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Exact matrix formula for the unmixed resultant in three variables

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Abstract

We give the first exact determinantal formula for the resultant of an unmixed sparse system of four Laurent polynomials in three variables with arbitrary support. This follows earlier work by the author on exact formulas for bivariate systems and also uses the exterior algebra techniques of Eisenbud and Schreyer. Along the way we will prove an interesting new vanishing theorem for the sheaf cohomology of divisors on toric varieties. This will also allow us to describe some supports in four or more variables for which determinantal formulas for the resultant exist.
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1. Introduction

The resultant of $n+1$ polynomials $f_1, \ldots, f_{n+1}$ in n variables is a single polynomial in the coefficients of the f_i which vanishes when the f_i have a common root. The resultant can therefore be used to eliminate n variables from $n+1$ equations. Originally resultants were defined for generic polynomials of fixed total degrees. More recently a *sparse resultant* has been defined which exploits the monomial structure of the given polynomials. The foundational work was laid by Kapranov et al. [11]. Sparse resultants are discussed in depth in the book [9].

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Formally, let $f_1, f_2, \ldots, f_{n+1} \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ be polynomials with the same Newton polytope Q. Let $A = Q \cap \mathbb{Z}^n$. We will assume that A affinely generates $\mathbb{Z}^n$.

We can write

$$f_i = \sum_{\alpha \in A} C_{i\alpha} x^{\alpha}.$$

We will treat the coefficients $C_{i\alpha}$ as independent variables throughout.

Definition 1. The *A-resultant* $\mathrm{res}_A(f_1, \ldots, f_{n+1})$ is the irreducible polynomial in the ring $\mathbb{Z}[C_{i\alpha}]$, unique up to sign, which vanishes whenever $f_1, \ldots, f_n$ have a common root in $(\mathbb{C}^*)^n$.

The problem of finding explicit formulas for resultants, and their cousins the discriminants, dates back to the 19th century with the work of Cayley, Sylvester, Bézout and others. With the recent increase in computing power there has been a renewed interest in computing resultants and new applications in fields such as computer graphics, machine vision, robotic inverse kinematics, and molecular structure [7,13,14].

Even in very small examples, the resultant can have millions of terms. Therefore most authors have looked for a more compact representation. A determinantal formula, following the classical formulas of Sylvester and Bézout, writes the resultant as the determinant of a matrix whose entries are easily computable polynomials of low degree. In the dense case, when all the polynomials have the same degree, determinantal formulas are known when $n = 1, 2$, or 3 and for a very few cases in more variables. In the sparse case, $n = 1$ is the same as the dense case and there are the classical Sylvester and Bézout formulas, determinantal formulas for $n = 2$ were found by the author in [12]. This paper gives a new exact formula when $n = 3$.

Given any lattice polytope Q, let $D_1, \ldots, D_s$ denote the facets (codimension 1 faces) of Q. Given a subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, s\}$, let $D_I = \{D_{i_1}, \ldots, D_{i_k}\}$ be the corresponding subset of facets. Let $\overline{D_I}$ be the set of facets of Q not in D_I. $Q - D_I$ will refer to the set of all points in Q but not on any facet on D_I. More generally, $kQ - D_I$ is the set of all points in the Minkowski sum of k copies of Q but not on any of the facets corresponding to D_I. Finally, given a set $S \subset \mathbb{R}^n$ let $l(S) = S \cap \mathbb{Z}^n$ be the set of *lattice* points in S. The main theorem is as follows:

Theorem 2. *Let* $f_1, f_2, f_3, f_4 \in \mathbb{C}[x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1}]$ *be four polynomials with common Newton polytope* $Q \subset \mathbb{R}^3$. *Suppose* $A = Q \cap \mathbb{Z}^3$ *affinely generates* $\mathbb{Z}^3$. *Pick a proper collection of the facets of Q,* $D_I = (D_{i_1}, \ldots, D_{i_k})$, *such that the union of the facets in* D_I *is homeomorphic to a disk. There is a determinantal formula for the resultant* $\mathrm{res}_A(f_1, f_2, f_3, f_4)$ *of the following block form*:

$$\begin{pmatrix} B & L \\ \tilde{L} & 0 \end{pmatrix}.$$

The rows of B and L are indexed by the points in $l(2Q - D_I)$. *The columns of B and* $\tilde{L}$ *are indexed by* $l(2Q - \overline{D_I})$. *The rows of* $\tilde{L}$ *are indexed by four copies of* $l(Q - \overline{D_I})$, *and the columns of L are indexed by four copies of* $l(Q - D_I)$.

The entries of B are of Bézout type and are polynomials of degree 4 *in the coefficients* $C_{i\alpha}$. *The entries of L and* $\tilde{L}$ *are of Sylvester type, thus linear in the* $C_{i\alpha}$.

We will see how the entries of B can be filled in using a free resolution over an exterior algebra. Both the proof and the construction are based on techniques developed by Eisenbud and Schreyer, which have been adapted for sparse resultants (toric varieties).

The paper is organized as follows. Section 2 discusses the background on toric varieties, exterior algebras, and the Tate resolution of Eisenbud–Schreyer. Section 3 uses these techniques along with some sheaf cohomology vanishing results to prove Theorem 2. In particular, Section 3 contains a new vanishing result for certain divisors on any projective toric variety. Section 4 shows how to actually construct the resultant matrix and gives some examples. Finally, Section 5 gives a different combinatorial perspective on the resultant matrix in terms of the Ehrhart polynomial and analyzes the size of the resultant matrix.

2. Notation and background

2.1. Toric varieties and Chow forms

Given a polytope $Q \subset \mathbb{R}^n$ and associated $A = Q \cap \mathbb{Z}^n$, let $N = |A|$. The *toric variety* $X_A \subset \mathbb{P}^{N-1}$ is defined as the Zarioki closure of the set $(x^{\alpha_1} : \cdots : x^{\alpha_N})$ where α_i ranges over the elements of A and $x \in (\mathbb{C}^*)^n$. It has dimension n. In terms of X_A, the polynomials f_i are hyperplane sections. The system $(f_1, f_2, \ldots, f_{n+1})$ defines a codimension $n+1$ plane. The set of all codimension $n+1$ planes meeting X_A defines a hypersurface in the Grasmannian $G(n+1, N)$. The A-resultant is identified with the equation of this hypersurface, also called the *Chow form* of X_A.

Proposition 3. *The resultant* $\mathrm{res}_A(f_1, \ldots, f_{n+1}) = 0$ *if and only if the hyperplanes* f_i *have a common intersection on* X_A.

Let Σ_Q be the normal fan of Q with $\Sigma_Q(1) = \{\eta_1, \ldots, \eta_s\}$ the inner normals to the facets. There is an associated normal toric variety X_{Σ_Q} (see [8, Chapter1]). Assuming A affinely spans $\mathbb{Z}^n$, X_{Σ_Q} is the normalization of X_A. This is essentially Proposition 4.9 in Chapter 5 of [9]. The results below are standard and can be found in [8].

Proposition 4. *The* η_i *are in 1-1 correspondence with the torus invariant prime Weil divisors on* X_{Σ_Q}. *Let* D_i *denote the divisor corresponding to* η_i, *and* $\mathcal{O}(D_i)$ *the corresponding rank 1 reflexive sheaf on* X_{Σ_Q}.

In the Introduction and in the statement of Theorem 2, D_i denoted a facet of Q. This facet will be identified with the corresponding prime divisor, also denoted D_i, as defined above.

