overline{x}_1) \\ & \vdots \\ R_k\overline{x}_k & \leftarrow \varphi_k(R_1, \dots, R_k, \overline{x}_k) \end{cases}$$

is a system of formulae in (simultaneous) IFP. On any structure $\mathfrak{A}$, a formula φ_i in S induces an operator

$$\begin{aligned} I_{\varphi_i}\colon \mathrm{Pow}(A^{r_1}) \times \cdots \times \mathrm{Pow}(A^{r_k}) &\to \mathrm{Pow}(A^{r_i}) \\ (R_1, \dots, R_k) &\mapsto R_i \cup \{\overline{a} : (\mathfrak{A}, R_1, \dots, R_k) \models \varphi_i[\overline{a}]\}. \end{aligned}$$

The stages S^α of an induction on such a system S of formulae are now k-tuples of sets $(R_1^\alpha, \dots, R_k^\alpha)$ defined as

$$R_i^\alpha := I_{\varphi_i}(R_1^{<\alpha}, \dots, R_n^{<\alpha}) = R_i^{<\alpha} \cup \{\overline{a} : (\mathfrak{A}, (R_j^{<\alpha})_{1\le j\le n}) \models \varphi_i[\overline{a}]\},$$

where $R_i^{<\alpha} := \bigcup_{\xi<\alpha} R_i^\xi$. For every structure $\mathfrak{A} := (A, \tau)$ and any tuple $\overline{a}$ from A, $\mathfrak{A} \models \psi[\overline{a}]$ if, and only if, $\overline{a} \in R_i^\infty$, where R_i^∞ denotes the i-th component of the simultaneous fixed point of the system S. The definition of simultaneous LFP is analogous.

It can be shown, that by increasing the arity of the involved fixed-point relations, any formula in IFP with simultaneous inductions can be transformed into an equivalent IFP formula without simultaneous fixed points. The same is true for LFP. See e.g. [3].

3. Comparing the stages of inductive definitions

In this section we introduce the stage comparison method, one of the most important tools to reason about fixed-point logics. The method will be essential for the explorations below. Let $\varphi(R, \overline{x})$ be a formula, e.g. in first-order logic. As mentioned above, the inflationary and—if it exists—the least fixed point of a formula $\varphi(R, \overline{x})$ can be obtained as the fixed point of the sequence of sets as defined in (1) or (2). We concentrate on such sequences of sets approximating least or inflationary fixed points.

Let $\mathfrak{A} := (A, \tau)$ be a τ-structure with universe A. By definition, the sequence of stages defined in (1) is increasing and thus there is an ordinal $\alpha < |A|^+$ such that $R^\alpha = R^{\alpha+1} = R^\infty$. Here $|A|^+$ denotes the least infinite cardinal greater than the cardinality of A. The individual sets occurring in the sequence induced by a formula φ are called the *stages of the induction on* φ. The set R^α is called the α-th stage of the induction. Sometimes we also write φ^α for R^α. As a final bit of notation, we write $R^{<\alpha}$ or $\varphi^{<\alpha}$ for the union of all stages up to α, i.e. $\varphi^{<\alpha} := \bigcup_{\beta<\alpha} \varphi^\beta$, and likewise for $R^{<\alpha}$.

We now define the stage comparison relations for least or inflationary fixed-point inductions.

Definition 3.1. Let $\varphi(R, \overline{x})$ be a formula and $\overline{a} \in A$. The *rank* $|\overline{a}|_\varphi$ *of* $\overline{a}$ *with respect to* φ is defined as the least ordinal α such that $\overline{a} \in \varphi^\alpha$ if such an ordinal exists and ∞ otherwise.

The *stage comparison relations* $\leq_\varphi$ *and* $\prec_\varphi$ are defined as

$$x \leq_\varphi y \Longleftrightarrow x, \qquad y \in \varphi^\infty \qquad \text{and} \qquad |x|_\varphi \leq |y|_\varphi,$$

and

$$x \prec_\varphi y \Longleftrightarrow x \in \varphi^\infty \qquad \text{and} \qquad |x|_\varphi < |y|_\varphi,$$

where we allow $|y|_\varphi = \infty$.

The proof of the following lemma is immediate from the definition.

Lemma 3.2. *Let* $\varphi(R, \overline{x})$ *be a formula. For all* $\overline{a} \in A$,

$$\begin{aligned} \overline{a} \in \varphi^\infty \quad & \textit{if, and only if,} \quad \overline{a} \leq_\varphi \overline{a} \\ & \textit{if, and only if,} \quad (\mathfrak{A}, \{\overline{u} : \overline{u} \prec_\varphi \overline{a}\}) \models \varphi[\overline{a}]. \end{aligned}$$

The next theorem shows that the stage comparison relations are themselves definable. It essentially goes back to Moschovakis, who proved the corresponding theorems for least and inflationary fixed-point inductions on first-order formulae. The extension of his proofs to the case of full LFP and IFP is immediate. See [9] and references therein for the case of LFP and [10] for the IFP-version.

Theorem 3.3 (Stage Comparison Theorem). (i) *Let* $\varphi(R, \overline{x})$ *be a formula in* LFP *positive in* R. *Then* $\leq_\varphi$ *and* $\prec_\varphi$ *are definable in LFP.*
(ii) *Let* $\varphi(R, \overline{x})$ *be a formula in* IFP. *Then* $\leq_\varphi$ *and* $\prec_\varphi$ *are definable in* IFP.

Proof. We only present the proof for Part (ii), as this case will be used in Section 4 below. The more complicated proof for the first part can be found in [9].

Let $\varphi(R, \overline{x})$ be a formula in IFP. W.l.o.g. we assume that φ is of the form $R\overline{x} \vee \varphi'$. We claim that the relations $\leq_\varphi$ and $\prec_\varphi$ can be obtained as the simultaneous fixed point of the following system S of formulae:

$$S := \begin{cases} \overline{x} \leq \overline{y} \longleftarrow \varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{y}) \wedge \varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{y}) \\ \overline{x} \prec \overline{y} \longleftarrow \varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{x}) \wedge \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}). \end{cases}$$

Here, $\varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{y})$ means that every occurrence of an atom $R\overline{u}$ in φ, for some tuple of terms $\overline{u}$, is replaced by the new atom $\overline{u} \prec \overline{y}$. Note that, strictly speaking, the simultaneous induction is unnecessary, as only the $\prec$-relation occurs on the right-hand side of the rules. We state the system here in the simultaneous form as it will be used as a starting point for the exploration in Section 4.1 below.

