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Existence and Uniqueness of Solutions of Boundary Value Problems for Lipschitz Equations

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1. INTRODUCTION

Let $f(t, y_1, y_2, \ldots, y_n)$ be a real valued function on $(a, b) \times R^n$. Then the boundary value problem

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}) \tag{1}$$

$$y^{(i)}(t_j) = c_{ji}, \qquad 0 \leqslant i \leqslant m_j - 1, \quad 1 \leqslant j \leqslant k, \tag{2}$$

where $2 \leqslant k \leqslant n$, $a < t_1 < t_2 < \cdots < t_k < b$, $m_j \geqslant 1$ for $1 \leqslant j \leqslant k$, and $\sum_{j=1}^{k} m_j = n$ will be called a k-point boundary value problem. During the past decade a number of papers have appeared which are concerned with conditions under which the existence of solutions of such problems is implied by the uniqueness of solutions. Hartman [1] and Klaasen [2] have given independent proofs of the following Theorem.

THEOREM 1. *Assume that with respect to equation* (1) *the following four conditions are satisfied*:

[A] $f(t, y_1, y_2, \ldots, y_n)$ is continuous on $(a, b) \times R^n$,

[B] Solutions of initial value problems for (1) are unique and all solutions extend to (a, b),

[C] If $[c, d]$ is a compact subinterval of (a, b) and if $\{y_k(t)\}$ is a sequence of solutions of (1) which is uniformly bounded on $[c, d]$, then there is a subsequence $\{y_{k_j}(t)\}$ such that $\{y_{k_j}^{(i)}(t)\}$ converges uniformly on $[c, d]$ for each $i = 0, 1, \ldots, n - 1$, and

[D] If $a < t_1 < t_2 < \cdots < t_n < b$ and if $y(t)$ and $z(t)$ are solutions of (1) such that $y(t_j) = z(t_j)$ for $1 \leqslant j \leqslant n$, then $y(t) \equiv z(t)$ on (a, b).

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Then it follows that for any $a < t_1 < t_2 < \cdots < t_n < b$ and any real numbers c_j, $1 \leqslant j \leqslant n$, the n-point boundary value problem for (1) with $y(t_j) = c_j$ for $1 \leqslant j \leqslant n$ has a solution.

Hartman [3] has proven that, if an equation (1) satisfies conditions [A], [B], and [D], and also satisfies the conclusion of Theorem 1, then all k-point boundary value problems, $2 \leqslant k \leqslant n$, for (1) have solutions which are unique.

In this paper we shall be concerned with equations (1) which satisfy condition Lipschitz condition:

[E] For each j with $1 \leqslant j \leqslant n$

$$k_j(t)(y_j - z_j) \leqslant f(t, y_1, ..., y_j, ..., y_n) - f(t, y_1, ..., z_j, ..., y_n) \leqslant l_j(t)(y_j - z_j)$$

for all points $(t, y_1, ..., y_j, ..., y_n)$ and $(t, y_1, ..., z_j, ..., y_n)$ in $(a, b) \times R^n$ with $y_j \geqslant z_j$ where $k_j(t)$, $l_j(t)$ are continuous functions on (a, b) with $k_j(t) < l_j(t)$ on (a, b) for each $1 \leqslant j \leqslant n$.

If an equation (1) satisfies conditions [A] and [E], then condition [B] is satisfied. Furthermore, since conditions [A] and [E] imply that n-point boundary value problems for (1) are uniquely solvable on small enough subintervals of (a, b), condition [C] is also satisfied. Thus, if an equation (1) satisfies conditions [A], [E], and [D] on a subinterval $(\alpha, \beta) \subset (a, b)$, then all k-point boundary value problems, $2 \leqslant k \leqslant n$, for (1) have unique solutions on (α, β).

The purpose of this paper is to characterize in terms of the Lipschitz coefficients $k_j(t)$, $l_j(t)$, $1 \leqslant j \leqslant n$, the subintervals (α, β) of (a, b) of maximal length on which condition [D] is satisfied for all equations (1) satisfying [A] and [E]. Such intervals will then be intervals on which all k-point boundary value problems, $2 \leqslant k \leqslant n$, will be uniquely solvable for all differential equations (1) satisfying conditions [A] and [E]. This will be accomplished by an application of control theory methods which is motivated by the work in [4]. In the remainder of the paper we will assume that we are dealing with an arbitrary but fixed equation (1) which satisfies conditions [A] and [E].

2. An Application of Control Theory Methods

Assume that $y(t)$ and $z(t)$ are distinct solutions of (1) on (a, b) and for $0 \leqslant j \leqslant n$ define the functions $h_j(t)$ by

$$h_0(t) = f(t, y(t), y'(t), ..., y^{(n-1)}(t)),$$
$$h_j(t) = f(t, z(t), z'(t), ..., z^{(j-1)}(t), y^{(j)}(t), ..., y^{(n-1)}(t))$$

for $1 \leqslant j \leqslant n - 1$, and

$$h_n(t) = f(t, z(t), z'(t), ..., z^{(n-1)}(t)).$$

Then define the functions $u_j(t)$, $1 \leqslant j \leqslant n$, by

$$u_j(t) = \begin{cases} \dfrac{h_{j-1}(t) - h_j(t)}{y^{(j-1)}(t) - z^{(j-1)}(t)} & \text{for} \quad y^{(j-1)}(t) \neq z^{(j-1)}(t) \\ k_j(t) \qquad \text{for} \quad y^{(j-1)}(t) = z^{(j-1)}(t). \end{cases}$$

It follows from the continuity of the functions involved and from condition [E], that, for each $1 \leqslant j \leqslant n$, $u_j(t)$ is measurable on (a, b) and

$$k_j(t) \leqslant u_j(t) \leqslant l_j(t) \tag{3}$$

on (a, b). Furthermore, the difference $x(t) = y(t) - z(t)$ is a solution of the linear equation

$$x^{(n)} = \sum_{j=1}^{n} u_j(t) x^{(j-1)} \tag{4}$$

on (a, b). Now assume that $y(t)$ and $z(t)$ are distinct solutions of (1) and that there are points $a < t_1 < t_2 < \cdots < t_n < b$ such that $y(t_j) = z(t_j)$ for $1 \leqslant j \leqslant n$. Then, if c and d are chosen so that $a < c \leqslant t_1$ and $t_n \leqslant d < b$, the difference $x(t) = y(t) - z(t)$ is a nontrivial solution of (4) having n distinct zeros on the compact interval $[c, d]$.