Given a general divisor $D = \sum a_i D_i$ on X_{Σ_Q}, we will denote by $\mathcal{O}_{X_A}(D)$ or, when there is no confusion just $\mathcal{O}(D)$, the push-forward of the sheaf $\mathcal{O}_{X_{\Sigma_Q}}(D)$ onto X_A via the normalization map. The linear equivalence classes of divisors are computed by the following exact sequence:

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\text{div}} \mathbb{Z}^s \xrightarrow{[\cdot]} Cl_X \longrightarrow 0,$$

where $\text{div}(u) = (\langle u, \eta_1 \rangle, \ldots, \langle u, \eta_s, \rangle)$ and Cl_X is the cokernel of this map. Given a divisor $D \in \mathbb{Z}^s$ we let $[D]$ be the image of D in Cl_X.

There is a nice combinatorial description of the global sections $H^0(X_A, \mathcal{O}(D))$. A divisor $D = \sum a_i D_i$ determines a convex polytope $P_D = \{m \in \mathbb{R}^n : \langle m, \eta_i \rangle \geqslant -a_i\}$. For any polytope P, let S_P denote the $\mathbb{C}$ vector space with basis the lattice points in P, i.e., $S_P = \mathbb{C}\{P \cap \mathbb{Z}^n\}$.

Proposition 5.

$$H^0(X_A, \mathcal{O}(D)) \cong S_{P_D}.$$

If we start with a polytope Q, then it determines an ample divisor on the toric variety X_{Σ_Q}. Write

$$Q = \{m \in \mathbb{R}^n \ \langle m, \eta_i \rangle \geqslant -a_i,\ i = 1, \ldots, s\}$$

for some $a_1, \ldots, a_s \in \mathbb{Z}$. Let $D_Q = \sum a_i D_i$ be the corresponding divisor. If X_A is the (possibly non-normal) toric variety above defined by the lattice points in Q, then the push-forward of D_Q yields the very ample divisor corresponding to the embedding of X_A into $\mathbb{P}^{N-1}$. On X_{Σ_Q}, D_Q will always be ample but not necessarily very ample. One final useful fact is that the sheaf $\mathcal{O}(-\sum_{i=1}^{s} D_i)$ is the canonical sheaf on the Cohen–Macaulay variety X_{Σ_Q}. This will be needed when we apply Serre duality below.

2.2. *Exterior algebra and the Tate resolution*

Eisenbud and Schreyer [6] have developed some powerful machinery to compute Chow forms using resolutions over an exterior algebra. Suppose $X \subset \mathbb{P}^{N-1}$ is a variety of dimension n. We are interested in the finding the Chow form of X.

The ambient projective space $\mathbf{P} = \mathbb{P}^{N-1}$ has the graded coordinate ring $R = \mathbb{C}[X_1, \ldots, X_N]$. If we let W be the $\mathbb{C}$ vector space spanned by the X_i, (identified with the degree 1 part of R), then $\mathbf{P}$ is the projectivization $\mathbb{P}(W)$. The ring R can also be identified with the symmetric algebra $\text{Sym}(W)$.

Now let $V = W^*$, the dual vector space, with a corresponding dual basis $e_1, \ldots e_N$. We will consider the *exterior algebra* $E = \bigwedge V$, also a graded algebra where the generators e_i have degree -1. We will use the standard notation $E(k)$ to refer the rank 1 free E-module generated in degree $-k$.

For any coherent sheaf $\mathcal{F}$ on $\mathbf{P}$, there is an associated exact complex of graded free E-modules, called the *Tate resolution*, denoted $T(\mathcal{F})$. The terms of $T(\mathcal{F})$ can be written in terms of the vector spaces of sheaf cohomology of twists of $\mathcal{F}$. Namely, we have

$$T^e(\mathcal{F}) = \oplus[H^j(\mathcal{F}(e-j)) \bigotimes_{\mathbb{C}} E(j-e)]. \tag{1}$$

Here e is any positive integer. In particular, this complex is infinite in both directions, although the terms themselves are finite dimensional free E-modules.

Now suppose that $\mathscr{F}$ is supported on X. Recall that the Chow form of X, also called the X-resultant and denoted res_X, is the defining equation of the set of codimension $n+1$-planes meeting X. Such a plane is specified by a $n+1$ dimensional subspace $W_f = \mathbb{C}\{f_1, \ldots, f_{n+1}\} \subset W$. Let $\mathbf{G}$ be the Grasmannian of codimension $n+1$-planes on $\mathbf{P}$. Let $\mathscr{T}$ be the tautological bundle on $\mathbf{G}$, that is to say the fiber at the point corresponding to f is just W_f. There is a functor, U_{n+1} from free E-modules to vector bundles on $\mathbf{G}$ which sends $E(p)$ to $\wedge^p \mathscr{T}$.

This functor when applied to the Tate resolution gives a *finite* complex of vector bundles on $\mathbf{G}$, $U_{n+1}(T(\mathscr{F}))$ that is fiberwise a finite complex of $\mathbb{C}$ vector spaces.

Theorem 6 [6].

$$\det(U_{n+1}(T(\mathscr{F}))) = \mathrm{res}_X^{\mathrm{rank}(\mathscr{F})}.$$

This is a determinant of a complex, which in general can be computed as a certain alternating product of determinants. We will be most interested in the special case where the complex in question has only two terms

$$0 \longrightarrow A \xrightarrow{\Psi} B \longrightarrow 0.$$

In this case, the determinant of the complex is just the determinant of the matrix of the map Ψ. Sheaves whose Tate resolutions yield such two term complexes for the Chow form are called *weakly Ulrich*. Determinantal formulas for the resultant correspond to finding a weakly Ulrich sheaf of rank 1 on the toric variety X_A.

Let $M = \bigoplus_{i \in \mathbb{N}} H^0(\mathscr{F}(i))$. This is a graded R-module. The *linear strand* of the Tate resolution is the subcomplex defined by the terms $M_e \otimes E(-e)$. The maps in the linear strand are completely canonical:

$$\phi_e \ : \ M_e \otimes E(-e) \to M_{e+1} \otimes E(-e-1),$$
$$m \otimes 1 \mapsto \sum_{i=1}^{N} m \cdot X_i \otimes e_i.$$

An extremely important fact is that for large enough e, anything larger than the *regularity* of M, all the higher cohomology vanishes and only the linear strand remains. For a definition and discussion on regularity see [1].

This suggests an algorithm to compute terms of the Tate resolution:

(1) Given $\mathscr{F}$ compute M.
(2) Pick $e = \mathrm{reg}(M) + 1$ and compute ϕ_e.
(3) Start computing a free resolution of ϕ_e over E.

Note: As a consequence we can read off the cohomology of twists of $\mathscr{F}$ as graded pieces of this resolution. As Eisenbud et al. [4] point out, in many cases this is the most efficient known way to compute sheaf cohomology.

3. Proof of Theorem 2

Suppose we are given f_1, f_2, f_3, f_4 with common Newton polytope $Q \subset \mathbb{R}^3$. To apply the exterior algebra construction we take $W = S_Q$, the $\mathbb{C}$ vector space with basis the lattice points in Q, and $V = S_Q^*$. The corresponding projective space is $\mathbf{P} = \mathbb{P}(W) \cong \mathbf{P}^{N-1}$, and the exterior algebra is $E = \bigwedge V$. Let $y_1, \ldots, y_N$ denote the basis of S_Q and $e_1, \ldots, e_N$ the corresponding dual basis of E.

We now show how Theorem 2 reduces to showing that an appropriate push-forward of a Weil divisor class onto X_A is a weakly Ulrich sheaf. This will require proving that certain cohomology groups vanish.

Let $I \subset \{1, \ldots, s\}$, thought of as a subset of the facets. Let $D_I = \sum_{i\in I} D_i$ and $\overline{D}_I = \sum_{i\notin I} D_i$ be formal sums of the corresponding divisors. The divisors we will be interested in are of the form $kD_Q - D_I$ where $k \in \mathbb{Z}$.