As before, let, for every ordinal α, $\leq^\alpha$ and $\prec^\alpha$ denote the relations $\leq$ and $\prec$ at stage α of the induction on S and let $\leq^{<\alpha}$ and $\prec^{<\alpha}$ be the union of all stages less than α, i.e. $\leq^{<\alpha} = \bigcup_{\beta<\alpha} \leq^\beta$ and $\prec^{<\alpha} = \bigcup_{\beta<\alpha} \prec^\beta$. We prove by induction that for all α and all pairs $(\overline{a}, \overline{b})$,

- $(\overline{a}, \overline{b}) \in \leq^\alpha$ if, and only if, $|\overline{b}|_\varphi \leq \alpha$ and $|\overline{a}|_\varphi \leq |\overline{b}|_\varphi$ and
- $(\overline{a}, \overline{b}) \in \prec^\alpha$ if, and only if, $|\overline{a}|_\varphi \leq \alpha$ and $|\overline{a}|_\varphi < |\overline{b}|_\varphi$.

From this, the theorem follows immediately. Let α be an ordinal and suppose that for all $\beta < \alpha$ the claim has already been proved, i.e. $(\overline{a}, \overline{b}) \in \leq^{<\alpha}$ if, and only if, $|\overline{a}|_\varphi \leq |\overline{b}|_\varphi < \alpha$ and likewise for $\prec^{<\alpha}$.

Suppose $\overline{b}$ is a tuple of elements of rank $\xi \leq \alpha$. Then, the set $\{\overline{u} : \overline{u} \prec^{<\alpha} \overline{b}\}$ contains precisely the elements of rank less than ξ. Thus, $\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{y})$ is satisfied by $\overline{b}$. Set $\overline{y} := \overline{b}$. A tuple $\overline{a}$ satisfies $\varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{y})$ if, and only if, the rank of $\overline{a}$ is at most ξ and therefore $|\overline{a}|_\varphi \leq |\overline{b}|_\varphi$.

On the other hand, if the rank of $\overline{b}$ is greater than α, then $\{\overline{u} : \overline{u} \prec^{<\alpha} \overline{b}\}$ is just $\varphi^{<\alpha}$ and therefore $\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{y})$ is not satisfied by $\overline{b}$. This proves the induction hypothesis for the first item above.

For $\prec$, let $\overline{a}$ be a tuple of elements of rank $\xi \leq \alpha$. Again, $\{\overline{u} : \overline{u} \prec^{<\alpha} \overline{a}\}$ contains all elements of rank less than ξ and therefore $\varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{x})$ is satisfied by $\overline{a}$. Obviously, if we set $\overline{x} := \overline{a}$, then $\neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x})$ is satisfied by those tuples $\overline{b}$ whose rank is greater than ξ and therefore greater than the rank of $\overline{a}$. Finally, if $\overline{a}$ is a tuple of rank greater than α, it does not satisfy $\varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{x})$. This proves the second item above and, with it, the claim.

Thus, the stage comparison relations $\leq_\varphi$ and $\prec_\varphi$ are defined by the IFP-formulae $[\mathbf{ifp} \leq : S](\overline{x}, \overline{y})$ and $[\mathbf{ifp} \prec : S](\overline{x}, \overline{y})$ respectively. □

4. Expressive equivalence of least and inflationary fixed-point logic

In this section, we establish the equivalence of least and inflationary fixed-point logic. As noted above, in the restriction to finite structures, the equivalence has already been proved by Gurevich and Shelah [5]. We first hint at their proof and explain where its extension to infinite structures fails.

4.1. Equivalence on finite structures

Consider again the proof of Theorem 3.3. As shown there, the stage comparison relations of any IFP-formula $\varphi(R, \overline{x})$ are definable by the formulae $[\mathbf{ifp} \leq : S](\overline{x}, \overline{y})$ and $[\mathbf{ifp} \prec : S](\overline{x}, \overline{y})$ respectively, where S is the system of formulae defined as

$$S := \begin{cases} \overline{x} \leq \overline{y} & \longleftarrow \quad \varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{y}) \wedge \varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{y}) \\ \overline{x} \prec \overline{y} & \longleftarrow \quad \varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{x}) \wedge \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}). \end{cases}$$

Now suppose $\varphi(R, \overline{x})$ is itself an LFP-formula but not necessarily positive in R. It was shown by Gurevich and Shelah, that in restriction to finite structures, the stage comparison relations for the inflationary induction on φ are definable in LFP. For this, they converted the system S above to an equivalent system T of formulae, which are positive in their free fixed-point variables. W.l.o.g. we assume that φ is of the form $R\overline{x} \vee \varphi'$. The problem to be solved is that if every atom $R\overline{u}$ in φ is replaced by a new atom involving $\prec$, then at all places where R is used negatively, also the new relation $\prec$ is used negatively. Therefore, we have to come up with a definition of the complement R^c of R by a formula positive in $\prec$ and $\leq$. For this, let $\mathfrak{A}$ be a finite structure of size n. Clearly, if k is the arity of R, then there is some $m \leq n^k$ such that the induction of φ on $\mathfrak{A}$ reaches its fixed point at stage m. Now consider the sequence of stages $(R^\alpha)_{\alpha \leq m}$ induced by φ on $\mathfrak{A}$. Let $\leq_\varphi$ and $\prec_\varphi$ be the stage comparison relations of φ. For every stage R^α, with $\alpha > 0$, there is a tuple $\overline{z}$ whose rank is precisely α, i.e. $\overline{z} \in R^\alpha - R^{\alpha-1}$. For any such tuple $\overline{z}$, $\{\overline{u} : \overline{u} \leq_\varphi \overline{z}\} = R^\alpha$ and $\{\overline{u} : \overline{z} \prec_\varphi \overline{u}\} = (R^\alpha)^c$. Thus the stage R^α as well as its complement $(R^\alpha)^c$ can be defined by a positive formula. This is used to define an induction process, positive in $\leq$ and $\prec$, defining the relations $\leq_\varphi$ and $\prec_\varphi$.