For an integer k with $1 \leqslant k \leqslant n - 1$ a solution $w(t)$ of (4) will be said to have an $(n - k, k)$ pair of zeros on the interval $[c, d]$ in case there exist t_1 and t_2 with $c \leqslant t_1 < t_2 \leqslant d$ such that $w(t)$ has a zero of order at least $n - k$ at t_1 and a zero of order at least k at t_2. In [9] Sherman has proven that, if for each k with $1 \leqslant k \leqslant n - 1$ there is no nontrivial solution of (4) with an $(n - k, k)$ pair of zeros on $[c, d]$, then (4) is disconjugate on $[c, d]$, that is, no nontrivial solution of (4) has n zeros on $[c, d]$ counting multiplicities of the zeros. Thus, since $x(t) = y(t) - z(t)$ is a nontrivial solution of (4) with n zeros on $[c, d]$, there is a k_0 with $1 \leqslant k_0 \leqslant n - 1$ such that (4) has a nontrivial solution with an $(n - k_0, k_0)$ pair of zeros on $[c, d]$.

Now let U be the set of all vector functions $u = (u_1(t), u_2(t), \ldots, u_n(t))$ such that the components $u_j(t)$ are Lebesgue measureable on (a, b) and satisfy inequalities (3) on (a, b), and consider the collection of all 2-point boundary value problems of the form

$$\begin{aligned} x^{(n)} &= \sum_{j=1}^{n} u_j(t) x^{(j-1)} \\ x^{(i)}(t_1) &= 0, \qquad 0 \leqslant i \leqslant n - k_0 - 1, \\ x^{(i)}(t_2) &= 0, \qquad 0 \leqslant i \leqslant k_0 - 1, \end{aligned} \tag{5}$$

where $(u_1(t), u_2(t), \ldots, u_n(t)) \in U$ and $c \leqslant t_1 < t_2 \leqslant d$. As remarked above there is a problem in this collection that has a nontrivial solution. This being the case, it follows from standard arguments that there is a boundary value problem in the collection (5) which has a nontrivial solution which is time optimal, that is, which is such that the spacing, $t_2 - t_1$, between its zeros is a minimum among all nontrivial solutions of boundary value problems in the collection (5).

For each $u \in U$ let $z' = A[u(t)]\, z$ be the first order vector system corresponding to the nth order scalar equation $x^{(n)} = \sum_{j=1}^{n} u_j(t)\, x^{(j-1)}$. Let $x(t)$ be a time optimal solution in the set of boundary value problems (5), let $u^* \in U$ be an associated time optimal control, and let $c \leqslant t_1 < t_2 \leqslant d$ be zeros of $x(t)$ of respective orders $n - k_0$ and k_0 such that $t_2 - t_1$ is a minimum. Then it follows from the Pontryagin Maximum Principle [5, p. 310] that the adjoint system

$$\psi' = -A^T[u^*(t)]\, \psi, \tag{6}$$

where A^T represents the transpose of A, has a nontrivial solution $\psi(t) = (\psi_1(t),\ldots, \psi_n(t))^T$ such that for almost all t with $t_1 \leqslant t \leqslant t_2$

$$\sum_{j=1}^{n} x^{(j)}(t)\, \psi_j(t) = (z'(t), \psi(t)) = \operatorname{Max}\{(A[u(t)]\, z(t), \psi(t)) \mid u \in U\} \tag{7}$$

where $z(t) = (x(t), x'(t),\ldots, x^{(n-1)}(t))$ and $(\cdot, \cdot)$ represents the inner product. Furthermore, $\psi(t)$ is such that $(z, \psi(t_1)) = 0$ for all vectors $z = (z_1, z_2,\ldots, z_n)$ with $z_j = 0$ for $1 \leqslant j \leqslant n - k_0$ and $(z, \psi(t_2)) = 0$ for all vectors z with $z_j = 0$ for $1 \leqslant j \leqslant k_0$. These conditions imply that

$$\psi_j(t_1) = 0 \qquad \text{for} \quad n - k_0 + 1 \leqslant j \leqslant n \qquad \text{and} \tag{8}$$

$$\psi_j(t_2) = 0 \qquad \text{for} \quad k_0 + 1 \leqslant j \leqslant n. \tag{9}$$

Since

$$(A[u(t)]\, z(t), \psi(t)) = \sum_{j=1}^{n-1} x^{(j)}(t)\, \psi_j(t) + \psi_n(t) \sum_{j=1}^{n} u_j(t) x^{(j-1)}(t),$$

the maximum condition (7) can be written as

$$\psi_n(t) \sum_{j=1}^{n} u_j^*(t) x^{(j-1)}(t) = \operatorname{Max}\{\psi_n(t) \sum_{j=1}^{n} u_j(t) x^{(j-1)}(t) \mid u \in U\}. \tag{10}$$

