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Toroidal automorphic forms for some function fields

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ABSTRACT

Zagier introduced toroidal automorphic forms to study the zeros of zeta functions: an automorphic form on GL_2 is toroidal if all its right translates integrate to zero over all non-split tori in GL_2, and an Eisenstein series is toroidal if its weight is a zero of the zeta function of the corresponding field. We compute the space of such forms for the global function fields of class number one and genus $g \leqslant 1$, and with a rational place. The space has dimension g and is spanned by the expected Eisenstein series. We deduce an "automorphic" proof for the Riemann hypothesis for the zeta function of those curves.

1. Introduction

Let X denote a smooth projective curve over a finite field $\mathbf{F}_q$ with q elements, $\mathbf{A}$ the adeles over its function field $F := \mathbf{F}_q(X)$, $G = \mathrm{GL}_2$, B its standard (upper-triangular) Borel subgroup, $K = G(\mathcal{O}_{\mathbf{A}})$ the standard maximal compact subgroup of $G_{\mathbf{A}}$, with $\mathcal{O}_{\mathbf{A}}$ the maximal compact subring of $\mathbf{A}$, and Z the center of G. Let $\mathscr{A}$ denote the space of unramified automorphic forms $f : G_F\backslash G_{\mathbf{A}}/KZ_{\mathbf{A}} \to \mathbf{C}$. We use the following notations for matrices:

$$\mathrm{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad [\![a, b]\!] = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

There is a bijection between quadratic separable field extensions E/F and conjugacy classes of maximal non-split tori in G_F via

$$E^\times = \mathrm{Aut}_E(E) \subset \mathrm{Aut}_F(E) \simeq G_F.$$

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If T is a non-split torus in G with $T_F \cong E^\times$, define the space of *toroidal automorphic forms for F with respect to T (or E)* to be

$$\mathsf{T}_F(E) = \left\{ f \in \mathscr{A} \;\middle|\; \forall g \in G_{\mathbf{A}}, \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} f(tg)\,dt = 0 \right\}. \tag{1}$$

The integral makes sense since $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}$ is compact, and the space only depends on E, viz., the conjugacy class of T. The space of *toroidal automorphic forms for F* is

$$\mathsf{T}_F = \bigcap_E \mathsf{T}_F(E),$$

where the intersection is over all quadratic separable E/F. The interest in these spaces lies in the following version of a formula of Hecke [5, Werke, p. 201]; see Zagier [15, pp. 298–299] for this formulation, in which the result essentially follows from Tate's thesis:

Proposition 1.1. *Let ζ_E denote the zeta function of the field E. Let $\varphi : \mathbf{A}^2 \to \mathbf{C}$ be a Schwartz–Bruhat function. Set*

$$f(g,s) = |\det g|_F^s \int\limits_{\mathbf{A}^\times} \varphi\big((0,a)g\big)|a|^{2s}\,d^\times a.$$

An Eisenstein series $E(s)$,

$$E(s)(g) := \sum_{\gamma \in B_F \backslash G_F} f(\gamma g, s) \quad \big(\mathrm{Re}(s) > 1\big)$$

satisfies

$$\int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} E(s)(tg)\,dt = c(\varphi, g, s)|\det g|^s \zeta_E(s)$$

for some holomorphic function $c(\varphi, g, s)$. For every g and s, there exists a function φ such that $c(\varphi, g, s) \neq 0$. In particular, $E(s) \in \mathsf{T}_F(E) \Leftrightarrow \zeta_E(s) = 0$.

Remark 1.2. Toroidal integrals of parabolic forms are ubiquitous in the work of Waldspurger ([13], for recent applications, see Clozel and Ullmo [1] and Lysenko [10]). Wielonsky and Lachaud studied analogues for GL_n, $n \geqslant 2$, and tied up the spaces with Connes' view on zeta functions [2,6,7,14].