As the inflationary fixed point can easily be obtained from the stage comparison relations (see Lemma 3.2), this shows that on finite structures, every inflationary fixed point of an LFP-formula can be obtained as a least fixed point also. By induction on the number of **ifp**-operators in the formulae, the equivalence of IFP and LFP on finite structures follows immediately.

Theorem 4.1 (Gurevich–Shelah [5]). *For every formula in* IFP *there is a* LFP-*formula equivalent to it on all finite structures.*

We aim at extending the equivalence of IFP and LFP to arbitrary, not necessarily finite structures. If $\mathfrak{A}$ is an infinite structure, the sequence of stages induced by an LFP-formula $\varphi(R, \overline{x})$ on $\mathfrak{A}$ is no longer guaranteed to be finite. The formulae used in the Gurevich–Shelah proof still define the correct stage comparison relations up to stage ω, i.e. for all finite stages. However, at stage ω—and all other infinite limit stages also—it is no longer true that there is a tuple $\overline{z}$ of rank less than ω such that $\overline{u} \leq \overline{z}$ defines $R^{<\omega}$. For, each such tuple $\overline{z}$ is itself of finite rank $\beta < \omega$ and therefore $\overline{u} \leq \overline{z}$ defines the stage $R^\beta \subsetneq R^\omega$. Thus, to extend the result to infinite structures, we have to treat the limit stages differently.

4.2. Equivalence of monotone and inflationary fixed-point logic

As mentioned in the introduction, least and inflationary inductions on infinite structures were already studied in the 1970s, mainly on the class of acceptable structures. The research was motivated by questions arising in descriptive set theory. Hence, there are significant differences in notation and type of questions addressed in the 1970s and in later work on fixed-point logics in computer science. We briefly recall some of the terminology and results. Our presentation follows [10].

Let $\mathfrak{A} := (A, \tau)$ be a structure. A *coding scheme on A* is a triple $(\mathcal{N}, \leq, <>)$, with $\mathcal{N} \subseteq A$, such that the structure $(\mathcal{N}, \leq)$ is isomorphic to $(\omega, \leq)$ and $<>$ is an injective map from $\bigcup_{n<\omega} A^n$ into A. The image a of $a_1, \ldots, a_n$ under $<>$ is called the *code of* $a_1, \ldots, a_n$. We associate with a coding scheme the relations *lh* giving the length of a coded sequence, $q(a, i)$ giving the i-th element of the sequence coded in a, and *seq* which is true for all codes of sequences. A coding scheme on A allows to code arbitrary finite sequences of elements into a single element. In particular it allows to code relations of arbitrary arity by monadic relations. A structure on which a coding scheme is definable is called *acceptable*.

A *second-order relation* $S := S(\overline{x}, \overline{R})$ on $\mathfrak{A}$ is a relation with elements $\overline{x}$ and relations $\overline{R}$ as arguments. Of particular interest to us are second-order relations with only one relation as argument, i.e. relations of the form $S(\overline{x}, R)$, where the arity k of R and $\overline{x}$ coincide. Relations of this form are called *operative* and they naturally induce an operator $F_S \colon A^k \rightarrow A^k$ taking any relation R of arity k to the set $\{\overline{a} \colon (\overline{a}, R) \in S\}$. As in Section 2, we can form the inflationary and, if the relation S is monotone, also the least fixed point of F_S. Inflationary fixed points were commonly referred to as *inductive fixed points* in the 1970s.

Let $\mathcal{F}$ be a class of second-order relations on the structure $\mathfrak{A}$. A k-ary relation R is called *$\mathcal{F}$-inductive* if there is an operative relation $S(\overline{x}, \overline{y}, R') \in \mathcal{F}$, where $\overline{x}$ is k-ary, such that there is a tuple of elements $\overline{a}$ in A and for all $\overline{b}$, $\overline{b} \in R$ if, and only if, $(\overline{b}, \overline{a}) \in S^\infty$, where S^∞ denotes the inflationary fixed point of S. The elements $\overline{a}$ are called *parameters*. Analogously, R is called *$\mathcal{F}$-monotone inductive*, if it can be obtained in this way as the least fixed point of a monotone relation in $\mathcal{F}$. Let $\mathcal{F}$-IND denote the class of $\mathcal{F}$-inductive and $\mathcal{F}^{\text{mon}}$-IND the class of $\mathcal{F}$-monotone inductive relations.

A line of research active in the 1970s aimed at classifying the classes $\mathcal{F}$-IND and $\mathcal{F}^{\text{mon}}$-IND according to structural properties of the underlying class $\mathcal{F}$. Of particular interest were classes $\mathcal{F}$ of operators definable in first-order logic or in prefix-classes of second-order logic.

A result related to the present paper is the next theorem due to Harrington and Kechris [6]. The following exposition on the Harrington–Kechris theorem and the consequences derived from it were pointed out to us by Wayne Richter. I am very grateful for his detailed comments.

Let $\neg\text{WF} \in \mathcal{F}$ be the statement that $\mathcal{F}$ contains a 0-ary relation $\neg\text{WF}(S)$ which is true for S if, and only if, S is not well-founded, i.e. contains an infinite descending chain of elements. Further, a class $\mathcal{F}$ of operators is called *adequate*, if it contains all the $\forall_1$ operators, is closed under $\wedge, \vee, \exists$ and trivial combinatorial substitutions and contains the relations and functions of a coding scheme on $\mathfrak{A}$, i.e. the relations *seq*, *lh* and *q* needed to code and decode a sequence of elements. Finally, by $\check{\mathcal{F}}$ we denote the class of operators whose complements are in $\mathcal{F}$.

