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On Simultaneous Chebyshev Approximation

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1. INTRODUCTION

Let $f_1, ..., f_n$ belong to $C[a, b]$, and let G be a nonempty subset of $C[a, b]$. The problem addressed here is that of approximating the n functions f_i simultaneously, in some sense, by an element of G.

This problem and variations on it have attracted much interest, and we have included a complete list of references known to us. Not included in the references are works concerning the related problem of *widths* of function classes, nor works concerned with approximating one function and its derivatives.

A number of natural definitions of a simultaneous approximant may be offered. For example, Dunham [4] and Diaz and McLaughlin [6] investigated the problem of selecting $g \in G$ to minimize the expression

$$\max_{1 \leqslant i \leqslant n} \max_{a \leqslant x \leqslant b} | f_i(x) - g(x)|.$$

Ling, in [19], considered (with $n = 2$) the problem of minimizing the expression $\max_{a \leqslant x \leqslant b} \sum_{i=1}^{n} | f_i(x) - g(x)|$. This is a special case of a more general situation in which a not-necessarily-finite number of functions f_i is given, and a summation or integration process is involved in the error criterion. It is this type of simultaneous approximation that is investigated here.

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2. The $\mathscr{L}_1 - \mathscr{L}_\infty$ Simultaneous Approximation

The finite set of univariate functions in Section 1 is now replaced by a single bivariate function $f(x, y)$. In order for everything to work smoothly we assume that x ranges over a compact Hausdorff space X, and y ranges over a measure space Y of measure 1. Furthermore, $f(x, y)$ is continuous in x for fixed y and integrable in y for fixed x. The *mean* of f is defined to be

$$\bar{f}(x) = \int_Y f(x, y)\, dy \tag{1}$$

and the supremum norm of an element g of $C(X)$ is defined by $\| g \|_\infty = \max_x | g(x)|$.

An element g^* in a prescribed class $G \subset C(X)$ is now termed a *best simultaneous approximation* of f if for all $g \in G$,

$$\max_x \int | f(x, y) - g^*(x)|\, dy \leqslant \max_x \int | f(x, y) - g(x)|\, dy.$$

The finite case is recovered, of course, by taking Y to be a measure space with only n points, each of mass $1/n$. But there are important cases, notably in the approximation of integral transforms, when the full generality of this setting is required.

Lemma. *If $g \in C(X)$ and $| \bar{f}(x) - f(x, y)| \leqslant \| \bar{f} - g \|_\infty$ for all x and y then* $\| \bar{f} - g \|_\infty = \max_x \int | f(x, y) - g(x)|\, dy$.

Proof. Let v be any element of $\mathscr{L}_\infty(Y)$ satisfying the equation $| v(y)| = 1$ almost everywhere. Let A be the set of y such that $v(y) = 1$, and B the set of y such that $v(y) = -1$. Then for all x,

$$\begin{aligned}
&\int v(y)[f(x, y) - g(x)]\, dy \\
&\quad = \int_A [f(x, y) - g(x)]\, dy + \int_B [g(x) - f(x, y)]\, dy \\
&\quad = \int_Y f(x, y)\, dy - \int_B f(x, y)\, dy - g(x)\, \mu(A) + g(x)\, \mu(B) - \int_B f(x, y)\, dy \\
&\quad = \bar{f}(x) - 2 \int_B f(x, y)\, dy - g(x)[1 - 2\mu(B)] \\
&\quad = 2 \int_B [\bar{f}(x) - f(x, y)]\, dy + [\bar{f}(x) - g(x)][1 - 2\mu(B)].
\end{aligned}$$

Here the equation $\mu(A) + \mu(B) = 1$ was used. If $\mu(B) \leqslant \frac{1}{2}$ then the preceding equation yields

$$\int v(y)[f(x, y) - g(x)]\, dx$$

$$\leqslant 2\mu(B)\, \|\bar{f} - g\|_\infty + \|\bar{f} - g\|_\infty\, [1 - 2\mu(B)] = \|\bar{f} - g\|_\infty\,.$$

If $\mu(B) > \frac{1}{2}$ then a similar line of reasoning leads to the same inequality with A in place of B. Hence we may take first a supremum for all v and then a supremum for all x to conclude that

$$\max_x \int |f(x, y) - g(x)|\, dy \leqslant \|\bar{f} - g\|_\infty = \max_x \left| \int [f(x, y) - g(x)]\, dy \right|$$

$$\leqslant \max_x \int |f(x, y) - g(x)|\, dy \quad \blacksquare$$

THEOREM. *If $g^* \in G$ and if for all $g \in G$ and $y \in Y$* $\max_x |\bar{f}(x) - f(x, y)| \leqslant \max_x |\bar{f}(x) - g^*(x)| \leqslant \max_x |\bar{f}(x) - g(x)|$ *then* $\max_x \int |f(x, y) - g^*(x)|\, dy \leqslant \max_x \int |f(x, y) - g(x)|\, dy$.