In our applications of (10) it will be the case that the time optimal solution $x(t)$ will be positive on (t_1, t_2) and the associated solution $\psi(t)$ of the corresponding adjoint system will be such that its nth component $\psi_n(t)$ will have no zeros on (t_1, t_2). In this case it follows from (10) that, if $\psi_n(t) < 0$ on (t_1, t_2), then

$$u_1^*(t) = k_1(t) \tag{11}$$

and for $2 \leqslant j \leqslant n$

$$u_j^*(t) = \begin{cases} k_j(t) & \text{when } x^{(j-1)}(t) \geqslant 0 \\ l_j(t) & \text{when } x^{(j-1)}(t) < 0. \end{cases} \tag{12}$$

Similarly, if $\psi_n(t) > 0$ on (t_1, t_2), then

$$u_1^*(t) = l_1(t) \tag{13}$$

and for $2 \leqslant j \leqslant n$

$$u_j^*(t) = \begin{cases} l_j(t) & \text{when } x^{(j-1)}(t) \geqslant 0 \\ k_j(t) & \text{when } x^{(j-1)}(t) < 0. \end{cases} \tag{14}$$

It follows that, if we define the differential operators L_1, L_2 by

$$L_1(t, x, x', \ldots, x^{(n-1)}) = k_1(t)x + \frac{1}{2} \sum_{j=2}^{n} [l_j(t) + k_j(t)] x^{(j-1)} \tag{15}$$

and

$$L_2(t, x', x'', \ldots, x^{(n-1)}) = \frac{1}{2} \sum_{j=2}^{n} [l_j(t) - k_j(t)] \mid x^{(j-1)} \mid, \tag{16}$$

then, under the assumptions that $x(t) > 0$ and $\psi_n(t) < 0$ on (t_1, t_2), the time optimal solution is a solution of

$$x^{(n)} = L_1(t, x, x', \ldots, x^{(n-1)}) - L_2(t, x', x'', \ldots, x^{(n-1)}), \tag{17}$$

and, under the assumptions that $x(t) > 0$ and $\psi_n(t) > 0$ on (t_1, t_2), the time optimal solution is a solution of

$$\begin{aligned} x^{(n)} = {} & [l_1(t) - k_1(t)]\, x + L_1(t, x, x', \ldots, x^{(n-1)}) \\ & + L_2(t, x', x'', \ldots, x^{(n-1)}). \end{aligned} \tag{18}$$

It should be noted that the differential equations (17) and (18) both satisfy conditions [A] and [E]. Of course it is still true that $u^* \in U$ whether defined by (11), (12) or by (13), (14) and the time optimal solution is a solution of a differential equation appearing in the collection of boundary value problems (5).

To conclude this section we consider the collection of boundary value problems

$$\begin{aligned} \psi' &= -A^T[u(t)]\, \psi \\ \psi_j(t_1) &= 0, \qquad n - k_0 + 1 \leqslant j \leqslant n, \\ \psi_j(t_2) &= 0, \qquad k_0 + 1 \leqslant j \leqslant n, \end{aligned} \tag{19}$$

where $u \in U$ and $c \leqslant t_1 < t_2 \leqslant d$. A solution vector $\psi(t)$ of one of these problems will be said to have a $(k_0, n - k_0)$ pair of zeros on $[c, d]$ with zero of order k_0 at t_1 and zero of order $n - k_0$ at t_2. Again it can be argued that, if there is a problem in the collection (19) with a nontrivial solution, then there will exist a problem in the collection which has a nontrivial time optimal solution. If $u^* \in U$ is an associated time optimal control vector and if the "time optimal" zeros are at t_1 and t_2 with $c \leqslant t_1 < t_2 \leqslant d$, then the Pontryagin Maximum Principle can be applied to conclude that the boundary value problem

$$\begin{aligned} x^{(n)} &= \sum_{j=1}^{n} u_j^*(t) x^{(j-1)} \\ x^{(i)}(t_1) &= 0, \qquad 0 \leqslant i \leqslant n - k_0 - 1, \\ x^{(i)}(t_2) &= 0, \qquad 0 \leqslant i \leqslant k_0 - 1, \end{aligned}$$

has a nontrivial solution. Thus the Maximum Principle associates with each time optimal solution in the collection (5) a time optimal solution in the collection (19), and conversely.

3. The (n-1, 1) Zero Boundary Value Problem

Let $[c, d]$ be a compact subinterval of (a, b), let $k_0 = 1$, and consider the corresponding collection of boundary value problems (5). Assume that there is a problem in the collection for which the solution $x(t)$ is time optimal and that the associated zeros are at t_1 and t_2 with $c \leqslant t_1 < t_2 \leqslant d$. Then from the concluding remarks of the last section it follows that, if $\psi(t)$ is the solution of the adjoint system associated with $x(t)$ by the Maximum Principle, then $\psi_n(t) \neq 0$ on (t_1, t_2). For if this were not the case $\psi(t)$ would not be time optimal. In this section we shall prove that $\psi_n(t) < 0$ on (t_1, t_2) and that the time optimal solution is a solution of (17) on $[t_1, t_2]$.

Theorem 2. *Assume that there is a subinterval $[t_1, t_2] \subset (a, b)$ and a solution $x(t)$ of equation (18) with $x^{(i)}(t_1) = 0$ for $0 \leqslant i \leqslant n - 2$, with $x(t_2) = 0$, and with $x(t) > 0$ on (t_1, t_2). Then there is a proper subinterval $[s_1, s_2] \subset [t_1, t_2]$ and a solution $v(t)$ of (17) such that $v^{(i)}(s_1) = 0$ for $0 \leqslant i \leqslant n - 2$, $v(s_2) = 0$, and $v(s) > 0$ on (s_1, s_2).*

Proof. Assume that no such solution $v(t)$ of (17) exists. Let $w(t, s)$ be the solution of the initial value problem

$$\begin{aligned} x^{(n)} &= L_1(t, x, x', \ldots, x^{(n-1)}) \\ x^{(i)}(s) &= 0, \qquad 0 \leqslant i \leqslant n - 2, \\ x^{(n-1)}(s) &= 1. \end{aligned}$$