Let $\mathscr{H} = C_0^\infty(K \backslash G_{\mathbf{A}}/K)$ denote the bi-K-invariant Hecke algebra, acting by convolution on $\mathscr{A}$. There is a correspondence between K-invariant $G_{\mathbf{A}}$-modules and Hecke modules; in particular, we have

Lemma 1.3. *The spaces $\mathsf{T}_F(E)$ (for each E with corresponding torus T) and T_F are invariant under the Hecke algebra $\mathscr{H}$, and*

$$\mathsf{T}_F(E) \subseteq \left\{ f \in \mathscr{A} \;\middle|\; \forall \Phi \in \mathscr{H}, \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} \Phi(f)(t)\,dt = 0 \right\}. \tag{2}$$

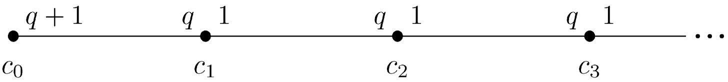

Fig. 1. The graph $\Gamma\backslash\mathscr{T}$ for $X = \mathbf{P}^1$.

Now assume F has class number one and there exists a place ∞ of degree one for F; let t denote a local uniformizer at ∞. Strong approximation implies that we have a bijection

$$G_F\backslash G_{\mathbf{A}}/KZ_\infty \xrightarrow{\sim} \Gamma\backslash G_\infty/K_\infty Z_\infty,$$

where $\Gamma = G(A)$ with A the ring of functions in F holomorphic outside ∞, and a subscript ∞ refers to the ∞-component. We define a graph $\mathscr{T}$ with vertices $V\mathscr{T} = G_\infty/K_\infty Z_\infty$. If $\sim$ denotes equivalence of matrices modulo $K_\infty Z_\infty$, then we call vertices in $V\mathscr{T}$ given by classes represented by matrices g_1 and g_2 adjacent, if $g_1^{-1}g_2 \sim [\![t, b]\!]$ or $[\![t^{-1}, 0]\!]$ for some $b \in \mathscr{O}_\infty/t$. Then $\mathscr{T}$ is a tree that only depends on q (the so-called Bruhat–Tits tree of $\mathrm{PGL}(2, F_\infty)$, cf. [11, Chapter II]).

The Hecke operator $\Phi_\infty \in \mathscr{H}$ given by the characteristic function of $K[\![t, 0]\!]K$ maps a vertex of $\mathscr{T}$ to its neighboring vertices. The action of Φ_∞ on the quotient graph $\Gamma\backslash\mathscr{T}$ can be computed from the orders of the Γ-stabilizers of vertices and edges in $\mathscr{T}$. When drawing a picture of $\Gamma\backslash\mathscr{T}$, we agree to label a vertex along the edge towards an adjacent vertex by the corresponding weight of a Hecke operator, as in the next example.

Example 1.4. In Fig. 1, one sees the graph $\Gamma\backslash\mathscr{T}$ for the function field of $X = \mathbf{P}^1$, with the well-known vertices representing $\{c_i = [\![\pi^{-i}, 0]\!]\}_{i \geqslant 0}$ and the weights of Φ_∞, meaning

$$\text{for } n \geqslant 1, \quad \Phi_\infty(f)(c_n) = qf(c_{n-1}) + f(c_{n+1}) \quad \text{and} \quad \Phi_\infty(f)(c_0) = (q+1)f(c_1). \tag{3}$$

2. The rational function field

First, assume $X = \mathbf{P}^1$ over $\mathbf{F}_q$, so F is a rational function field. Set $E = \mathbf{F}_{q^2}F$ the quadratic constant extension of F.

Theorem 2.1. $\mathsf{T}_F = \mathsf{T}_F(E) = \{0\}$.

Proof. Let T be a torus with $T_F = E^\times$, that has a basis over F contained in the constant extension $\mathbf{F}_{q^2}$. The integral defining $f \in \mathsf{T}_F(E)$ in Eq. (1) for the element $g = 1 \in G_{\mathbf{A}}$ becomes

$$\int\limits_{T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}} f(t)\,dt = \kappa \cdot \int\limits_{T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}/(T_{\mathbf{A}}\cap K)} f(t)\,dt = \kappa \cdot \int\limits_{E^\times \mathbf{A}_F^\times\backslash \mathbf{A}_E^\times/\mathscr{O}_{\mathbf{A}_E}^\times} f(t)\,dt = \kappa \cdot f(c_0),$$

with $\kappa = \mu(T_{\mathbf{A}} \cap K) \neq 0$. Indeed, by our choice of "constant" basis, we have $T_{\mathbf{A}} \cap K \cong \mathscr{O}_{\mathbf{A}_E}^\times$. For the final equality, note that the integration domain $E^\times\mathbf{A}_F^\times\backslash\mathbf{A}_E^\times/\mathscr{O}_{\mathbf{A}_E}^\times$ is isomorphic to the quotient of the class group of E by that of F, and that both of these groups are trivial, so map to the identity matrix c_0 in $\Gamma\backslash\mathscr{T}$.