As in the statement of Theorem 2, we pick a proper subset $I \subset \{1, \ldots, s\}$ such that the union of the facets in D_I is homeomorphic to a disk. In Section 4, while describing the algorithmic construction of the matrix of 2, we also show how to pick such D_I as a partial shelling of the facets of Q. We will consider the sheaf $\mathscr{F} = \mathscr{O}(2D_Q - D_I)$. As before this is a divisor on the normal toric variety X_{Σ_Q} pushed forward onto X_A. The main fact we will need is the following cohomology vanishing theorem. For simplicity, and when there is no confusion, we will often write $H^i(\mathscr{O}(D))$ instead of $H^i(X_A, \mathscr{O}(D))$.

Theorem 7. *Let $X = X_A$ be a projective toric variety of dimension n arising from a polytope Q with corresponding ample divisor D_Q. Let D_I be a proper subset of the facets such that the unions of the facets in D_I is a topological manifold with no reduced homology. Then*

$$\begin{aligned} &H^0(\mathscr{O}(kD_Q - D_I) \cong S_{kQ-D_I}, && \\ &H^i(\mathscr{O}(kD_Q - D_I) \cong 0, && i = 1, \ldots, n-1 \\ &H^n(\mathscr{O}(kD_Q - D_I) \cong S^*_{-kQ-\overline{D}_I}, && \end{aligned}$$

for all $k \in \mathbb{Z}$.

In the case Q is a 3-polytope the only 2-manifold with no reduced homology is the disk. The proof is postponed until Section 3.1. But note that plugging this into the description of the Tate resolution using $\mathscr{F}(k) = \mathscr{O}((k+2)D - D_I)$ gives us:

Corollary 8. *The Tate resolution of $\mathscr{F}$ has terms*

$$\begin{aligned} &T^e(\mathscr{F}) \cong S^*_{(1-e)Q-\overline{D_I}} \otimes E(3-e) \quad \text{for } e < -1, \\ &T^{-1}(\mathscr{F}) \cong S^*_{2Q-\overline{D_I}} \otimes E(4) \oplus S_{Q-D_I} \otimes E(1), \\ &T^0(\mathscr{F}) \cong S^*_{Q-\overline{D_I}} \otimes E(3) \oplus S_{2Q-D_I} \otimes E, \\ &T^e(\mathscr{F}) \cong S_{(e+2)Q-D_I} \otimes E(-e) \quad \text{for } e > 0. \end{aligned}$$

Finally, to get the Chow form we need to apply the functor U_4 which sends $E(p)$ to $\wedge^p \mathscr{T}$. But, $\mathscr{T}$ is a vector bundle of rank 4, so by the above proposition only $T^{-1}(\mathscr{F})$ and $T^0(\mathscr{F})$ survive the application of U_4. Therefore, $\mathscr{F}$ is weakly Ulrich and the matrix of the resulting two term complex is exactly the matrix of Theorem 2 which we restate here in the language of this section.

Corollary 9. *The resultant of* $f_1, \ldots, f_4$ *is the determinant of the two term complex below*:

$$
\begin{array}{ccccccc}
 & & S^*_{2Q-\overline{D_I}} \otimes \bigwedge^4 \mathcal{T} & \xrightarrow{\widetilde{L}} & S^*_{Q-\overline{D_I}} \otimes \bigwedge^3 \mathcal{T} & & \\
0 & \longrightarrow & \oplus & \searrow^{B} & \oplus & \longrightarrow & 0 \\
 & & S_{Q-D_I} \otimes \bigwedge^1 \mathcal{T} & \xrightarrow{L} & S_{2Q-D_I} \otimes \bigwedge^0 \mathcal{T} & &
\end{array}
$$

Theorem 7 can be used to give exact determinantal formulas for resultants in dimension 4 and above for some cases of polytopes.

Theorem 10. *Let* $Q \subset \mathbb{R}^4$ *be a polytope such that* $A = Q \cap \mathbb{Z}^4$ *affinely spans* $\mathbb{Z}^4$. *There is a determinantal formula for* Res_A *if* Q *has no interior points and there is some facet* D_i *of* Q *with no relative interior points*.

Proof. Take $\mathscr{F} = \mathscr{O}(2D_Q - D_i)$. Going through the Tate resolution machinery using our vanishing theorem, we get a three term complex whose left most term is $S^*_{Q-\overline{D_i}}$. The points here are exactly the interior points of Q together with the relative interior points of D_i. So under the given hypothesis, this term is zero and we have a two term complex. □

In the case of $X_Q = \mathbb{P}^4$ we recover the formulas for resultants of 5 homogeneous polynomials of degree less than or equal to 3. We can make a similar statement in dimension 5 and higher but the hypotheses get stricter.

Theorem 11. *Let* $Q \subset \mathbb{R}^n$ *and* $A = Q \cap \mathbb{Z}^n$ *affinely spans* $\mathbb{Z}^n$ *for* $n \geqslant 5$. *Let* $k_1 = \lfloor \frac{n+1}{2} \rfloor - 2$ *and* $k_2 = \lceil \frac{n+1}{2} \rceil - 2$. *There is a determinantal formula for* res_A *if there is a collection of facets* D_I *of* Q *forming a manifold without homology such that* $k_1 Q$ *and* $k_2 Q$ *have no interior lattice points*, D_I *has no relative interior lattice points in* $k_2 Q$ *and* $\overline{D_I}$ *has no relative interior points in* $k_1 Q$.

Proof. Take $\mathscr{F} = \mathscr{O}(\lfloor \frac{n+1}{2} \rfloor Q - D_I)$. The result follows from the Theorem 7 and counting lattice points. □

For example when $n = 5$ and Q is the coordinate simplex we recover the determinantal formula for 6 homogeneous polynomials of degree 2. For $n = 6$ or greater we only get a resultant formula for $d = 1$. It would be interesting to classify all polytopes of arbitrary shape satisfying these conditions. It may be that there is only be a finite list for $n = 6$ or greater.

We do not claim that these theorems generate all determinantal resultant formulas. For example, by Proposition 2.6 of [6] if Q_1 and Q_2 (of any dimension) have resultant formulas with sheaves $\mathscr{F}_1$ and $\mathscr{F}_2$, then $\mathscr{F}_1 \otimes \mathscr{F}_2$ will give a determinantal formula for $Q_1 \times Q_2$. In any case polytopes satisfying Theorem 11 together with all products of such polynomials is at least a start towards classifying exact resultant formulas in higher dimension.

3.1. Cohomology vanishing

In this section we will prove Theorem 7. So we will need to compute the cohomology of $\mathcal{O}(kD_Q - D_I)$ for all $k \in \mathbb{Z}$. We already know the global sections $H^0(X_A, \mathcal{O}(\cdot))$. The next proposition shows how to compute the top cohomology $H^n(X_A, \mathcal{O}(\cdot))$.