Then $w(t_2, s) > 0$ for all $t_1 \leqslant s < t_2$. To see this assume that $w(t_2, s_0) \leqslant 0$ for some s_0 with $t_1 \leqslant s_0 < t_2$. Then there exist t_3, t_4 such that $s_0 \leqslant t_3 < t_4 \leqslant t_2$, $w(t_4, t_3) = 0$, and $w(t_4, s) > 0$ for all $t_3 < s < t_4$. Let $v(t)$ be the solution of (17) satisfying the intial conditions $v^{(i)}(t_3) = 0$, $0 \leqslant i \leqslant n-2$, and $v^{(n-1)}(t_3) = 1$. Then

$$v(t_4) = w(t_4, t_3) - \int_{t_3}^{t_4} w(t_4, s) L_2(s, v'(s), v''(s), \ldots, v^{(n-1)}(s)) \, ds$$

from which we conclude $v(t_4) < 0$. Thus $v(t) = 0$ for some t with $t_3 < t < t_4$ which contradicts our assumption that there are no such solutions $v(t)$. From this contradiction we conclude that $w(t_2, s) > 0$ for $t_1 \leqslant s < t_2$. However, this fact and the following representation for $x(t_2)$,

$$x(t_2) = w(t_2, t_1) x^{(n-1)}(t_1) + \int_{t_1}^{t_2} w(t_2, s) \{[l_1(s) - k_1(s)] \, x(s) + L_2(s, x'(s), \ldots, x^{(n-1)}(s)\} \, ds,$$

leads to the conclusion $x(t_2) > 0$ which contradicts $x(t_2) = 0$. From this final contradiction we conclude the existence of solutions of (17) of the specified type.

THEOREM 3. *Assume that $x(t)$ is a time optimal solution of the $(n-1, 1)$ zero boundary value problem*

$$x^{(n)} = \sum_{j=1}^{n} u_j(t) x^{(j-1)}$$

$$x^{(i)}(t_1) = 0, \qquad 0 \leqslant i \leqslant n-2,$$

$$x(t_2) = 0$$

where $a < c \leqslant t_1 < t_2 \leqslant d < b$ and $u \in U$ and assume $x(t) > 0$ on (t_1, t_2). If $\psi(t)$ is the associated time optimal solution of the $(1, n-1)$ zero boundary value problem for the adjoint system, then $\psi_n(t) < 0$ on (t_1, t_2) and $x(t)$ is a solution of (17) *on* $[t_1, t_2]$.

Proof. Since $x(t)$ can be replaced by $-x(t)$, there is no loss in generality in assuming $x(t) > 0$ on (t_1, t_2). From the fact that the solution $\psi(t)$ of the $(1, n-1)$ zero boundary value problem for the adjoint system associated with $x(t)$ by the Maximum Principle is time optimal, we conclude that $\psi_n(t) \neq 0$ on (t_1, t_2). Thus $x(t)$ is a solution of (17) on $[t_1, t_2]$ or is a solution of (18) on $[t_1, t_2]$. If $x(t)$ is a solution of (18) on $[t_1, t_2]$, then it follows from Theorem 2 that there is a nontrivial solution of (17) with an $(n-1, 1)$ pair of zeros on a proper subinterval of $[t_1, t_2]$. This contradicts the time optimality of $x(t)$. Hence, $\psi_n(t) < 0$ on (t_1, t_2) and $x(t)$ is a solution of (17) on $[t_1, t_2]$.

4. $(n\text{-}k, k)$ Zero Boundary Value Problems with $2 \leq k \leq n\text{-}1$

Assume that the compact interval $[c, d] \subset (a, b)$ is such that for each control vector $u \in U$ the corresponding differential equation $x^{(n)} = \sum_{j=1}^{n} u_j(t) x^{(j-1)}$ has no nontrivial solution with an $(n-j, j)$ pair of zeros on $[c, d]$ for any j with $1 \leqslant j \leqslant k-1$ where k is a fixed integer satisfying $2 \leqslant k \leqslant n-1$. Assume that there is a control $u \in U$ such that $x^{(n)} = \sum_{j=1}^{n} u_j(t)\, x^{(j-1)}$ does have a nontrivial solution with an $(n-k, k)$ pair of zeros on $[c, d]$. Then there is a control which produces a time optimal such solution $x(t)$ with corresponding time optimal zeros at t_1, t_2 with $c \leqslant t_1 < t_2 \leqslant d$. In this section we prove that in this case $x(t)$ is either a solution of (17) on $[t_1, t_2]$ or a solution of (18) on $[t_1, t_2]$.

Theorem 4. *Assume that the conditions stated in the above paragraph are satisfied on the compact interval $[c, d] \subset (a, b)$ and assume $u \in U$ is a control such that $x^{(n)} = \sum_{j=1}^{n} u_j(t)\, x^{(j-1)}$ has a time optimal solution $x(t)$ with an $(n-k, k)$ pair of zeros at the respective points t_1 and t_2 with $c \leqslant t_1 < t_2 \leqslant d$. Then $x(t) \neq 0$ on (t_1, t_2).*

Proof. The conclusion of this Theorem is an immediate consequence of Lemma 4 in [6].