The theorem follows at once from the lemma. It states that if the functions $f(\cdot, y)$ are closer to their mean, $\bar{f}$, than $\bar{f}$ is to G, then the best simultaneous approximation is nothing but the best approximation to $\bar{f}$ in the usual sense. In many applications, the functions in the family are close to one another and the hypothesis is fulfilled.

An interesting special case is that in which Y is a set of n points, each of mass $1/n$, and X is an interval $[a, b]$ on the real line. Ling [19] proves the above theorem in this setting for the case $n = 2$.

EXAMPLE 1. Take X and Y to be unit intervals $[0, 1]$, and let G be π_1, the set of linear polynomials. Let

$$f(x, y) = x^{1/2}(1 + \epsilon \sin 2\pi y).$$

It is easily verified that the hypotheses of the above theorem hold when $|\epsilon| \leqslant \frac{1}{8}$ and $g^*(x) = x + \frac{1}{8}$. Thus g^* is the best simultaneous approximation to $f(x, y)$.

An example showing that the theorem does not encompass all cases is as follows.

EXAMPLE 2. Let X be a two-point space, $X = \{p, q\}$, and Y a five-point space, $Y = \{1, 2, \ldots, 5\}$. Let f be given by the table

x	p	q	p	q	p	q	p	q	p	q
y	1	1	2	2	3	3	4	4	5	5
$f(x, y)$	1	0	0	1	1	1	-1	0	0	-1

Let G consist of all functions g on X such that $g(p) = g(q)$. Then $\bar{f} = (.2, .2) \in G$, while the best simultaneous approximation of $f(x, y)$ is 0.

3. Characterization of Best Simultaneous Approximation

In this section, several mild hypotheses are imposed upon the simultaneous approximation problem of Section 2 in order that a characterization can be established for best approximations.

It is assumed that $f: X \times Y \to \mathbb{R}$, where X is a compact topological space and Y is a measure space. Furthermore, these conditions are imposed:

(1) $\{f(\cdot, y): y \in Y\}$ is an equicontinuous family of functions from X to $\mathbb{R}$.

(2) For each $x \in X$, $f(x, \cdot)$ is an integrable function from Y to $\mathbb{R}$.

(3) For each x, the zeros of $f(x, \cdot)$ form a set of measure zero in Y.

As in Section 2, a simultaneous approximation of the family $\{f(\cdot, y): y \in Y\}$ is sought, by an element g of a prescribed subset G in $C(X)$. It is assumed now that the set G is convex. An element of $g \in G$ is a best simultaneous approximation to f if it minimizes the expression

$$\sup_{x\in X} \int_Y |f(x, y) - g(x)|\, dy.$$

Theorem. *In order that a particular element $g^* \in G$ shall minimize the above expression, it is necessary and sufficient that each element $g \in G$ shall satisfy the inequality*

$$\inf_{x\in K} [g(x) - g^*(x)] \int_Y \operatorname{sgn}[f(x, y) - g^*(x)]\, dy \leqslant 0,$$

where K is the critical set, i.e., the set of x in X for which

$$\int_Y |f(x, y) - g^*(x)|\, dy = \sup_t \int_Y |f(t, y) - g^*(t)|\, dy.$$

Proof. This follows from the main theorem of [29], if the family $\mathscr{F}$ of that paper is taken to be the set of all functions $f(x, y) - g(x)$ as g ranges over G. ∎

A special case of the theorem arises when X is an interval and G is an n-dimensional Haar subspace in $C(X)$. In this case, an element $g^* \in G$ is a best simultaneous approximation of f if and only if the function

$$\phi(x) \equiv \int_Y \operatorname{sgn}[f(x, y) - g^*(x)]\, dy$$

alternates in sign $n + 1$ times on the critical set. That is, there should exist points $x_0 < x_1 < \cdots < x_n$ in X such that

$$\int_Y |f(x_i, y) - g^*(x_i)|\, dy = \sup_t \int_Y |f(t, y) - g^*(t)|\, dy \qquad (1)$$

and

$$\phi(x_i)\, \phi(x_{i-1}) < 0 \qquad (1 \leqslant i \leqslant n). \qquad (2)$$

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