Hence we first of all find $f(c_0) = 0$. For $\Phi = \Phi_\infty^k$, this equation transforms into $(\Phi_\infty^k f)(c_0) = 0$ (cf. (2)), and with (3) this leads to a system of equations for $f(c_i)$ $(i = 1, 2, \ldots)$ that can easily be shown inductively to only have the zero solution $f = 0$. $\square$

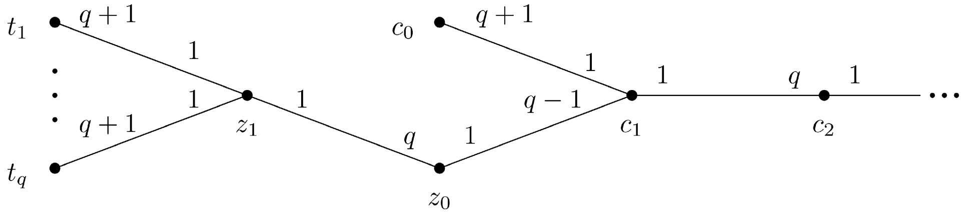

Fig. 2. The graph $\Gamma\backslash\mathscr{T}$ for F_q $(q=2,3,4)$.

3. Three elliptic curves

Now assume that F is not rational, has class number one, a rational point ∞ and genus $\leqslant 1$. In this paper, we focus on such fields F, since it turns out that the space T_F can be understood elaborating only on existing structure results about the graph $\Gamma\backslash\mathscr{T}$.

The Hasse–Weil theorem implies that there are only three possibilities for F, which we conveniently number as follows: $\{F_q\}_{q=2}^4$ with F_q the function field of the projective curve $X_q/\mathbf{F}_q$ $(q=2,3,4)$ are the respective elliptic curves

$$y^2+y=x^3+x+1, \qquad y^2=x^3-x-1 \quad \text{and} \quad y^2+y=x^3+\alpha$$

with $\mathbf{F}_4=\mathbf{F}_2(\alpha)$. Let $F_q^{(2)}=\mathbf{F}_{q^2}F_q$ denote the quadratic constant extension of F_q.

The graph $\Gamma\backslash\mathscr{T}$ for F_q $(q=2,3,4)$ with the Φ_∞-weights is displayed in Fig. 2, cf. Serre [11, 2.4.4 and Exercise 3b) + 3c), p. 117] and/or Takahashi [12] for these facts.

Further useful facts: One easily calculates that $X_q(\mathbf{F}_{q^2})$ is cyclic of order $2q+1$; let Q denote any generator. We will use later on that the vertices t_i correspond to classes of rank-two vector bundles on $X_q(\mathbf{F}_q)$ that are pushed down from line bundles on $X_q(\mathbf{F}_{q^2})$ given by multiples $Q, 2Q, \dots, qQ$ of Q, cf. Serre [11]. For a representation in terms of matrices, one may refer to [12]: if $iQ=(\ell,*)\in X_q(\mathbf{F}_{q^2})$, then $t_i=[\![t^2, t^{-1}+\ell t]\!]$.

We denote a function f on $\Gamma\backslash\mathscr{T}$ by a vector

$$f=\big[f(t_1),\dots,f(t_q)\,\big|\,f(z_0),f(z_1)\,\big|\,f(c_0),f(c_1),f(c_2),\dots\big].$$