If in the adjoint system $\psi' = -A^T[u(t)]\, \psi$ corresponding to a fixed $u \in U$ we reverse the order of the components of ψ, that is, define the vector $y = (y_1, \dots, y_n)^T$ by setting $y_j = \psi_{n+1-j}$ for $1 \leqslant j \leqslant n$, we obtain a system

$$y' = B[u(t)]\, y \tag{20}$$

which is of the type studied by Hinton in [7]. We will say that a solution $y(t) = (y_1(t), \dots, y_n(t))^T$ has an $(n-k, k)$ pair of zeros at the respective points $t = t_1$ and $t = t_2$ in case $y_j(t_1) = 0$ for $1 \leqslant j \leqslant n-k$ and $y_j(t_2) = 0$ for $1 \leqslant j \leqslant k$. Thus, if a solution $\psi(t)$ of $\psi' = -A^T[u(t)]\, \psi$ has an $(n-k, k)$ pair of zeros at $t = t_1$ and $t = t_2$ respectively as defined earlier, then the corresponding solution $y(t)$ of (20) also has an $(n-k, k)$ pair of zeros at $t = t_1$ and $t = t_2$ as defined above.

For solution vectors $y^1(t), \dots, y^p(t)$ of (20) let $W(y^1, \dots, y^p)$ be the pth order determinant in which the ith row, $1 \leqslant i \leqslant p$, consists of the respective ith components of the solutions $y^1(t), \dots, y^p(t)$. Then Theorem 2.1 of [7] can be formulated in the following way.

THEOREM 5. *Assume that $y^1(t),\dots, y^n(t)$ are linearly independent solutions of* (20) *and that $y^0(t)$ is also a solution of* (20). *Let $Y_0 = y_1{}^0(t)$ and for $1 \leqslant i \leqslant n$ let $Y_i = W(y^1,\dots, y^i, y^0)$ and $W_i = W(y^1,\dots, y^i)$. Then, for each i such that W_i does not have a zero on the interval $J \subset (a, b)$, we have*

$$a_i W_{i-1} Y_i = W_i^2 (Y_{i-1}/W_i)' \tag{21}$$

on J where $W_0 \equiv 1$, $a_i = -1$ for $1 \leqslant i \leqslant n-1$ and $a_n = +1$.

THEOREM 6. *Assume that for a fixed $u \in U$ and a fixed integer k with $2 \leqslant k \leqslant n-1$ and $k \leqslant n-k$ the system* (20) *and the interval $J \subset (a, b)$ are such that there is no nontrivial solution of* (20) *with a $(j, n-j)$ pair of zeros on J for any j with $1 \leqslant j \leqslant k-1$. Assume that there is a nontrivial solution $y^0(t)$ of* (20) *with a zero of order k at t_1 and a zero of order $n-k$ at t_2 with $t_1 < t_2$ and $t_1, t_2 \in J$ and assume that there is no nontrivial solution of* (20) *with a zero of order $n-k$ at $t = t_2$ and a zero of order k at a point in (t_1, t_2). Then $y_1{}^0(t) \neq 0$ on (t_1, t_2).*

Proof. For each i with $1 \leqslant i \leqslant n$ let $y^i(t)$ be the solution of (20) with $y^i(t_2) = (\delta_{1i},\dots, \delta_{ni})^T$ where δ_{ij} is the Kronecker delta. Then the solutions $y^1(t),\dots, y^n(t)$ are linearly independent and, since $y^0(t)$ has a zero of order $n-k$ at $t = t_2$, there are constants c_j, $n-k+1 \leqslant j \leqslant n$, such that

$$y^0(t) = c_{n-k+1} y^{n-k+1}(t) + \cdots + c_n y^n(t). \tag{22}$$

Since no nontrivial solution of (20) has a $(j, n-j)$ pair of zeros on J for any j with $1 \leqslant j \leqslant k-1$, it follows that for each j with $1 \leqslant j \leqslant k-1$, $W(y^{n-j+1},\dots, y^n) \neq 0$ for all $t \in J$ with $t < t_2$. Furthermore, since $y^0(t)$ does not have a zero of order k between t_1 and t_2, it follows that $W(y^{n-k+1},\dots, y^n) \neq 0$ on (t_1, t_2).

Now in Theorem 5 let us change notation to fit the present situation, that is, set $Y_0 = y_1{}^0(t)$ and for $1 \leqslant i \leqslant k$ set $Y_i = W(y^n,\dots, y^{n-i+1}, y^0)$ and $W_i = W(y^n,\dots, y^{n-i+1})$. Then as observed above $W_i \neq 0$ on (t_1, t_2) for each i with $1 \leqslant i \leqslant k$. Now assume that $y_1{}^0(t_3) = 0$ for some t_3 with $t_1 < t_3 < t_2$. Then applying Theorem 5 we conclude that Y_1 has a zero at some t_4 with $t_3 < t_4 < t_2$. A second application of Theorem 5 yields the existence of a zero of Y_2 at some point in (t_4, t_2). After repeated applications of Theorem 5 we reach the conclusion that there is a t_0 with $t_1 < t_0 < t_2$ such that Y_{k-1} has a zero at $t = t_0$. In view of (22) this implies that

$$c_{n-k+1} W(y^{n-k+1},\dots, y^n) = 0$$

at $t = t_0$. Since it was assumed that (20) has no nontrivial solution with a $(k-1, n-k+1)$ pair of zeros on J, it follows that $c_{n-k+1} \neq 0$. Therefore

$W(y^{n-k+1},..., y^n) = 0$ at $t = t_0$ which implies that (20) has a nontrivial solution with a zero of order $n - k$ at $t = t_2$ and a zero of order k at $t_0 \in (t_1, t_2)$ which contradicts our hypotheses. We conclude that $y_1^0(t) \neq 0$ on (t_1, t_2).