Theorem 4.2 (Harrington, Kechris). *Let $\mathfrak{A}$ be structure and let $\mathcal{F}$ be an adequate class of operators on $\mathfrak{A}$. If $\neg\text{WF} \in \mathcal{F}$ and $\check{\mathcal{F}} \subseteq \mathcal{F}^{\text{mon}}$-IND, then*

$$\mathcal{F}^{\text{mon}}\text{-IND} = \mathcal{F}\text{-IND}.$$

Now let $\mathfrak{A}$ be acceptable and take $\mathcal{F}$ as the class of second-order relations definable in monotone fixed-point logic MFP. As $\mathfrak{A}$ is acceptable, it is clear that MFP has all the closure properties required by the Harrington–Kechris theorem. Thus the theorem states that $\mathcal{F}^{\text{mon}}\text{-IND} = \mathcal{F}\text{-IND}$, i.e. the monotone and the inflationary closure of $\mathcal{F}$ coincide. Clearly, any relation definable by a monotone fixed point of a relation in MFP is already definable in MFP, as the logic is closed under taking least fixed points of monotone formulae. It follows that any inflationary fixed point of a MFP-formula is definable in MFP itself and therefore MFP and IFP are equivalent on $\mathfrak{A}$. Thus we get the following corollary.

Corollary 4.3. MFP = IFP *on acceptable structures.*

In an appendix to their paper [5], Gurevich and Shelah give a proof for the equivalence of IFP and MFP on finite structures. As explained in [2, pp. 70–71], this can be extended to infinite structures, generalising the theorem above. Note that the theorem only gives the equivalence of *monotone* and inflationary fixed-point logic on acceptable structures and not the equivalence of *least* and inflationary fixed-point logic. On acceptable structures, the equivalence of IFP and LFP could be derived from the following theorem by Moschovakis (see [10, Theorem 15, p. 60]).

Theorem 4.4 (Moschovakis). *If $\mathcal{F}$ is a typical, nonmonotone class of second-order relations on a structure $\mathfrak{A}$, then $\mathcal{F}$-*IND *is the smallest $\mathcal{F}$-compact spector class on $\mathfrak{A}$ such that every relation in $\mathcal{F}$ is Δ on Δ.*

We refrain from giving precise definitions for the notions mentioned in the theorem. Note, though, that if $\mathfrak{A}$ is acceptable, then the class of first-order definable second-order relations is 'typical, nonmonotone'and so is the class of relations definable in IFP. Clearly, for $\mathcal{F} :=$ IFP the inductive closure $\mathcal{F}$-IND is again IFP. Thus, one possibility to show the equivalence of LFP and IFP is by proving that LFP is an IFP-compact spector class such that every relation in IFP is Δ on Δ. A description of this approach can be found in [2, pp. 70–71]. However, proofs of the equivalence of IFP and MFP or LFP on acceptable structures based on the Harrington–Kechris and Moschovakis result do not immediately give a constructive translation of formulae of IFP into MFP or LFP. In particular, in the proofs of these theorems parameters from the structure are used in the formulae. Thus the resulting LFP and MFP-formulae may vary with different structures.

Therefore, we will not follow this approach but give a direct translation of IFP-formulae into LFP-formulae. A consequence of our proof is that LFP is indeed the smallest IFP-compact spector class such that every relation in IFP is Δ on Δ. As we will see in Proposition 5.8, we really need nested fixed points for this. In particular, if $\mathcal{F}$ is the class of first-order definable operators, then $\mathcal{F}^{\text{mon}}$-IND is a proper subset of $\mathcal{F}$-IND. In [1], Aczel gives an example showing this latter fact.

In the next section we will establish the equivalence of LFP and IFP by giving an explicit transformation of IFP-formulae into equivalent LFP-formulae. In particular, the transformation is independent of a given structure and puts no constraints on the admissible structures.

Note that the Harrington–Kechris Theorem and the equivalence proof given below are somewhat incomparable. Our proof establishes the equivalence of LFP and IFP on arbitrary classes of structures. In one way, this is more general than the Harrington–Kechris result as the equivalence of LFP and IFP implies the equivalence of MFP and IFP and we do not require the structures to be acceptable.

On the other hand, the theorem by Harrington and Kechris is true for arbitrary classes of operators—as long as they have some mild closure properties. Thus it applies not only to the case of MFP-definable operators but also to classes of operators definable in fragments of second-order logic and even to operators which do not arise from any particular logic. In this sense the Harrington–Kechris result is more general than our result which is only true for LFP and IFP and does not easily transfer to other cases.

4.3. Equivalence in the general case

In this section we aim at establishing the equivalence of IFP and LFP on arbitrary structures. Towards this, let $\varphi'(R, \overline{x})$ be in LFP, not necessarily positive in R, and consider the formula $\varphi := R\overline{x} \vee \varphi'(\overline{x})$. Clearly, φ and φ' have the same inflationary fixed point. Fix φ for the rest of the section.

We aim at defining the stage comparison relation $\prec_\varphi$ for φ in LFP. Consider again the proof of the stage comparison Theorem 3.3 above. We showed that $\prec_\varphi$ can be defined by the inflationary fixed point of the formula

$$\varphi'(\prec, \overline{x}, \overline{y}) := \varphi(\overline{x}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}) \\ \wedge \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}),$$

where $\prec$ is a second-order variable of appropriate arity.

To turn this into a formula in LFP we have to replace the formula $\neg\overline{u} \prec \overline{x}$ by a definition positive in $\prec$. Essentially, we define a second formula $\vartheta(\ll, \overline{x}, \overline{y})$, with free second-order variables $\ll$ and $\prec$, such that ϑ is negative in $\prec$ and if $\prec$ is interpreted by a given stage $\prec^\alpha$, for some ordinal α, then the least fixed point $\ll^\infty$ of ϑ is just $\prec^\alpha$. We can then use $[\mathbf{lfp}_{\ll,\overline{x},\overline{y}}\,\vartheta]$ negatively to get the desired positive definition of $\prec$.