Proposition 12. *Let $Q \subset \mathbb{R}^n$ be a lattice polytope of dimension n with facets $D_1, \ldots, D_s$ and $A = Q \cap \mathbb{Z}^n$ affinely generating $\mathbb{Z}^n$. Let X_A be the corresponding toric variety, and $D = \sum a_i D_i$ a Weil divisor on the normalization X_{Σ_Q} which pushes forward as before to a sheaf on X_A. Then*

$$H^n(X_A, \mathcal{O}(D)) \cong H^0\left(X_A, \mathcal{O}\left(-D - \sum_{i=1}^{s} D_i\right)\right)^* .$$

Proof. As per our earlier discussion all of the cohomology can be computed on the associated normal toric variety $X = X_{\Sigma_Q}$. This is Cohen–Macaulay with dualizing sheaf $\omega_X = \mathcal{O}(-\sum_{i=1}^s D_i)$. If D were Cartier the statement would follow immediately from Serre duality. In the general Weil divisor case we have to be a little bit more careful. So we compute

$$\begin{aligned}
H^n(X, \mathcal{O}(D))^* &\cong \mathrm{Hom}(\mathcal{O}(D), \omega_X)\\
&\cong \mathrm{Hom}\left(\mathcal{O}(D), \mathscr{H}om\left(\mathcal{O}\left(\sum_{i=1}^{s} D_i\right), \mathcal{O}_X\right)\right)\\
&\cong \mathrm{Hom}\left(\mathcal{O}(D) \otimes \mathcal{O}\left(\sum_{i=1}^{s} D_i\right), \mathcal{O}_X\right)\\
&\cong \mathrm{Hom}\left(\mathcal{O}_X, \left(\mathcal{O}(D) \otimes \mathcal{O}\left(\sum_{i=1}^{s} D_i\right)\right)^*\right)\\
&\cong H^0\left(X, \left(\mathcal{O}(D) \otimes \mathcal{O}\left(\sum_{i=1}^{s} D_i\right)\right)^*\right).
\end{aligned}$$

The first isomorphism is Serre duality. The second uses that Weil divisors are reflexive sheaves and $\mathscr{H}om(\mathcal{O}(D), \mathcal{O}_X) \cong \mathcal{O}(D)^* \cong \mathcal{O}(-D)$. The third and fourth steps are by the adjointness of $\mathscr{H}om$ and $\otimes$, and the last step is the definition of global sections. Finally, by Corollary 2.1 in [10], the dual of any coherent sheaf is reflexive. So,

$$\left(\mathcal{O}(D) \otimes \mathcal{O}\left(\sum_{i=1}^{s} D_I\right)\right)^* \cong \left(\mathcal{O}(D) \otimes \mathcal{O}\left(\sum_{i=1}^{s} D_I\right)\right)^{***} .$$

However $(\mathcal{O}(D) \otimes \mathcal{O}(E))^{**}$ is always isomorphic to $\mathcal{O}(D+E)$ even if D and E are not locally free. Hence we get

$$\left(\mathcal{O}(D) \otimes \mathcal{O}\left(\sum_{i=1}^{s} D_I\right)\right)^{*} \cong \mathcal{O}\left(-D-\sum_{i=1}^{s} D_i\right),$$

as desired. □

It remains to show that the "middle cohomology" always vanishes under the given conditions. The proof is broken up into three parts, showing $H^i(\mathcal{O}(kD_Q - D_I)) = 0$ when $k > 0$, $k = 0$, and $k < 0$. The first two follow fairly easily from results of Mustaţă [5,15]. The case $k < 0$ requires more work and will be quite interesting in its own right.

Proposition 13. *Let Q be a polytope and X_A the toric variety as in Proposition 12. Let D_I be the sum of any collection of facets as before. $H^i(\mathcal{O}(kD_Q - D_I)) = 0$ for all $i > 0$ and all $k > 0$.*

Proof. Since kD_Q is ample, this is just [15, Corollary 2.5(iii)]. □

In general, the cohomology of all divisors can be grouped into a single object $H^i_*(\mathcal{O}_X)$ which has a $\mathbb{Z}^s$ fine grading

$$H^i_*(\mathcal{O}_X) = \bigoplus\nolimits_p H^i_*(\mathcal{O}_X)_p.$$

where $p \in \mathbb{Z}^s$.

The cohomology of a particular divisor class $[D]$ can now be recovered as

$$H^i(\mathcal{O}_X(D)) = \sum_p H^i_*(\mathcal{O}_X)_p$$

where the sum is over all p such that $\left[\sum p_i D_i\right] = [D]$.

The next lemma can be viewed as a reformulation of a result of [15] yielding a topological formula for computing these graded pieces. It shows that in the case of a projective toric variety sheaf cohomology can be computed in terms of the ordinary homology of pure cell complexes.

Lemma 14. *Let $p \in \mathbb{Z}^s$. Let $J = \operatorname{neg}(p) \subset \{1, \ldots, s\}$ be the set of coordinates for which p is strictly negative. Let $|D_J|$ be the topological space consisting of the union of all the facets D_j with $j \in J$ of the polytope Q of X.*

$$H^i_*(\mathcal{O}_X)_p \cong \tilde{H}^{i-1}(|D_J|).$$

The latter is the ordinary reduced cohomology of $|D_J|$.

Proof. Let Y_J be the union of all cones in the fan Σ having all edges in the complement of J. Theorem 2.7 in [5] shows that for $i \geqslant 1$:

$$H^i_*(\mathcal{O}_X)_d \cong \tilde{H}^{i-1}(\mathbb{R}^n \backslash Y_J).$$

The latter is isomorphic to $\tilde{H}^{i-1}(S^{n-1} \backslash S^{n-1} \cap Y_J)$ (excision) which is further isomorphic to $\tilde{H}_{n-i-1}(S^{n-1} \cap Y_J)$ by topological Alexander duality. This is a subcomplex of the boundary complex of a polytope polar dual to Q. The combinatorial Alexander dual is the set of faces of Q whose dual is not in Y_I. But this is precisely all of those faces of Q contained in some facet D_I. The underlying topological space is $|D_I|$. So, by combinatorial Alexander duality:

$$\tilde{H}_{n-i-1}(S^{n-1} \cap Y_J) \cong \tilde{H}^{i-1}(|D_J|).$$

as desired. □

We now tackle the case $k = 0$, the proof of this next proposition was given to me in a personal communication with Mircea Mustaţă.

Proposition 15. *If the union of the collection of facets in D_I is non-empty and homologically trivial, then $H^i(\mathcal{O}(-D_I)) = 0$ for all i. More generally, $H^i(\mathcal{O}(-D_I)) \cong \tilde{H}^{i-1}(|D_I|)$.*

Proof (*Due to Mustaţă*). $H^0(\mathcal{O}(-D_I)) = 0$ as the corresponding polytope is empty. Let p_I be such that $(p_I)_i = -1$ if $i \in I$ and $(p_I)_i = 0$, otherwise. Clearly, $\text{neg}(p_I) = I$ and $\sum (p_I)_i D_i = -D_I$. By Lemma 14, $H^i_*(\mathcal{O}_X)_{p_I} = \tilde{H}^{i-1}(|D_I|)$.

We now show that if q is such that $\left[\sum q_i D_i\right] = [-D_I]$, but $q \neq p_I$, then $H^i_*(\mathcal{O}_X)_q = 0$ for all i. Indeed, by linear equivalence $q = p_I + \text{div}(u)$, for some $u \in \mathbb{Z}^d \neq 0$. Let $J = \text{neg}(q)$. It is clear that

$$J = \{i \,|\, \langle u, \eta_i \rangle < 0 \text{ or } \langle u, \eta_i \rangle = 0 \text{ and } i \in I\}.$$

Now the above implies there is a hyperplane $H \subset \mathbb{R}^s$ which separates the edges of Σ_Q indexed by J and $\overline{J}$. By [5, Proposition 2.6] this forces $H^i_*(\mathcal{O}_X)_q = 0$ for $i \geqslant 1$. □

To complete the proof of Theorem 7 we need to consider the case $k < 0$. This will require a new vanishing theorem which has intrinsic interest. Therefore, we state it in somewhat more generality than necessary.

Theorem 16. *Let X be a projective toric variety of dimension n, and D a nef and big line bundle on X. Let $D_I = \sum_{i \in I} D_i$ be a sum of prime torus invariant divisors. If the union of the facets D_i of Q with $i \in I$ is a topological manifold with boundary then $H^i(X, \mathcal{O}(-D - D_I)) = 0$ for all $0 \leqslant i < n$.*

Proposition 3.3 in [15] states that the fan of X refines the normal fan of P_D and $\mathcal{O}(D)$ is the pull-back of an ample divisor, thus we can reduce to the case that D is ample.