COROLLARY. *Assume that the compact interval* $[c, d] \subset (a, b)$ *and the integer* k *with* $2 \leqslant k \leqslant n - 1$ *are such that for each control* $u \in U$ *and each integer* j *with* $1 \leqslant j \leqslant k - 1$ *the equation* $x^{(n)} = \sum_{j=1}^{n} u_j(t)\, x^{(j-1)}$ *has no nontrivial solution with an* $(n - j, j)$ *pair of zeros on* $[c, d]$. *Assume that there is a* $u \in U$ *such that the corresponding differential equation does have a nontrivial solution with an* $(n - k, k)$ *pair of zeros on* $[c, d]$. *Then, if* $x(t)$ *is a time optimal such solution with zero of order* $n - k$ *at* $t = t_1$ *and zero of order* k *at* $t = t_2$ *with* $c \leqslant t_1 < t_2 \leqslant d$, *it follows that for the associated solution* $\psi(t) = (\psi_1(t),..., \psi_n(t))^T$ *of the corresponding adjoint system we have* $\psi_n(t) \neq 0$ *on* (t_1, t_2). *Then, since by Theorem* 4 $x(t) \neq 0$ *on* (t_1, t_2), *we can assume* $x(t) > 0$ *on* (t_1, t_2) *and it follows that* $x(t)$ *is either a solution of* (17) *or a solution of* (18) *on* $[t_1, t_2]$ *depending on whether* $\psi_n(t) < 0$ *or* $\psi_n(t) > 0$ *on* (t_1, t_2).

THEOREM 7. *Assume that in the collection of boundary value problems*

$$x^{(n)} = \sum_{j=1}^{n} u_j(t) x^{(j-1)}$$

$$x(t_1) = 0$$

$$x^{(i)}(t_2) = 0, \qquad 0 \leqslant i \leqslant n - 2,$$

where $u \in U$ *and* $c \leqslant t_1 < t_2 \leqslant d$, $x(t)$ *is a time optimal solution. Then* x *is a solution of* (17) *on* $[t_1, t_2]$ *if* n *is even and is a sulution of* (18) *on* $[t_1, t_2]$ *if* n *is odd.*

Proof. If $x(t)$ is a solution of $x^{(n)} = \sum_{j=1}^{n} u_1(t)\, x^{(j-1)}$ with $x(t_1) = 0$, $x^{(i)}(t_2) = 0$ for $0 \leqslant i \leqslant n - 2$, and $x(t) > 0$ on (t_1, t_2), then $y(t) \equiv x(-t)$ is a solution of

$$y^{(n)}(t) = \sum_{j=1}^{n} (-1)^{n+j-1} u_j(-t)\, y^{(j-1)}(t) \tag{23}$$

on $-b < t < -a$ with $y^{(i)}(-t_2) = 0$ for $0 \leqslant i \leqslant n - 2$, $y(-t_1) = 0$, and $y(t) > 0$ on $(-t_2, -t_1)$. Hence, if $x(t)$ is time optimal for a $(1, n - 1)$ pair of zeros on $[c, d] \subset (a, b)$, then $y(t) = x(-t)$ is time optimal for an $(n - 1, 1)$ pair of zeros on $[-d, -c] \subset (-b, -a)$, and conversely.

For the equation (23) the inequalities satisfied by the controls depend on whether n is even or odd. When n is even, we have

$$k_j(-t) \leqslant (-1)^{n+j-1}\, u_j(-t) \leqslant l_j(-t) \qquad \text{for odd } j$$

and $$-l_j(-t) \leqslant (-1)^{n+j-1}\, u_j(-t) \leqslant -k_j(-t) \qquad \text{for even } j,$$

and when n is odd, we have

$$-l_j(-t) \leqslant (-1)^{n+j-1} u_j(-t) \leqslant -k_j(-t) \qquad \text{for odd } j$$

and
$$k_j(-t) \leqslant (-1)^{n+j-1} u_j(-t) \leqslant l_j(-t) \qquad \text{for even } j.$$

Applying Theorem 3 on the interval $(-b, -a)$ we conclude that, if $y(t)$ is a time optimal solution of (23) with an $(n-1, 1)$ pair of zeros respectively at $t = -t_2$ and $t = -t_1$, then when n is even $y(t)$ is a solution of

$$y^{(n)} = k_1(-t)y + \tfrac{1}{2} \sum_{j=2}^{n} (-1)^{j+1}[l_j(-t) + k_j(-t)]\, y^{(j-1)} - \tfrac{1}{2} \sum_{j=2}^{n} [l_j(-t) - k_j(-t)]\, | y^{(j-1)} |$$

on $[-t_2, -t_1]$ and, when n is odd, $y(t)$ is a solution of

$$y^{(n)} = -l_1(-t)y + \tfrac{1}{2} \sum_{j=2}^{n} (-1)^{j}\, [l_j(-t) + k_j(-t)]\, y^{(j-1)} - \tfrac{1}{2} \sum_{j=2}^{n} [l_j(-t) - k_j(-t)]\, | y^{(j-1)} |$$

on $[-t_2, -t_1]$. When these equations are translated back in terms of $x(t) = y(-t)$ we obtain the desired conclusion.

5. Third and fourth Order Differential Equations

In this section we use the results of the previous sections to obtain subintervals of (a, b) on which all k-point boundary value problems, $2 \leqslant k \leqslant n$, for Eq. (1) have unique solutions in the cases where (1) is of order three or of order four.