Unfortunately, by definition, the relation defined by such a formula must increase with increasing stages $\prec^\alpha$. On the other hand, as ϑ is supposed to be negative—and therefore antitone—in $\prec$, the relation defined by ϑ must decrease with increasing stages $\prec^\alpha$. Thus, in general, we cannot hope for such a formula to exist. Instead we will use a formula defining a slightly different relation. But it might be helpful to keep the original idea in mind.

Consider the following formula

$$\chi(\overline{x}, \overline{y}) := [\mathbf{lfp}_{\prec,\overline{x},\overline{y}}\chi'(\overline{x}, \overline{y})](\overline{x}, \overline{y}),$$

where

$$\begin{aligned} \chi'(\overline{x}, \overline{y}) := {} & \varphi(\overline{x}, R\overline{u}/u \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \lhd \overline{x}) \\ & \wedge \forall\overline{u}(\overline{u} \prec \overline{x} \vee \neg\overline{u} \lhd \overline{x}) \\ & \wedge \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \lhd \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}) \end{aligned} \tag{3}$$

and

$$\overline{x} \lhd \overline{y} := [\mathbf{lfp}_{\ll,\overline{x},\overline{y}}\,\vartheta(\ll, \overline{x}, \overline{y})](\overline{x}, \overline{y})$$

where

$$\begin{aligned} \vartheta(\overline{x}, \overline{y}) := {} & \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\}) \\ & \wedge \neg\exists\overline{u}(\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})) \\ & \wedge \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})). \end{aligned}$$

Obviously, the formula χ' is positive in $\prec$ and is itself a formula in LFP. Thus the least fixed point (with respect to $\prec$) of χ' exists. We claim that this fixed point defines the stage comparison relation $\prec_\varphi$ of φ. Before proving this we first have to establish some facts about the sub-formula ϑ. Recall from the beginning of this section that φ is supposed to be of the form $R\overline{x} \vee \varphi'$. This is important for the proofs below as it ensures that whenever a tuple $\overline{x}$ satisfies φ at a stage α, it satisfies φ at all higher stages also.

Lemma 4.5. *Consider the fixed-point induction on ϑ where $\prec$ is interpreted by $\prec^{<\alpha}$, i.e. $\overline{x} \prec \overline{y}$ if, and only if, $\overline{x} \in \varphi^{<\alpha}$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$.*

(i) *If $\overline{x} \in \varphi^\alpha$ or $\overline{y} \in \varphi^\alpha$, then $(\overline{x}, \overline{y}) \in \vartheta^\infty$ if, and only if, $|\overline{x}| < |\overline{y}|$.*
(ii) *For all $\overline{y}$ such that $|\overline{y}| > \alpha$ there is an $\overline{x}$ such that $|\overline{x}| = \alpha$ and $(\overline{x}, \overline{y}) \in \vartheta^\infty$.*
(iii) *If the fixed-point of $\prec$ has already been reached, i.e. if $\prec^\alpha = \prec^{<\alpha}$, then $\vartheta^\infty = \prec^\alpha$.*

Proof. Throughout this proof, the variable $\prec$ will always be interpreted by the set $\prec^{<\alpha}$. Therefore we drop the index and write $\prec$ for $\prec^{<\alpha}$.

1. We prove by induction on β that for all $\beta < \alpha$, $(\overline{x}, \overline{y}) \in \vartheta^{\beta}$, i.e.

$$(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \vartheta(\overline{x}, \overline{y}), \text{ if, and only if, } \overline{x} \in \varphi^{\beta} \text{ and } |\overline{x}|_{\varphi} < |\overline{y}|_{\varphi}.$$

Again we omit the index most of the time and write $\ll$ for $\ll^{<\beta}$.

Suppose that for all $\gamma < \beta$ the claim has been proved. We distinguish between the case where $\overline{x} \in \varphi^{\beta}$ and $\overline{x} \notin \varphi^{\beta}$.

- Suppose $\overline{x} \in \varphi^{\beta}$. We show that $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \vartheta(\overline{x}, \overline{y})$, if, and only if, $|\overline{x}|_{\varphi} < |\overline{y}|_{\varphi}$.

  By induction hypothesis, if $\overline{x} \in \varphi^{\beta}$ then for all $\overline{u}$, $\overline{u} \ll \overline{x}$ if, and only if, $|\overline{u}|_{\varphi} < |\overline{x}|_{\varphi}$ and, as $\beta < \alpha$, $\neg\overline{u} \prec \overline{x}$ if, and only if, $|\overline{u}|_{\varphi} \geq |\overline{x}|_{\varphi}$. Thus,

$$(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\}).$$

  Now consider $\overline{y}$. If $|\overline{y}|_{\varphi} > |\overline{x}|_{\varphi}$, then $\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ reduces to $\neg\overline{u} \ll \overline{x}$. As $\beta < \alpha$, $\neg\overline{u} \ll^{<\beta} \overline{x}$ is equivalent to $\neg\overline{u} \prec^{<\alpha} \overline{x}$. Therefore there is no $\overline{u}$ satisfying $(\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$ and the second conjunct in ϑ is satisfied. Further, $\overline{y}$ does not satisfy $\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$ as otherwise $|\overline{y}|_{\varphi} \leq |\overline{x}|_{\varphi}$. Thus, $(\overline{x}, \overline{y}) \in \vartheta^{\beta}$.

  On the other hand, if $|\overline{y}|_{\varphi} < |\overline{x}|_{\varphi}$, then $(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ in the second conjunct reduces to $\overline{u} \prec \overline{y}$ and thus there is a $\overline{u}$ satisfying $\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$, $\overline{u} := \overline{y}$ for instance.

  Finally, suppose $|\overline{x}|_{\varphi} = |\overline{y}|_{\varphi}$. By the same argument as above we get that in this case

$$(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$$

  and thus ϑ is not satisfied.

- Suppose $\overline{x} \notin \varphi^{\beta}$. We show that $\varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\})$ is not satisfied. By induction hypothesis, $\overline{u} \ll \overline{x}$ defines the set $M := \varphi^{<\beta}$. Clearly, as $\overline{x} \notin \varphi^{\beta}$,

$$\mathfrak{A} \not\models \varphi(\overline{x}, R\overline{u}/\overline{u} \in M, \neg R\overline{u}/\overline{u} \in M^{c}).$$

  Now consider the set $N := \{\overline{u} : \neg\overline{u} \prec \overline{x}\}$. As $\overline{x} \notin \varphi^{\beta}$, we get $M^{c} \supseteq N$, where M^{c} denotes the complement of M.