Theorem 7 gives general vanishing conditions for all $k \in \mathbb{Z}$ but the results in this section show that the vanishing theorem can be refined using different hypotheses for different cases of the integer k. When $k > 0$, all higher cohomology vanishes for any subset D_I. When $k = 0$ we need the toplogical space $|D_I|$ to have no reduced homology in which case all cohomology vanishes. Finally for $k < 0$, when $|D_I|$ is a manifold, all cohomology vanishes except at the top.

Proof. By the remark above assume that D is ample. As before we will need to compute $H^i_*(\mathcal{O}_X)_p$ for $\sum p_i D_i$ linearly equivalent to $-D - D_I$. Let p_I be defined as in the proof of Proposition 15. Any p as above is of the form $q - p_I$ where $\sum q_i D_i$ is linearly equivalent to $-D$. Write $D = \sum a_i D_i$, in which case $q_i = \langle u, \eta_i \rangle - a_i$ for some $u \in \mathbb{Z}^n$.

Therefore,

$$\mathrm{neg}(q) = \{i \,|\, \langle u, \eta_i \rangle < a_i\}$$

and

$$\mathrm{neg}(p) = \mathrm{neg}(q) \cup \{i \,|\, \langle u, \eta_i \rangle = a_i \text{ and } i \in I\}$$

Let $J' = \mathrm{neg}(q)$ and $J = \mathrm{neg}(p)$ with $|D_{J'}|$ and $|D_J|$ the corresponding unions of facets. Since D is an ample divisor, $H^i(\mathcal{O}(-D)) = 0$ for $i < n$, derived for example by Proposition 13 and Serre duality. We need to show that under the given hypotheses $H^i(\mathcal{O}(-D-D_I))=0$. We already know by Lemma 14 $\tilde{H}^i(|D_{J'}|) = 0$ for $i < n-1$, but we will need to prove that $\tilde{H}^i(|D_J|) = 0$.

We have three cases for u:

Case 1 $\langle u, \eta_i \rangle < a_i$ *for all* i: Equivalently, $-u \in \mathrm{int}(P_D)$. In this case $|D_J|$ is the entire boundary of P_D which is an $n-1$ sphere and only has reduced homology at the top.

Case 2 $\langle u, \eta_i \rangle \leqslant a_i$ *for all i and* $\langle u, \eta_i \rangle = a_i$ *for some* i: This means that $-u$ is on the boundary of P_D. Since D is ample, P_D has the same normal fan as Q and so has parallel faces to Q. The set of all facets D_j for which $\langle u, \eta_j \rangle = a_j$ cuts out a face f of Q. Moreover, since $-D$ is Cartier there is a corresponding function ψ_{-D} on the fan Σ, defined to be a_i on the rays η_i and extended linearly in each cone. Since the linear functional $\langle u, \cdot \rangle$ agrees with ψ_{-D} on a spanning set of the cone corresponding to f it agrees with ψ_{-D} on all of this cone. Therefore, $\langle u, \eta_i \rangle = a_i$ for all facets D_i containing f and so $|D'_J|$ is the union of all facets of Q not containing f. If f is not a face of a facet in D_I then none of the D_j containing f are part of D_I, in which case $\mathrm{neg}(p) = \mathrm{neg}(q)$ and therefore $H^i(|D_J|) = H^i(|D_{J'}|)$.

Next, assume that f is a face of some facet in D_I. The facets D_I define a cell complex, also denoted D_I, realizing the manifold $|D_I|$. The star $st(f)$ is the union of all of the relatively open faces of D_I that have f as a face and the link $lk(f)$ is $\overline{st(f)} - st(f)$.

The key observation is that $|D_J| = |D_{J'}| \cup \overline{st(f)}$ and $lk(f) = |D_{J'}| \cap \overline{st(f)}$. So, we have a Mayer–Vietoris sequence

$$\cdots \to \tilde{H}^{a-1}(lk(f)) \to \tilde{H}^a(|D_J|) \to \tilde{H}^a(\overline{st(f)}) \oplus \tilde{H}^a(|D_{J'}|) \to \cdots .$$

We know that $\overline{st(f)}$ is contractible (it is star shaped!) and from above $\tilde{H}^a(|D_{J'}|) = 0$ for $a < n-1$. It remains to show that $\tilde{H}^{a-1}(lk(f)) = 0$ for $a < n-1$. This is where we use that $|D_I|$ is a manifold.

Start with the cell complex D_I and perform a stellar subdivision at the face f. This induces a subdivision of D_I, which we call D_I^f, with a new vertex v_f corresponding to the face f. Furthermore the star and link $st(v_f)$ and $lk(v_f)$ in D_I^f are the same as $st(f)$ and $lk(f)$ in D_I. So it now suffices to show that $\tilde{H}^{a-1}(lk(v_f)) = 0$ for $a < n-1$.

Since $|D_I|$ is a manifold with boundary, the local cohomology of $|D_I|$ at v_f, $H^a_{v_f}(|D_I|)$, vanishes for $a \neq n-1$ if v_f is an interior point of $|D_I|$, and for all a if v_f is on the boundary. This local cohomology can also be computed from the triangulation as the relative cohomology $H^i(\overline{st(v_f)}, lk(v_f))$. The long exact sequence in relative cohomology yields:

$$\cdots \to \tilde{H}^{a-1}(st(v_f)) \to \tilde{H}^{a-1}(lk(v_f)) \to H^a(st(v_f), lk(v_f)) \to \cdots .$$

Since $st(v_f)$ is contractible, $H^a(st(v_f), lk(v_f)) \cong \tilde{H}^{a-1}(lk(v_f)) = 0$ for $a < n-1$ as desired.

Case 3 $\langle u, \eta_i \rangle > a_i$ *for some i*: In this case $-u$ is outside the polytope P_D. A point *p* in P_D is *visible* from $-u$ if the straight line from *p* to $-u$ meets P_D first in *p*. It is easy to see that if a visible point *p* is in the relative interior of a face *f* then the whole face is visible and any subface of a visible face is visible. Therefore visibility is a property of whole faces. A face *f* of P_D will be called *degenerate* if $-u$ is in the affine span of *f*. In particular P_D itself is a degenerate face. A face is invisible if it is not visible or degenerate. Any facet containing an invisible face must be invisible or degenerate and if every facet containing some face *f* is degenerate then *f* itself is degenerate. Clearly, a facet *f* is visible if and only if $-u$ is on the opposite side of *f* as P_D. Therefore, $D_{J'}$ is the set of invisible facets. D_J is the union of D'_J with some degenerate facets. So it will suffice to prove the following proposition taking $P = P_D$ and $v = -u$.

Proposition 17. *Let* $P \subset \mathbb{R}^n$ *be a polytope of any dimension. If v is any point in the affine span of P but outside of P, then the union of the invisible facets of P together with any collection of degenerate faces is homologically trivial.*

Proof. We proceed by induction on the number of degenerate faces and the dimension of *P*. If *f* is a degenerate face of *P*, so that v is in the affine span of *f*, we can talk about the visible, invisible, and degenerate faces of *f* regarded as a polytope in its own right. It is immediate from the definitions that a face of *f* is visible (invisible, degenerate) if and only if it visible (invisible, degenerate) as a face of *P*.

To apply the induction we need to show that the intersection of a degenerate face *f* with the union of the invisible facets and some degenerate faces of *P* is precisely the union of the invisible facets of *f* and some degenerate subfaces.

We first consider the intersection of a degenerate face *f* with the union of the invisible facets of *P*. Any invisible facet of *f* is an invisible face of *P* and hence contained in an invisible facet of *P*. For the converse, let *H* be the affine span of *f*. Suppose f' is a face of *f* contained in an invisible facet *F* of *P*. Since *u* is on the same side of *F* as *P*, it is on the same side of the intersection of *F* and *H* as *f*. In particular there must be some facet of *f* containing f' invisible from *u*. Hence, the union of the invisible facets of *P* intersects *f* precisely in the union of its invisible facets.