THEOREM 8. *Assume that the equation*

$$y''' = f(t, y, y', y'') \tag{24}$$

satisfies conditions [A] *and* [E] *on* $(a, b) \times R^3$. *Assume that* $[c, d] \subset (a, b)$ *is such that for any* $c \leqslant t_0 < d$ *the solution* $x(t)$ *of the third order equation* (17) *with* $x(t_0) = x'(t_0) = 0$ *and* $x''(t_0) = 1$ *satisfies* $x(t) > 0$ *on* $(t_0, d]$ *and for any* $c < t_0 \leqslant d$ *the solution* $x(t)$ *of the third order equation* (18) *with* $x(t_0) = x'(t_0) = 0$ *and* $x''(t_0) = 1$ *satisfies* $x(t) > 0$ *on* $[c, t_0)$. *Then all* 2*-point and all* 3*-point boundary value problems for* (24) *on* (c, d) *have solutions and these solutions are unique.*

Proof. As observed in the Introduction it suffices to show that condition [D] with $n = 3$ is satisfied on (c, d). If condition [D] is not satisfied on (c, d), then as noted in Section 2 there is a $u \in U$ such that $x''' = \sum_{j=1}^{3} u_j(t)\, x^{(j-1)}$ has a nontrivial solution with three zeros on $[c, d]$. This implies that the same differential equation has a nontrivial solution with a (2, 1) pair of zeros on $[c, d]$ or a nontrivial solution with a (1, 2) pair of zeros on $[c, d]$. Thus there is either a time optimal solution with a (2, 1) pair of zeros on $[c, d]$ or a time optimal solution with a (1, 2) pair of zeros on $[c, d]$. In the first case the time optimal solution is a solution of (17) and in the second case is a solution of (18). Both of these cases are ruled out by the hypotheses of the Theorem.

COROLLARY. *Assume that $f(t, y, y', y'')$ is continuous and satisfies the Lipschitz condition*

$$| f(t, y, y', y'') - f(t, z, z', z'')| \leqslant K | y - z | + L | y' - z' | + M | y'' - z'' |$$

on $(a, b) \times R^3$ where K, L, and M are positive constants. Let $x(t)$ be the solution of the initial value problem

$$\begin{aligned} x''' &= -Kx - L | x' | - M | x'' | \\ x(0) &= x'(0) = 0, \qquad x''(0) = +1. \end{aligned} \tag{25}$$

Let $t = h$ be the first zero of $x(t)$ to the right of $t = 0$. Then on any open subinterval of (a, b) of length less than h all 2*-point and all* 3*-point boundary value problems for* (24) *have unique solutions. This corollary is essentially contained in Theorem* 4 *of reference* [4].

Proof. In the case of the specified Lipschitz condition the corresponding third order forms of equations (17) and (18) are respectively

$$x''' = -Kx - L | x' | - M | x'' |$$

and

$$x''' = Kx + L | x' | + M | x'' |.$$

Furthermore, since these equations are autonomous, in applying Theorem 8 we need only consider the solutions of the initial value problems (25) and

$$\begin{aligned} x''' &= Kx + L | x' | + M | x'' | \\ x(0) &= x'(0) = 0, \qquad x''(0) = 1. \end{aligned} \tag{26}$$

For the solution of (25) we are concerned with the first zero to the right of $t = 0$ and for the solution of (26) we are concerned with the first zero to the left of $t = 0$. However, if $x(t)$ is a solution of the initial value problem (26), $y(t) =$

$x(-t)$ is a solution of the initial value problem (25). Hence the interval length for solvability of boundary value problems in this case is determined by the first zero to the right of $t = 0$ of the solution of (25).

Since equations (17) and (18) themselves satisfy the Lipschitz condition [E], the results stated in Theorem 8 and its Corollary are best possible for differential equations (24) in which the function $f(t, y, y', y'')$ satisfies the stated Lipschitz condition. If $f(t, y, y', y'')$ satisfies the Lipschitz condition of the Corollary of Theorem 8 on $(a, b) \times R^3$, it is known that 2-point boundary value problems for (24) have unique solutions on subintervals of (a, b) of length less than h where h is the positive root of the equation

$$\frac{2}{81} Kh^3 + \frac{1}{6} Lh^2 + \frac{2}{3} Mh = 1,$$

see [8] for example. In the case $K = L = M = 1$ this yields $h = 1.1284$ as compared to the best possible result $h = 2.7353$ obtained from the Corollary.

THEOREM 9. *Assume that the equation*

$$y^{(4)} = f(t, y, y', y'', y''') \tag{27}$$

satisfies conditions [A] *and* [E] *on* $(a, b) \times R^4$. *Assume that the interval* $[c, d] \subset (a, b)$ *is such that*

(1) *For any* $c \leqslant t_0 < d$ *the solution* $x(t)$ *of the fourth order equation* (17) *with* $x(t_0) = x'(t_0) = x''(t_0) = 0$ *and* $x'''(t_0) = +1$ *satisfies* $x(t) > 0$ *on* $(t_0, d]$,

(2) *For any* $c < t_0 \leqslant d$ *the solution* $x(t)$ *of* (17) *with* $x(t_0) = x'(t_0) = x''(t_0) = 0$ *and* $x'''(t_0) = -1$ *satisfies* $x(t) > 0$ *on* $[c, t_0)$, *and*

(3) *There is no nontrivial solution of* (18) *with a* (2, 2) *pair of zeros on* $[c, d]$. *Then all k-point boundary value problems,* $2 \leqslant k \leqslant 4$, *for equation* (27) *have solutions on* (c, d) *and these solutions are unique.*

Proof. As was remarked in the proof of Theorem 8 it suffices to show that condition [D] is satisfied on (c, d). Then again as noted in Section 2, if this were not so, there would exist a $u \in U$ such that the associated equation $x^{(4)} = \sum_{j=1}^{4} u_j(t)\, x^{(j-1)}$ has a nontrivial solution with either a (3, 1), a (2, 2), or a (1, 3) pair of zeros on $[c, d]$. In any one of these cases there would exist a time optimal solution with the same type of pair of zeros on $[c, d]$. This being the case conditions (1) and (2) of Theorem 9 and Theorems 3 and 7 rule out the possibility of a nontrivial solution having a (3, 1) or a (1, 3) pair of zeros on $[c, d]$ for solutions of any equation $x^{(4)} = \sum_{j=1}^{4} u_j(t)\, x^{(j-1)}$ with $u \in U$.