Proposition 3.1. *A function $f\in\mathsf{T}_{F_q}(F_q^{(2)})$ $(q=2,3,4)$ belongs to the Φ_∞-stable linear space $\mathscr{S}$ of functions*

$$\mathscr{S}:=\big\{[T_1,\dots,T_q \mid Z_0,Z_1 \mid C_0,C_1,C_2,\dots]\big\} \tag{4}$$

with $C_0=-2(T_1+\dots+T_q)$ and for $k\geqslant 0$,

$$C_k=\begin{cases}\lambda_k Z_0+\mu_k(T_1+\dots+T_q) & \text{if } k \text{ even},\\ \nu_k Z_1 & \text{if } k \text{ odd},\end{cases} \tag{5}$$

for some constants λ_k,μ_k,ν_k. In particular,

$$\dim\mathsf{T}_{F_q}\big(F_q^{(2)}\big)\leqslant\dim\mathscr{S}=q+2,$$

and $\dim\mathsf{T}_{F_q}$ *is finite.*

Proof. We choose arbitrary values T_j at t_j $(j=1,\dots,q)$ and Z_j at z_j $(j=1,2)$, and set $\tau=T_1+\dots+T_q$. We have

$$\int_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} f(t)\,dt = C_0 + 2\tau.$$

Indeed, by the same reasoning as in the proof of Theorem 2.1, the integration area maps to the image of

$$\operatorname{Pic}\bigl(X_q(\mathbf{F}_{q^2})\bigr) \big/ \operatorname{Pic}\bigl(X_q(\mathbf{F}_q)\bigr) = X_q(\mathbf{F}_{q^2})/X_q(\mathbf{F}_q) = X_q(\mathbf{F}_{q^2})$$

(the final equality since X_q is assumed to have class number one) in $\Gamma \backslash \mathscr{T}$, and these are exactly the vertices c_0 and t_j (the latter with multiplicity two, since $\pm Q \in E(\mathbf{F}_{q^2})$ map to the same vertex). The integral is zero exactly if $C_0 = -2\tau$. Applying the Hecke operator Φ_∞ to this equation (cf. (2)) gives $C_1 = -2Z_1$, then applying Φ_∞ again gives $C_2 = -(q+1)Z_0$. The rest follows by induction. If we apply Φ_∞ to Eqs. (5) for $k \geqslant 2$, we find by induction for k even that

$$C_{k+1} = \lambda_k C_1 + \bigl(\lambda_k q + \mu_k q(q+1) - q\nu_{k-1}\bigr) Z_1$$

and for k odd that

$$C_{k+1} = (\nu_k - q\lambda_{k-1}) Z_0 + (\nu_k - q\mu_{k-1})\tau. \qquad \square$$

Lemma 3.2. *The space $\mathscr{S}$ from (4) has a basis of $q+2$ Φ_∞-eigenforms, of which exactly $q-1$ are cusp forms with eigenvalue zero and support in the set of vertices $\{t_j\}$, and three are non-cuspidal forms with respective eigenvalues $0, q, -q$.*

Proof. With $\tau = T_1 + \cdots + T_q$, the function

$$f = [T_1, \ldots, T_q \mid Z_0, Z_1 \mid -2\tau, C_1, C_2, \ldots]$$

is a Φ_∞-eigenform with eigenvalue λ if and only if

$$\lambda T_j = (q+1)Z_1, \qquad \lambda Z_1 = \tau + Z_0, \qquad \lambda Z_0 = qZ_1 + C_1, \qquad \lambda(-2\tau) = (q+1)C_1, \quad \text{etc.}$$

We consider two cases:

(a) if $\lambda = 0$, we find q forms

$$f_k = [0, \ldots, 0, 1, 0, \ldots, 0 \mid 0, -1 \mid -q, \ldots]$$

with $T_j = 1 \Leftrightarrow j = k$.

(b) if $\lambda \neq 0$, we find $\lambda = \pm q$ with eigenforms

$$f_\pm = \bigl[q+1, \ldots, q+1 \bigm| -q, \pm q \bigm| -2q(q+1), \mp 2q^2, \ldots\bigr].$$

Since we found $q+2$ eigenforms, they span $\mathscr{S}$. From the fact that a cusp form satisfies $f(c_i) = 0$ for all i sufficiently large (cf. Harder [4, Theorem 1.2.1]), one easily deduces that a basis of cusp forms in $\mathscr{S}$ consists of $f_k - f_1$ for $k = 2, \ldots, q$. $\square$

Corollary 3.3. *The Riemann hypothesis is true for ζ_{F_q} $(q = 2, 3, 4)$.*

Proof. From Lemma 3.2, we deduce that the only possible Φ_∞-eigenvalue of a toroidal Eisenstein series is $\pm q$ or 0, but on the other hand, from Proposition 1.1, we know this eigenvalue is $q^s + q^{1-s}$ where $\zeta_{F_q}(s) = 0$. We deduce easily that s has real part $1/2$. □

Remark 3.4. One may verify that this proves the Riemann hypothesis for the fields F_q without actually computing ζ_{F_q}: it only uses the expression for the zeta function by a Tate integral. Using a sledgehammer to crack a nut, one may equally deduce from Theorem 2.1 that $\zeta_{\mathbf{P}_1}$ does not have any zeros. At least the above corollary shows how enough knowledge about the space of toroidal automorphic forms does allow one to deduce a Riemann hypothesis, in line with a hope expressed by Zagier [15].