Next let *f* be the intersection of two degenerate faces. Let *H* be the intersection of the corresponding two affine spans. So *H* contains both v and *f* and moreover $H \cap P = f$. Let H' be the affine span of *f*, a subspace of *H*. Each facet of *P* defines a half-space containing *P*. The intersection of all of these half-spaces for the facets containing *f* is the convex hull of *P* and H'. Intersecting with *H* yields just H'. One can instead take all of the opposite half-spaces and it remains true that the intersection with *H* is H'. Now if none of these facets are invisible from v, then v lies in all of the opposite half spaces as above, which means that v lies in H' and thus f is degenerate. In conclusion, the intersection of two degenerate faces must either be degenerate or contained in an invisible facet.

We can now proceed with the induction on the number of degenerate faces. Let P_0 be the union of all the invisible facets of P. This has no reduced cohomology since it is the negative support of a negative ample divisor as before and therefore has no cohomology below $\tilde{H}^{n-1}$. Since v is outside of P there is at least one visible facet and so the set of invisible facets is not the whole $n-1$-sphere.Therefore, $\tilde{H}^{n-1}$ is also 0.

Assume now that P_i, the union of P_0 with i degenerate faces, is cohomologically trivial. Let f be a new degenerate face. The Mayer–Vietoris sequence gives us:

$$\cdots \to \tilde{H}^{a-1}(f \cap P_i) \to \tilde{H}^a(f \cup P_i) \to \tilde{H}^a(f) \oplus \tilde{H}^a(P_i) \to \cdots .$$

As f itself is contractible and P_i is homologically trivial by induction, it suffices to show that $f \cap P_i$ is homologically trivial. However, the above arguments show that v is in the affine span of f and $f \cap P_i$ is a union of all of the invisible facets of f and some degenerate faces of f. Therefore, its cohomology vanishes by induction on dimension. The base case is when P is one dimensional, in which case for v in the line containing P but not in P, there is exactly one invisible facet (a single point) and no degenerate facets. □

Note, that this proposition, and hence all of Case 3, holds for arbitrary D_I and does not use that D_I is a manifold. Theorem 7 is an easy consequence of all of the above results. □

4. Constructing the resultant matrix

4.1. Partial shellings

In this section we show how to choose the D_i to form a topological ball (disk in dimension 2). Of course one can always choose a single facet for D_i, but as we shall see this does not usually yield the smallest matrices.

Definition 18. An ordering of the facets $D_1, \ldots, D_s$ of an n-dimensional polytope Q, is called a shelling if for $i = 2, \ldots, s$, $(D_1 \cup \cdots \cup D_{i-1}) \cap D_i$ is $n-2$ dimensional and is itself the union of an initial sequence of facets (codimension 2 faces in Q) of a shelling of D_i. A *partial shelling* is a proper sequence of facets, say $D_1, \ldots, D_t$ with $1 \leqslant t < s$, satisfying the same property above.

When Q has dimension 2, a partial shelling is the same as a connected set of edges. In our setting, where Q has dimension 3, being a partial shelling simply means that the intersection of each D_i with the union of the previous D_j is a connected set of edges of D_i.

Proposition 19. *Let Q be a polytope of dimension* 3. *The space $|D_I|$ is homeomorphic to a disk if and only if the facets in D_I can be arranged into a partial shelling of the boundary of Q.*

Proof. It is a standard result that any partial shelling of the boundary of a polytope is homemorphic to a ball. In the case of a 3-dimensional polytope it is actually a consequence of the Jordan curve theorem. Conversely, every topological disk is shellable in dimension two. This last statement fails in dimension three and higher. □

It is very easy to actually construct partial shellings for polytopes. A simple algorithm is to pass to the polar polytope Q° of Q. Facets of Q correspond to vertices of Q°. Next, pick a generic vector in $\mathbb{R}^n$. This will induce a linear functional on Q° which by genericity induces a linear order on the vertices. One can show that any initial segment of this linear ordering corresponds to a partial shelling of the facets of Q. Shellings arising this way are called line shellings.

4.2. Filling in entries

To actually construct our resultant formula we need to fill in the entries of the matrices B, L, and $\tilde{L}$. We saw above how these arise from a map in a Tate resolution. Therefore, we must compute appropriate terms and maps in the Tate resolution following the algorithm in Section 2.2 adapted to this situation.

Algorithm 1. (1) *Pick a partial shelling D_I. As we shall see in the next section, in order to get a smaller matrix we should pick D_I to have as many boundary points as possible.*

(2) *Compute the lattice points in $3Q - D_I$ and $4Q - D_I$, respectively.*

(3) *Construct the linear map $\phi_2 \;:\; S_{3Q-D_I} \otimes E \to S_{4Q-D_I} \otimes E$. In light of Theorem 8 this is precisely the differential $T^1(\mathscr{F}) \to T^2(\mathscr{F})$.*

Recall that y_i represents a basis element of S_Q hence a point in $A = Q \cap \mathbb{Z}^3$. So, for every basis element m of S_{3Q-D_I}, let the multiplicative notation $m \cdot y_i$ denote the basis element of S_{4Q-D_I} obtained by adding the two points. This can of course be extended linearly to all of S_{3Q-D_I}. Now the map ϕ_2 is explicitly defined by $\phi_2(m \otimes 1) = \sum_{i=1}^{N} (my_i \otimes e_i)$.

(4) *Compute two steps of a graded minimal free resolution, over E, of the cokernel of ϕ_2.*

$$T^{-1} \xrightarrow{\phi_0} T^0 \xrightarrow{\phi_1} T^1 \xrightarrow{\phi_2} T^2.$$

Since this minimal free resolution is precisely the Tate resolution, the map we are interested in is ϕ_0. Let M_0 be the corresponding matrix over E. The entries of this matrix will be either linear or of degree 4.

(5) *Apply the functor U_4 to ϕ_0, and therefore M_0. This is done by replacing each degree* 4 *term of the form $e_{i_1}e_{i_2}e_{i_3}e_{i_4}$ by the "bracket variable" $[i_1i_2i_3i_4]$ which represents the 4×4 determinant*:

$$\det \begin{bmatrix} C_{1i_1} & C_{1i_2} & C_{1i_3} & C_{1i_4} \\ C_{2i_1} & C_{2i_2} & C_{2i_3} & C_{2i_4} \\ C_{3i_1} & C_{3i_2} & C_{3i_3} & C_{3i_4} \\ C_{4i_1} & C_{4i_2} & C_{4i_3} & C_{4i_4} \end{bmatrix}.$$

Here C_{ij} is the coefficients of f_i corresponding to the monomial representing the point $y_j \in A$. These entries make up the submatrix B from Theorem 2. *The remaining rows and columns have linear entries, and correspond to L and $\tilde{L}$. Replace each such row* (*or column*) *by* 4 *rows* (*or columns*). *The entry e_i is replaced by C_{1i} in the first copy, C_{2i} in the second*

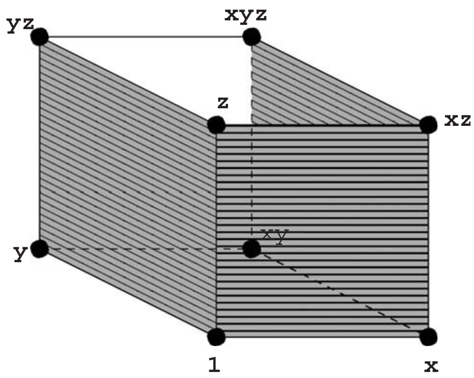

Fig. 1. Newton polytope of Example 20.

copy, and so on. This procedure is illustrated in the examples below. It is a consequence of Lemma 4.2 *in* [12]. *This results in a matrix M which is precisely the matrix of Theorem* 2.