  By monotonicity of φ in M and M^{c} it follows that

$$(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \not\models \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\}.$$

We get that for any pair $(\overline{x}, \overline{y})$, $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\beta}) \models \vartheta(\overline{x}, \overline{y})$ if, and only if, $\overline{x} \in \varphi^{\beta}$ and $|\overline{x}|_{\varphi} < |\overline{y}|_{\varphi}$.

2. Part 1 implies that $(\overline{x}, \overline{y}) \in \vartheta^{<\alpha}$ if, and only if, $\overline{x} \in \varphi^{<\alpha}$ and $|\overline{x}|_{\varphi} < |\overline{y}|_{\varphi}$. Thus, $\ll^{<\alpha} = \prec^{<\alpha}$. Now consider the next induction step. Again we distinguish between $\overline{x} \in \varphi^{\alpha}$ and $\overline{x} \notin \varphi^{\alpha}$.

- Suppose $\overline{x} \in \varphi^{\alpha}$. Obviously,

$$(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\alpha}) \models \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}).$$

If $|\overline{y}|_\varphi \geq |\overline{x}|_\varphi$, then $(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ reduces to $\overline{u} \prec \overline{x}$ and thus $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\alpha}) \models \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$ if, and only if, $|\overline{y}|_\varphi > |\overline{x}|_\varphi$.

Now suppose $|\overline{y}|_\varphi < |\overline{x}|_\varphi$. Then $|\overline{y}|_\varphi < \alpha$ and there is a $\overline{u}$ satisfying $\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$, again $\overline{y}$ being itself a witness for this. Thus $\vartheta(\overline{x}, \overline{y})$ is not satisfied.

- Now assume $\overline{x} \notin \varphi^\alpha$. Then

$$(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\alpha}) \not\models \varphi(\overline{x}, R\overline{u}/\overline{u} \ll \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x}\})$$

as $\overline{u} \ll \overline{x}$ defines the set $\overline{u} \in \varphi^{<\alpha}$ and $\neg\overline{u} \prec \overline{x}$ its complement.

It follows, that ϑ^α contains all pairs $(\overline{x}, \overline{y})$ such that $\overline{x} \in \varphi^\alpha$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$. This proves part (ii) because if there is a tuple $\overline{y}$ of rank greater than α there must also be a tuple $\overline{x}$ of rank exactly α and this pair would be in ϑ^α.

Further, if the fixed point of $\prec$ has already been reached, i.e. $\prec^\alpha = \prec^{<\alpha}$, then there are no tuples $\overline{x}$ of rank exactly α. In this case, all tuples $(\overline{x}, \overline{y}) \in \vartheta^\alpha$ already occur in $\vartheta^{<\alpha}$ and the fixed point of ϑ has been reached. This proves part (iii) of the lemma. Thus, from now on, we assume that $\prec^{<\alpha} \subsetneq \prec^\alpha$.

3. We show now that at no stage $\gamma > \alpha$ can a pair $(\overline{x}, \overline{y})$ with $\overline{x}, \overline{y} \in \varphi^\alpha$ and $|\overline{y}|_\varphi \leq |\overline{x}|_\varphi$ enter the fixed point. Towards a contradiction let γ be the smallest such stage and let $(\overline{x}, \overline{y})$ be as described. Then the same argument as in the first item of step 1 yields a contradiction.
4. What is left to be shown is that for no $\overline{x} \notin \varphi^\alpha$ and $\overline{y} \in \varphi^\alpha$ the pair $(\overline{x}, \overline{y})$ enters the fixed point at some higher stage. Towards a contradiction, let γ be the least such stage, i.e. the least stage such that there is a pair $(\overline{x}, \overline{y}) \in \vartheta^\gamma$ with $\overline{x} \notin \varphi^\alpha$ and $\overline{y} \in \varphi^\alpha$. In particular, $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\gamma}) \models \vartheta(\overline{x}, \overline{y})$.

   Now, as $\overline{x} \notin \varphi^\alpha$, $\overline{u} \prec \overline{x}$ defines just $\varphi^{<\alpha}$ and, as γ was chosen minimal, we get that $\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ defines the set of tuples $\overline{u}$ such that $|\overline{u}|_\varphi \geq |\overline{y}|_\varphi$. Thus, if $|\overline{y}|_\varphi < \alpha$ then there is a tuple $\overline{u}$ satisfying $\overline{u} \prec \overline{x} \wedge \neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y})$ and thus ϑ is not satisfied by $(\overline{x}, \overline{y})$. On the other hand, if $|\overline{y}|_\varphi = \alpha$, then $(\mathfrak{A}, \prec^{<\alpha}, \ll^{<\alpha}) \models \varphi(\overline{y}, R\overline{u}/\overline{u} \prec \overline{x}, \neg R\overline{u}/\neg(\overline{u} \ll \overline{x} \wedge \overline{u} \ll \overline{y}))$ and again ϑ is not satisfied.

This finishes the proof of the lemma. □

We now prove a technical lemma which will establish the induction step in the proof that the fixed point of χ' defines $\prec_\varphi$.

Lemma 4.6. *For all ordinals α, $(\mathfrak{A}, \prec^{<\alpha}) \models \chi'(\overline{x}, \overline{y})$, if, and only if, $\overline{x} \in \varphi^\alpha$ and $|\overline{x}|_\varphi < |\overline{y}|_\varphi$.*

Proof. We distinguish between the cases where $\overline{x} \in \varphi^\alpha$ and $\overline{x} \notin \varphi^\alpha$.