Theorem 3.5. *For $q = 2, 3, 4$, T_{F_q} is one-dimensional, spanned by the Eisenstein series of weight s equal to a zero of the zeta function ζ_{F_q} of F_q.*

Remark 3.6. Note that the functional equation for $E(s)$ implies that $E(s)$ and $E(1-s)$ are linearly dependent, so it does not matter which zero of ζ_{F_q} is taken.

Proof. By Lemma 3.2, T_{F_q} is a Φ_∞-stable subspace of the finite dimensional space $\mathscr{S}$, and Φ_∞ is diagonalizable on $\mathscr{S}$. By linear algebra, the restriction of Φ_∞ is also diagonalizable on T_{F_q} with a subset of the given eigenvalues, hence T_{F_q} is a subspace of the space of automorphic forms for the corresponding eigenvalues of Φ_∞. By [8, Theorem 7.1], it can therefore be split into a direct sum of a space of Eisenstein series $\mathscr{E}$, a space of residues of Eisenstein series $\mathscr{R}$, and a space of cusp forms $\mathscr{C}$ (note that in the slightly different notations of [8], “residues of Eisenstein series” are called “Eisenstein series,” too). We treat these spaces separately.

$\mathscr{E}$: By Proposition 1.1, $\mathsf{T}_{F_q}(F_q^{(2)})$ contains exactly two Eisenstein series, one corresponding to a zero s_0 of ζ_{F_q}, and one corresponding to a zero s_1 of

$$L_q(s) := \zeta_{F_q^{(2)}}(s)/\zeta_{F_q}(s).$$

Now consider the torus $\widetilde{T}$ corresponding to the quadratic extension $E_q = F_q(z)/F_q$ of genus two defined by $x = z(z+1)$. Set

$$\widetilde{L}_q(s) := \zeta_{E_q}(s)/\zeta_{F_q}(s)$$

and $T = q^{-s}$. One computes immediately that $L_q = qT^2 + qT + 1$ but

$$\widetilde{L}_2 = 2T^2 + 1, \qquad \widetilde{L}_3 = 3T^2 + T + 1 \quad \text{and} \quad \widetilde{L}_4 = 4T^2 + 1.$$

Since L_q and $\widetilde{L}_q$ have no common zero, the $\widetilde{T}$-integral of the Eisenstein series of weight s_1 is non-zero, and hence it does not belong to T_{F_q}. Hence $\mathscr{E}$ is as expected.

$\mathscr{R}$: Elements in $\mathscr{R}$ have Φ_∞-eigenvalues $\neq 0, \pm q$, so cannot even occur in $\mathscr{S}$: since the class number of F_q is one, $\mathscr{R}$ is spanned by the two forms

$$r_\pm := [1, \ldots, 1 \mid \pm 1, 1 \mid 1, \pm 1, 1, \pm 1, \ldots]$$

with $r(c_i) = (\pm 1)^i$, and this is a Φ_∞-eigenform with eigenvalue $\pm(q+1)$. (In general, the space is spanned by elements of the form $\chi \circ \det$ with χ a class group character, cf. [3, p. 174].)

$\mathscr{C}$: By multiplicity one, $\mathscr{C}$ has a basis of simultaneous $\mathscr{H}$-eigenforms. From Lemma 3.2, we know that potential cusp forms in T_{F_q} have support in the set of vertices $\{t_i\}$. To prove that $\mathscr{C} = \{0\}$, the following therefore suffices:

Proposition 3.7. *The only cusp form which is a simultaneous eigenform for the Hecke algebra $\mathscr{H}$ and has support in $\{\mathfrak{t}_i\}$ is $f = 0$.*

Proof. Let f denote such a form. Fix a vertex $\mathfrak{t} \in \{\mathfrak{t}_i\}$. It corresponds to a point $P = (\ell, *)$ on $X_q(\mathbf{F}_{q^2})$, which is a place of degree two of $\mathbf{F}_q(X_q)$. Let Φ_P denote the corresponding Hecke operator. We claim that

Lemma 3.8. $\Phi_P(c_0) = (q+1)c_2 + q(q-1)\mathfrak{t}$.