Step 4 above requires computing part of a graded minimal resolution over the exterior algebra. This can be done using Gröbner bases but may be quite time consuming. On the other hand this computation needs only be done once to compute the resultant of any system with a fixed support. One might hope to eliminate the expensive Gröbner basis computations by finding explicit formulas for the non-trivial maps in the resolution. This was done for the two-dimensional resultant in [12] but remains open in the three-dimensional case.

4.3. Examples

Example 20. Consider the multilinear system:

$$
\begin{aligned}
f_1 &= C_{11} + C_{12}x + C_{13}y + C_{14}z + C_{15}xy + C_{16}xz + C_{17}yz + C_{18}xyz,\\
f_2 &= C_{21} + C_{22}x + C_{23}y + C_{24}z + C_{25}xy + C_{26}xz + C_{27}yz + C_{28}xyz,\\
f_3 &= C_{31} + C_{32}x + C_{33}y + C_{34}z + C_{35}xy + C_{36}xz + C_{37}yz + C_{38}xyz,\\
f_4 &= C_{41} + C_{42}x + C_{43}y + C_{44}z + C_{45}xy + C_{46}xz + C_{47}yz + C_{48}xyz.
\end{aligned}
$$

The Newton polytope Q of this system is the unit cube in Fig. 1. In order to apply the resultant algorithm we must choose a partial shelling. So, for example, we can pick the left, front, and, right faces as shown. Now $l(Q - D_I)$ and, by symmetry, $l(Q - \overline{D_I})$ are empty while $l(2Q - D_I)$ consists of the 6 monomials $\{xy, xyz, xy^2, xyz^2, xy^2z, xy^2z^2\}$, and $l(2Q - \overline{D_I})$ consists of the 6 monomials $\{z, xz, yz, x^z, xyz, x^2yz\}$. By Theorem 2 the resultant is the determinant of a 6×6 pure Bézout matrix. To explicitly compute it, we construct the linear map $S_{2Q-D_I} \otimes E(1) \to S_{3Q-D_I} \otimes E$ and compute one step of a free resolution over E. The matrix turns out to be the one shown in Table 1 .

Note that the size of the matrix depends heavily on the choice of the partial shelling. If, on the other hand, we were to choose D_I to consist of the left, front, and top facets, then $\#l(Q - D_I) = \#l(Q - \overline{D_I}) = 1$, and $\#l(2Q - D_I) = \#l(2Q - \overline{D_I}) = 8$. Hence, the matrix

Table 1
Resultant matrix for Example 20

[1234]	[1236] − [1245]	[1237]	[1256]	[1238] + [1257]	[1258]
[1346] − [1247]	[2346] − [1248] −[1267] − [1456]	[2347] − [1367]	−[2456] − [1268]	[2348] − [1368] −[1567] − [2457]	−[1568] − [2458]
[1345]	[2345] − [1356]	−[1357]	−[2356]	−[2357] − [1358]	−[2358]
[1467]	[2467] + [1468]	[3467]	[2468]	[3468] − [4567]	−[4568]
[1457] + [1348]	[1368] + [2348] +[2457] − [1567]	[1378] − [3457]	[2368] − [2567]	[1578] + [2378] +[3458] − [3567]	[2578] − [3568]
−[1478]	−[1678] − [2478]	−[3478]	−[2678]	[4578] − [3678]	[5678]

from Theorem 2 would be a 12×12 matrix with an 8×8 block B, a 8×4 block L, a 4×8 block $\tilde{L}$, and a 4×4 block of zeroes.

If instead we tried the top and bottom facets, not homeomorphic to a disk, we would still have $l(Q - D_I)$ and $l(Q - \overline{D_I})$ empty. However, this time $l(2Q - D_I)$ would consist of 9 points, while $l(2Q - \overline{D_I})$ would have only 3 points. A closer look at the vanishing theorems shows that we can still get a 9×9 square resultant matrix as there is only other non-vanishing cohomology term $H^1(\mathcal{O}(-D_I)) = \tilde{H}_0(|D_I|) = \mathbb{C}$ tensored with $\bigwedge^2 \mathcal{T}$, a vector bundle of rank 6. Indeed one can show in general that if $|D_I|$ is a disjoint union of disks we still get an exact matrix formula.

Example 21. Our next example is the following system:

$$\begin{aligned}
f_1 &= C_{11} + C_{12}x + C_{13}y + C_{14}z + C_{15}x^{-1} + C_{16}y^{-1} + C_{17}z^{-1},\\
f_2 &= C_{21} + C_{22}x + C_{23}y + C_{24}z + C_{25}x^{-1} + C_{26}y^{-1} + C_{27}z^{-1},\\
f_3 &= C_{31} + C_{32}x + C_{33}y + C_{34}z + C_{35}x^{-1} + C_{36}y^{-1} + C_{37}z^{-1},\\
f_4 &= C_{41} + C_{42}x + C_{43}y + C_{44}z + C_{45}x^{-1} + C_{46}y^{-1} + C_{47}z^{-1}.
\end{aligned}$$

The Newton polytope Q is the octahedron of Fig. 2. As our set of facets (partial shelling) we choose the x, y, z facet and the three other facets adjoined to it by an edge. The chosen facets are shaded in the figure. Now we can see that there are 10 points in $l(2Q - D_I)$ and also by symmetry in $l(2Q - \overline{D_I})$. There is a single point in $l(Q - D_I)$ (respectively, $l(Q - \overline{D_I})$). By Theorem 2 the resultant is therefore the determinant of a 14×14 matrix shown in Table 2 . This matrix was found following the algorithm of Section 4.2 by starting with the map $S_{3D_Q - D_I} \otimes E(1) \to S_{4D_Q - D_I} \otimes E$ and computing a free resolution.

If we were to choose a non-partial shelling such as two facets meeting at a single point then the corresponding resultant complex would have non-trivial middle cohomology. Indeed, $H^2(-D - D_I) = H^2(-2D - D_I) = \mathbb{C}$. $|D_I|$ is still homologically trivial but it is not a disk. The complex arising from the Tate resolution has three terms.

5. Ehrhart polynomials and sizes of resultant matrices

The results of Section 3 show that the determinant of the matrix of Theorem 2 is the resultant. In particular, it must be square and the degree of its determinant is equal to that of the resultant. In this section, we give an alternate combinatorial proof of these facts. This

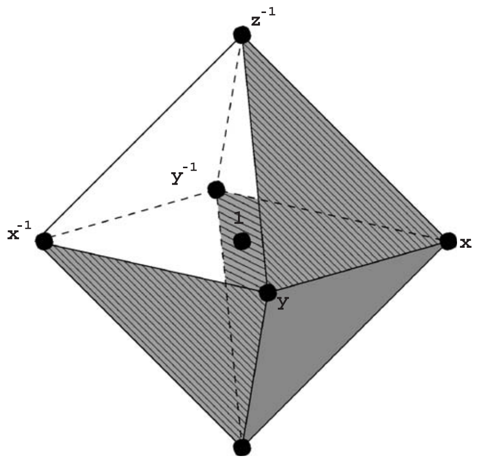

Fig. 2. Newton polytope of Example 21.