- Suppose $\overline{x} \in \varphi^\alpha$. By assumption, $\overline{u} \prec^{<\alpha} \overline{x}$ defines the set $\{\overline{u} : |\overline{u}|_\varphi < |\overline{x}|_\varphi\}$ and, by part (i) of Lemma 4.5, $\neg\overline{u} \lhd \overline{x}$ defines its complement. Thus, $(\mathfrak{A}, \prec^{<\alpha}) \models \varphi(\overline{x}, R\overline{u}/u \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \lhd \overline{x})$ and all $\overline{u}$ satisfy $\overline{u} \prec \overline{x} \vee \neg\overline{u} \lhd \overline{x}$.

  Now, $(\mathfrak{A}, \prec^{<\alpha}) \models \neg\varphi(\overline{y}, R\overline{u}/\overline{u} \lhd \overline{x}, \neg R\overline{u}/\neg\overline{u} \prec \overline{x})$ if, and only if, $|\overline{y}|_\varphi > |\overline{x}|_\varphi$.

  Thus, $(\mathfrak{A}, \prec^{<\alpha}) \models \chi'(\overline{x}, \overline{y})$ if, and only if, $|\overline{y}|_\varphi > |\overline{x}|_\varphi$.

- Suppose $\overline{x} \notin \varphi^\alpha$. Then $\overline{u} \prec \overline{x}$ defines the set $\{\overline{u} : \overline{u} \in \varphi^{<\alpha}\}$. If $\varphi^{<\alpha} = \varphi^\alpha$, i.e. if the fixed point of φ has been reached, then, by part (iii) of Lemma 4.5, we get $\lhd = \prec$ and $(\mathfrak{A}, \prec^{<\alpha}) \not\models \varphi(\overline{x}, R\overline{u}/u \prec \overline{x}, \neg R\overline{u}/\neg\overline{u} \lhd \overline{x})$ and therefore χ' is not satisfied.

  Otherwise, i.e. if $\varphi^{<\alpha} \subsetneq \varphi^\alpha$, then, by part (ii) of Lemma 4.5, there is a tuple $\overline{a}$ of rank α with $\overline{a} \lhd \overline{x}$. Thus, the conjunct $\forall \overline{u}(\overline{u} \prec \overline{x} \vee \neg\overline{u} \lhd \prec \overline{x})$ is not satisfied as $\overline{a} \lhd \overline{x}$ but $\overline{a} \not\prec \overline{x}$.

This finishes the proof of the lemma. □

As corollary we get that the relation $\prec_\varphi$ is definable in LFP.

Corollary 4.7. *Let* $\varphi(R, \overline{x})$ *be a formula in* LFP. *Then the stage comparison relation* $\prec_\varphi$ *of the inflationary fixed point of* φ *is definable in* LFP.

Proof. A simple induction on the stages using the previous lemma shows that $\prec_\varphi$ is defined by the formula χ above. □

The equivalence of LFP and IFP follows immediately.

Theorem 4.8. *For every formula in* IFP *there is an equivalent formula in* LFP.

Proof. By Corollary 4.7, for every $\varphi(R, \overline{x}) \in$ LFP the relation $\prec_\varphi$ is definable in LFP. Thus, for all $\overline{x}$, $\overline{x} \in \varphi^\infty$ if, and only if, $\mathfrak{A} \models \varphi(\overline{x}, R\overline{u}/\chi(\overline{u}, \overline{x}))$, where χ is the formula defining $\prec_\varphi$. Thus, the inflationary fixed point of an LFP-formula can be defined in LFP.

For arbitrary formulae $\varphi \in$ IFP, the theorem follows by induction on the number of inflationary fixed points in φ converting them to least fixed points from the inside out. □

The theorem shows that also on infinite structures, least and inflationary fixed-point logic have the same expressive power. But, contrary to the case of finite structures where the translation of IFP-formulae to equivalent LFP-formulae does not alter the fixed-point structure, in the general case their structure in terms of alternations between **lfp**-operators and negation and the nesting depth of fixed-point operators becomes more complicated. It might be possible to reduce the increase in the number of alternations of the resulting LFP-formulae. However, we will show below that an increase in the number of alternations cannot be avoided.

5. Normal forms and hierarchies

There are various natural parameters that may influence the expressive power of fixed-point logics. In this section we are particularly interested in two such parameters: the number of fixed-point operators that are nested within each other and the number of alternations between fixed-point operators and negation symbols. For this we first need some technical definitions.

Definition 5.1 (Alternation and Nesting-depth Hierarchy). Let $\varphi \in$ LFP be a formula such that no fixed-point variable is bound twice in it and let $X_1, \ldots, X_k$ be the fixed-point variables occurring in φ. Let for all i, φ_i be the formula binding X_i in φ, i.e. $\varphi_i := [\mathbf{lfp}_{X_i, \overline{x}_i}\, \varphi_i'](\overline{t}_i)$ for suitable $\overline{x}_i$, $\overline{t}_i$ and φ_i'.

We define a partial order $\sqsubseteq_\varphi$ on the variables $X_1, \dots, X_k$ as

$$X_i \sqsubseteq_\varphi X_j \text{ if, and only if, } \varphi_i \text{ is a sub-formula of } \varphi_j.$$

- The *nesting-depth of* φ is defined as the maximal cardinality of a subset of $\{X_1, \dots, X_k\}$ linearly ordered by $\sqsubseteq_\varphi$.
- The *alternation-level of* φ is defined as the maximal cardinality of a subset $\mathcal{M}$ of $\{X_1, \dots, X_k\}$, linearly ordered by $\sqsubseteq_\varphi$, such that in addition for all $X_i, X_j \in \mathcal{M}$, if X_i is a direct predecessor of X_j with respect to $\sqsubseteq_\varphi$, then φ_i occurs negative in φ_j.

The n-th level of the *alternation hierarchy* $(\text{LFP}^a_n)_{n\in\omega}$ consists of all formulae of LFP with alternation-level n. Analogously, the n-th level of the *nesting-depth hierarchy* $(\text{LFP}^d_n)_{n\in\omega}$ of LFP is defined as the class of formulae in LFP of nesting-depth n. Finally, by $(p\text{LFP}^d_n)_{n\in\omega}$ we denote the *positive nesting-depth hierarchy*, consisting of formulae with nesting-depth n but with only positive applications of the fixed-point operators.

The hierarchies for IFP are defined analogously.