Given this claim, we finish the proof as follows: we assume that f is a Φ_P-eigenform with eigenvalue λ_P. Then

$$0 = \lambda_P f(c_0) = \Phi_P f(c_0) = q(q-1) f(\mathfrak{t}) + (q+1) f(c_2) = q(q-1) f(\mathfrak{t})$$

since $f(c_2) = 0$, hence $f(\mathfrak{t}) = 0$ for all $\mathfrak{t}$. □

Proof of Lemma 3.8. As in [3, 3.7], the Hecke operator Φ_P maps the identity matrix (= the vertex c_0) to the set of vertices corresponding to the matrices $m_\infty := \operatorname{diag}(\pi, 1)$ and $m_b := \left(\begin{smallmatrix} 1 & b \\ 0 & \pi \end{smallmatrix}\right)$, where $\pi = x - \ell$ is a local uniformizer at P and b runs through the residue field at P, which is

$$\mathbf{F}_q[X_q]/(x-\ell) = \mathbf{F}_q[y]/F(\ell, y) \cong \mathbf{F}_{q^2}$$

if $F(x, y) = 0$ is the defining equation for X_q. Hence we can represent every such b as $b = b_0 + b_1 y$. We now reduce these matrices to a standard form in $\Gamma \backslash \mathscr{T}$ from [12, §2]. By right multiplication with $[\![1, -b_0]\!]$, we are reduced to considering only $b = b_1 y$.

If $b_1 = 0$, then the matrix is $m_b = \operatorname{diag}(1, \pi) \sim \operatorname{diag}(\pi^{-1}, 1)$. Recall that $t = x/y$ is a uniformizer at ∞, so $x - \ell = t^{-2} \cdot A$ for some $A \in \mathbf{F}_q[\![t]\!]^*$. Hence right multiplication by $\operatorname{diag}(A^{-1}, 1)$ gives that this matrix reduces to c_2. The same is true for m_∞.

On the other hand, if $b_1 \neq 0$, multiplication on the left by $\operatorname{diag}(1, b_1)$ and on the right by $\operatorname{diag}(1, b_1^{-1})$ reduces us to considering m_y. By multiplication on the right with

$$\operatorname{diag}\big((x-\ell)^{-1} \cdot A, (x-\ell)^{-1}\big),$$

we get $m_y \sim [\![t^2, y/(x-\ell)]\!]$. Now note that

$$\frac{y}{x-\ell} = \frac{y}{x} \cdot \left(1 + \frac{\ell}{x} + \left(\frac{\ell}{x}\right)^2 + \cdots\right) = t^{-1} + \ell t + \beta(t) t^2$$

for some $\beta \in \mathbf{F}_q[\![t]\!]$. Hence right multiplication with $[\![1, -\beta]\!]$ gives $m_y \sim [\![t^2, t^{-1} + \ell t]\!]$, and this is exactly the vertex $\mathfrak{t}$. □

Remark 3.9. Using different methods, more akin the geometrical Langlands programme, the second author has proven the following results [9]. For a general function field F of genus g and class number h, one may show that T_F is finite dimensional. Its Eisenstein part is of dimension at least $h(g-1)+1$ (in the above examples, the dimension is shown to be exactly equal to this number). For a general function field, residues of Eisenstein series are never toroidal. For general elliptic function fields over a field with q elements with q odd or number of rational places different from $q+1$, there are no toroidal cusp forms. For a general function field, the analogue of a result of Waldspurger [13, Proposition 7] implies that the cusp forms in T_F are exactly those having vanishing central L-value.

However, in [9], there is no method to in general compute the Hecke eigenvalues (as in Lemma 3.2), nor to conclude that the Riemann hypothesis holds for a general function field (as in Corollary 3.3), nor to compute the exact space of toroidal forms (as in Theorem 3.5).

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