On the other hand, if there is a $u \in U$ such that the equation $x^{(4)} = \sum_{j=1}^{4} u_j(t)\, x^{(j-1)}$ has a nontrivial solution with a (2, 2) pair of zeros on $[c, d]$, then there is a time optimal such solution $x(t)$ with associated zeros of order two at

t_1 and t_2, $c \leqslant t_1 < t_2 \leqslant d$. Then by Theorem 4 $x(t) \neq 0$ on (t_1, t_2) and we may assume that $x(t) > 0$ on (t_1, t_2). Furthermore, if $\psi(t) = (\psi_1(t),..., \psi_4(t))^T$ is the associated solution of the adjoint system paired with $x(t)$ by the Maximum Principle, then by the Corollary of Theorem 6 $\psi_4(t) \neq 0$ on (t_1, t_2). The Maximum Principle also asserts that $\sum_{j=1}^4 x^{(j)}(t)\, \psi_j(t)$ is a non-positive constant on $[t_1, t_2]$. By the condition (1) of Theorem 9 the order of the zero of $x(t)$ at $t = t_1$ is exactly 2 and by the condition (2) of Theorem 9 the order of the zero of $\psi(t)$ at $t = t_1$ is also exactly 2. Hence

$$\sum_{j=1}^{4} x^{(j)}(t_1)\, \psi_j(t_1) = x''(t_1)\, \psi_2(t_1) \neq 0$$

and, therefore, since $\sum_{j=1}^4 x^{(j)}(t_1)\, \psi_j(t_1)$ is non-positive it follows that $x''(t_1)\, \psi_2(t_1) < 0$. Then, since $x(t) > 0$ on (t_1, t_2), $x''(t_1) > 0$ and $\psi_2(t_1) < 0$. Finally, referring to the adjoint system one sees easily that $\psi_2(t_1) < 0$, $\psi_3(t_1) = 0$, and $\psi_4(t_1) = 0$ implies that $\psi_4(t) > 0$ on (t_1, t_2). Thus the time optimal solution $x(t)$ is a slution of (18) on $[t_1, t_2]$ but condition (3) of Theorem 9 rules this out. It follows that equation (27) satisfies condition [D] on (c, d) and the proof of Theorem 9 is complete.

COROLLARY. *Assume that Eq.* (27) *satisfies condition* [A] *and the Lipschitz condition*

$$\begin{aligned} &| f(t, y, y', y'', y''') - f(t, z, z', z'', z''')| \\ &\quad \leqslant K\,| y - z | + L\,| y' - z' | + M\,| y'' - z'' | + N\,| y''' - z''' | \end{aligned}$$

on $(a, b) \times R^4$. *Let* $x(t)$ *be the solution of the initial value problem*

$$x^{(4)} = -Kx - L\,| x' | - M\,| x'' | - N\,| x''' |$$

$$x(0) = x'(0) = x''(0) = 0, \qquad x'''(0) = 1.$$

If $x(t)$ *has a positive zero, let* $t = d_1$ *be its smallest positive zero; otherwise, let* $d_1 = +\infty$. *If the boundary value problem*

$$x^{(4)} = Kx + L\,| x' | + M\,| x'' | + N\,| x''' |$$

$$x(0) = x'(0) = 0, \qquad x(d) = x'(d) = 0$$

has a nontrivial solution for some $d > 0$, *let* d_2 *be the smallest* $d > 0$ *for which it has a nontrivial solution. If the boundary value problem has no nontrivial solution, let* $d_2 = +\infty$. *Then on any open subinterval of* (a, b) *of length less than* $d_3 = \operatorname{Min}\{d_1, d_2\}$ *all k-point boundary value problems,* $2 \leqslant k \leqslant 4$, *for Eq.* (27) *have solutions which are unique.*

Again the results contained in Theorem 9 and its Corollary are the best that can be obtained in terms of the Lipschitz coefficients.

To extend the results of Theorems 8 and 9 to equations of arbitrary order n we must determine for each integer k with $2 \leqslant k \leqslant n-2$ the sign of $\psi_n(t)$ where $\psi_n(t)$ is the last component of the solution $\psi(t)$ of the adjoint system associated with a solution $x(t)$ of $x^{(n)} = \sum_{j=1}^{n} u_j(t)\, x^{(j-1)}$ having a time optimal $(n-k, k)$ pair of zeros. The method for doing this used for the (2, 2) pair of zeros in the proof of Theorem 9 breaks down for equations of order greater than four. For example, if the equation is of order five and if $x(t)$ is a solution with a time optimal (3, 2) pair of zeros, the first zero in the (2, 3) pair of zeros of the associated solution $\psi(t)$ of the adjoint system might actually be of order 3 instead of 2.

However, it seems reasonable to conjecture that, if for a fixed integer k with $2 \leqslant k \leqslant n-2$ the interval $[c, d] \subset (a, b)$ has been determined so that for each $u \in U$ and each integer j with $1 \leqslant j \leqslant k-1$ the equation

$$x^{(n)} = \sum_{j=1}^{n} u_j(t) x^{(j-1)}$$

has no nontrivial solution with an $(n-j, j)$ or a $(j, n-j)$ pair of zeros on $[c, d]$ and if there is a $u \in U$ such that the corresponding equation has a nontrivial solution with an $(n-k, k)$ pair of zeros on $[c, d]$, then a time optimal such solution will be a solution of [17] if k is odd and will be a solution of (18) if k is even. This is equivalent to saying that between the $(k, n-k)$ pair of zeros of the solution $\psi(t)$ of the associated adjoint system sign $\psi_n(t) = (-1)^k$ where $\psi_n(t)$ is the last component of $\psi(t)$ and the time optimal solution $x(t)$ is assumed to be positive between its $(n-k, k)$ pair of zeros.

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