By definition, LFP^a_1 consists of all LFP-formulae where no **lfp**-operator occurs negatively, whereas LFP^a_0 and LFP^d_0 are just the class of first-order formulae.

The following theorem, due to Moschovakis, shows that in LFP, nested positive fixed points can be eliminated. See [9, Theorem 1C.3] for a proof. The presentation given here follows [3, Lemma 8.2.6 on p. 182].

Theorem 5.2 (Transitivity Theorem). *Let $\varphi(R, Q, \overline{x})$ and $\psi(R, Q, \overline{y})$ be first-order formulae positive in R and Q such that no free first-order variables of ψ are bound in $\chi := [\mathbf{lfp}_{R,\overline{x}}\, \varphi(Q\overline{u}/[\mathbf{lfp}_{Q,\overline{y}}\, \psi](\overline{u}))](\overline{x})$. Then χ is equivalent to a formula with only one application of an* **lfp**-*operator.*

An immediate consequence of the theorem is the following.

Corollary 5.3. *For all n, $p\,\text{LFP}^d_n = p\text{LFP}^d_1$, i.e. every formula $\varphi \in$ LFP in which all fixed-point operators occur only positively is equivalent to a formula with only one application of a fixed-point operator, i.e. the positive nesting-depth hierarchy for* LFP *collapses.*

Obviously, the nesting-depth hierarchy is finer than the alternation hierarchy in the sense that a formula with alternation depth n also has nesting-depth at least n. Using simple diagonalisation arguments it can be shown that in general the nesting-depth hierarchy is strict, i.e. there is no constant $k < \omega$ such that every LFP formula is equivalent to a formula with nesting depth at most k.

On the other hand, Theorem 5.2 implies that for LFP the nesting-depth hierarchy collapses to the alternation hierarchy. An immediate consequence of this is, that in general the alternation hierarchy for LFP is strict. (See e.g. [9, Chapter 5] for a proof of this.)

Theorem 5.4. *The alternation hierarchy for* LFP *is strict.*

The proof of this theorem uses a diagonalisation argument which relies on structures being infinite. And indeed, as proved by Immerman [7], the nesting depth and therefore also the alternation hierarchy for LFP collapses on finite structures.

We now turn towards alternation and nesting of inflationary fixed points. As the following theorem shows, alternation between inflationary fixed points and negation does not result in an increase in expressive power.

Theorem 5.5. *Every formula in* IFP *is equivalent to a formula where negation occurs only in front of atoms.*

Proof. The theorem is proved by induction on the structure of the formula. For the case of the **ifp**-operator note that a formula $\neg[\mathbf{ifp}_{R,\overline{x}}\,\varphi](\bar{t})$ is equivalent to the simultaneous fixed point $[\mathbf{ifp}\,Q : S](\bar{t})$ of the system

$$S := \begin{cases} R\overline{x} \;\leftarrow\; \varphi(R,\overline{x}) \\ Q\overline{x} \;\leftarrow\; \forall\,\overline{y}\,(\varphi(R,\overline{y}) \rightarrow\; R\overline{y}) \wedge \neg R\overline{x}. \end{cases}$$

On structures with at least two elements this is equivalent to the inflationary fixed point of a single formula whereas on structures with only one element, IFP collapses to FO anyway and the theorem is trivial. □

Theorem 5.5 above shows that the alternation hierarchy of IFP collapses to level one. Again, simple diagonalisation arguments show that the nesting depth hierarchy for IFP is strict in general and therefore, unlike for least fixed points, also the positive nesting-depth hierarchy is strict for IFP.

Using the results from Section 4, we can establish a close correspondence between the strictness of the alternation hierarchy for LFP on a structure $\mathfrak{A}$ and the strictness of the nesting-depth hierarchy for IFP on $\mathfrak{A}$.

Theorem 5.6. *For every* $n \geq 0$,

$$\mathrm{LFP}^a_n \leq \mathrm{IFP}^d_n \leq \mathrm{LFP}^a_{3^n}.$$

Proof. Let $\varphi \in \mathrm{LFP}_n$ be a formula with alternation depth n. By the transitivity theorem 5.2, nested **lfp**-operators which all occur positively can be contracted to a single **lfp**-operator increasing the arity. Thus, every formula in LFP_n is equivalent to a formula with n nested fixed points and therefore equivalent to an IFP formula with nesting depth n.

Towards the second containment, note that using the method of Theorem 4.8 to convert an IFP-formula to an equivalent LFP-formula, the translation of each individual **ifp**-operator at most triples the alternation depth. The theorem now follows by induction. □

We immediately get the following corollaries.

Corollary 5.7. *For any structure, the alternation depth hierarchy for* LFP *collapses if, and only if, the nesting depth hierarchy for* IFP *collapses.*

An example of a class of structures where the hierarchies are strict is the class of acceptable structures (see [9] for instance).

It is open whether there are infinite structures on which the alternation and nesting depth hierarchies for LFP and IFP collapse but where LFP is still more expressive than FO.

The previous theorem implies that every LFP-formula with alternation-depth n can be converted into an IFP-formula with nesting depth n. We show next that the converse does not hold, i.e. there are IFP-formulae with n fixed-point operators which are not equivalent to any LFP-formula of nesting-depth at most n. In particular, this shows that the Harrington–Kechris Theorem 4.2 cannot be strengthened to positive inductions, i.e. the *inductive closure* of FO does not coincide with the *positive* monotone closure.

Proposition 5.8. *For every* $n > 1$, $\mathrm{LFP}_n^a \lneq \mathrm{IFP}_n^d$.

Proof. Suppose $\mathrm{IFP}_n^d = \mathrm{LFP}_n^a$ for some n. For every formula $\varphi \in \mathrm{LFP}_n^a$, φ and $\neg\varphi$ are equivalent to formulae ψ and $\neg\psi$ in IFP_n^d. As by assumption $\mathrm{IFP}_n^d = \mathrm{LFP}_n^a$, the formulae $\neg\psi$ and ψ are both equivalent to a formula in LFP_n^a and therefore LFP_n^a is closed under complementation contradicting the strictness of the alternation hierarchy for LFP. □

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