Table 2
Resultant matrix for Example 21

0	0	0	0	0	0	0	[2345]	[2346]	[2347]	C_{11}	C_{21}	C_{31}	C_{41}
0	0	0	0	0	0	0	0	0	0	C_{12}	C_{22}	C_{32}	C_{42}
0	0	0	0	0	0	0	0	0	0	C_{13}	C_{23}	C_{33}	C_{43}
0	0	0	0	0	0	0	0	0	0	C_{14}	C_{24}	C_{34}	C_{44}
0	−[2356]	−[2357]	0	0	0	0	[1235]	0	0	C_{15}	C_{25}	C_{35}	C_{45}
0	−[2456]	0	[2467]	0	0	0	0	−[1246]	0	C_{16}	C_{26}	C_{36}	C_{46}
0	0	[3457]	[3467]	0	0	0	0	0	[1347]	C_{17}	C_{27}	C_{37}	C_{47}
−[2567]	[1256]	0	0	0	0	0	−[2356]	−[2456]	0	0	0	0	0
−[3567]	0	−[1357]	0	0	0	0	−[2357]	0	[3457]	0	0	0	0
−[4567]	0	0	[1467]	0	0	0	0	[2467]	[2467]	0	0	0	0
C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	0	0	0	0	0	0	0
C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	0	0	0	0	0	0	0
C_{31}	C_{32}	C_{33}	C_{34}	C_{35}	C_{36}	C_{37}	0	0	0	0	0	0	0
C_{41}	C_{42}	C_{43}	C_{44}	C_{45}	C_{46}	C_{47}	0	0	0	0	0	0	0

will also allow us to analyze the size of the resultant matrix in order to choose the smallest matrices.

Consider the Hilbert function of X_A, which turns out to be an honest polynomial $p(x)$ where the value $p(k)$, for $k \in \mathbb{N}$, counts the number of lattice points in the polytope kQ. This polynomial is associated to the polytope Q and is called the *Ehrhart polynomial* of Q. There is a very pretty duality theorem involving Ehrhart polynomials. See [8] for details.

Proposition 22. *Let Q be a lattice polytope of dimension n with Ehrhart polynomial p. Then,* $(-1)^n p(-k)$ *is the number of interior lattice points in kQ.*

Given a collection of facets D_I we are interested in counting the number of lattice points in $kQ - D_I$. A result of Stanley [16] extends Ehrhart polynomials and duality in this setting.

Proposition 23 (*Stanley[16, Proposition 8.2]*). *Let Q be a lattice polytope of dimension n, and* D_I *a collection of facets. Suppose* $|D_I|$ *is homeomorphic to a manifold. Then, there is* *a polynomial p_I of degree n, such that $p_I(k)$ for $k > 0$ is the number of points in $kQ - D_I$, $(-1)^n p_I(-k)$ for $k > 0$ is the number of lattice points in $kQ - \overline{D_I}$, and $p_I(0) = 1 - \chi(|D_I|)$. Here, $\chi(|D_I|)$ is the Euler characteristic of the manifold $|D_I|$. In particular, if $|D_I|$ is a disk, then $p_I(0) = 0$.*

The difference $p(k) - p_I(k)$, the number of lattice points on the facets D_I in kQ, is itself a polynomial of degree $(n-1)$.

Going back to resultants, we consider the two term complex appearing in Corollary 9

$$\begin{array}{ccccc} S^*_{2Q-\overline{D_I}} \otimes \bigwedge^4 \mathscr{T} & \xrightarrow{\tilde{L}} & S^*_{Q-\overline{D_I}} \otimes \bigwedge^3 \mathscr{T} & & \\ \oplus & \searrow^{B} & \oplus & \longrightarrow & 0 \\ S_{Q-D_I} \otimes \bigwedge^1 \mathscr{T} & \xrightarrow{L} & S_{2Q-D_I} \otimes \bigwedge^0 \mathscr{T} & & \end{array}$$

Let $p_I(k)$ be the Ehrhart polynomial of $kQ - D_I$. This is a cubic polynomial, thus the fourth difference is 0. In particular:

$$p_I(2) - 4p_I(1) + 6p_I(0) - 4p_I(-1) + p_I(-2) = 0.$$

Since $|D_I|$ homeomorphic to a disk, $p_I(0) = 0$, and the equation can be rewritten as $p_I(2) - 4p_I(-1) = -p_I(-2) + 4p_I(1)$. Indentifying the dimension of the terms in the diagram above using Proposition 23, this says precisely that the matrix is square.

The total degree is computed by taking $4\#l(Q - D_I)$ entries from L, $4\#l(Q - D_{\bar{I}})$ entries from $\tilde{L}$ and $\#l(2Q - D_I) - 4\#l(Q - D_I)$ entries from B. The entries of L and $\tilde{L}$ are of degree 1, while those of B are of degree 4. So the total degree is $4p_I(1) - 4p_I(-1) + 4(p_I(2) - 4p_I(1)) = 4(p_I(2) - 3p_I(1) + 3p_I(0) - p_I(-1))$. This is 4 times the third difference of p_I which is the same as 4 times 3! times the leading coefficient of p_I. This is the same as the leading coefficient of the Ehrhart poynomial of Q which is just the Euclidean volume. Hence, the degree in question is 4 times the normalized volume which is also the degree of the resultant.

This leads to a technique to analyze the size of the resultant matrices. The Ehrhart polynomial of Q is of the form $p(x) = Ax^3 + Bx^2 + Cx + 1$. The leading term A is the degree of the toric variety X_A divided by 3!, which is the Euclidian volume of Q. Moreover $p(1)$ is the number of lattice points in Q, and $p(-1)$ is the negative of the number of interior points. So the number of boundary points in Q is $p(1) + p(-1) = A + B + C + 1 - A + B - C + 1 = 2B + 2$. Let $B_Q = 2B + 2$ denote this number. Next, for any partial shelling D_I, we write the corresponding quadratic Ehrhart polynomial as $q_I(x) = ax^2 + bx + 1$. This time $q(-1)$ is equal to the number of relative interior points, so the number of boundary points is $q(1) - q(-1) = a + b + 1 - a + b - 1 = 2b$. Let $B_I = 2b$ denote this number.

Taking $p_I(x) = p(x) - q_I(x)$ as above, then the total size of the resultant matrix is

$$\begin{aligned} p_I(2) - 4p_I(-1) &= p(2) - q_I(2) - 4p(-1) + 4q_I(-1) \\ &= 8A + 4B + 2C + 1 - (4a + 2b + 1) - 4(-A + B - C + 1) \\ &\quad + 4(a - b + 1) \\ &= 12A + 6C - 6b. \end{aligned}$$

Let $i_Q = -p(-1) = A - B + C - 1$ be the total number of interior points of Q. So $C = i_Q + 1 + B - A$. Hence, we can rewrite the above as:

$$\begin{aligned} 12A + 6C - 6b &= 12A + 6(i_Q + B - A + 1) - 6b \\ &= 6A + 3(2B + 2 - 2b) + 6i_Q \\ &= V + 3(B_Q - B_I) + 6i_Q. \end{aligned}$$

Here, V denotes the normalized volume of Q which is 6 times the Euclidian volume A. Therefore, in order to minimize the size of the matrix we must maximize B_I which is the number of relative boundary points of the union of the facets D_I. This gives an obvious lower bound of V for the size of the resultant matrix. A more sophisticated argument would give an upper bound of $3V$ when i_Q is at least 1.

6. Conclusion

In this article we showed how the resultant of an unmixed system in three variables with arbitrary support can be computed as the determinant of a matrix. Combined with the authors earlier results [12], we have now generalized the formulas for the resultant of homogeneous systems in dimensions 2 and 3. However, it is still unknown how to make the dimension 3 formula completely explicit instead of in terms of a free resolution as presented here.

For dimension 4 and higher, no general exact formula is known. We do give some special cases, although still without an explicit closed form formula, and it would be nice to finish this classification. A second approach is to allow complexes with more than two terms, yielding resultant formulas with extraneous factors. In the case of projective spaces [2] and products of projective spaces [3] the extraneous factors have been identified. It is still open how to do this for general toric varieties.

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