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# A New Discretization Algorithm of Linear Canonical Transform

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**Abstract**

In order to improve the computing accuracy of Linear Canonical Transform (LCT), a new algorithm is proposed in this paper to compute the LCT of a function by using the eigenfunctions of the LCT. The proposed algorithm is easily understanding and implementing. In addition, this algorithm has an approximation results of the continuous LCT.



*Keyword*: Possibility theory; linear canonical transform; hermite function; time-frequency analysis

## 1. Introduction

The linear canonical transform (LCT) is a three-parameter family of integral transform. It was first introduced in 1970s [1]. The LCT is a unitary, additive, affine and invertible transform. Many operations, such as the Fourier transform (FT), fractional Fourier transform (FRFT), Fresnel transform FST and chirp multiplication are all the special cases of the LCT. These integral transforms are of great importance in electromagnetic, acoustic, and other wave propagation problems. The LCT is widely applied in wave propagation problems and optimal filtering [2]. It is also useful for radar system analysis, filter design, phase retrieval, and many other applications [2-4]. Therefore, the accurate and efficient digital computation of the LCT is of great interest for many applications. Discrete counterparts of continuous transforms are important for approximately computing the samples of continuous transforms.

Recently some theories about the definition and fast computation of the discrete LCT have been derived. In general there are two basic approaches to derive the fast LCT. The first kind of algorithms to

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compute the DLCT was proposed [5], [6] by an approach similar to that used in deriving the FFT from the DFT. It had the same efficiency as the FFT in computing the FT. But this kind of algorithms of LCT does slightly complicate the implementation of the algorithm on a computer. The other method to compute the DLCT of a function was proposed [8]. This method is based on decomposition of the LCT into basic operations of scaling, FT, chirp multiplication, FRFT. This method gave two decomposition algorithms. The first algorithm decomposed the LCT into scaling, FT and chirp multiplication. The second decomposed the LCT into FRFT, scaling and chirp multiplication. Both algorithms take $N \log N$ time, where N is the number of the sample of the original function. However these methods might require sampling rates that are higher than the Nyquist rate [7-9], depending on the parameters and particular decomposition employed.

There remains much to be worked out in the theory of the DLCT. However, the accuracy of computing the LCT need to be improved. In this paper, we will introduce a definition of DLCT and discuss an algorithm for numerically computing the continuous LCT by using the eigenfunctions of LCT. The algorithm is easily understanding and implementing. In addition, it has an approximation results of the continuous LCT. This paper is organized as follows. In Section 2 we briefly review the LCT and the eigenfunctions of the LCT. In Section 3 we derive a new algorithm of LCT by using the eigenfunctions introduced in Section 2. Numerical examples to demonstrate the accuracy of the algorithm are given in Section 4. Finally we offer a conclusion in Section 5.

## 2. The linear canonical transform

The linear canonical transform of a signal $x(t)$ with parameter matrix $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as

$$\mathcal{L}^{a,b,c,d}[x(t)] = \int_{-\infty}^{\infty} K(t,u)x(t)dt = \begin{cases} \sqrt{\frac{1}{j2\pi b}} e^{\frac{jd}{2b}u^2} \int_{-\infty}^{\infty} e^{-\frac{j}{b}ut} e^{\frac{ja}{2b}t^2} x(t)dt, b\neq 0 \\ \sqrt{d}\, e^{\frac{jcd}{2}u^2} x(du) \qquad ,b=0 \end{cases} \tag{1}$$

where $a,b,c,d$ are real parameters independent of $t$ and $u$ and $ad-bc=1$. The unit-determinant matrix $M$ belongs to the class of unimodular matrices. Therefore, only three parameters are free. The LCT kernel $K(t,u)$ is defined as $K(t,u) = \sqrt{1/j2\pi b}\, \exp(j(at^2 - 2tu + du^2)/2b)$.

A number of important and familiar transforms are special cases of the LCT. These include the Fourier Transform (FT), the Fractional Fourier Transform (FRFT) and the Fresnel Transform (FST), as well as scaling and chirp multiplication. The LCT can extend their utilities and applications and can solve some problems that cannot be solved well by these operations.

In [10], S. C. Pei and J. J. Ding discussed the eigenfunctions of the LCT for the case where $|a+d|<2$. The LCT has the eigenfunctions

$$\phi_n^{(\sigma,\tau)}(t)=\frac{1}{\sqrt{\sigma\cdot 2^n n!\sqrt{\pi}}}\exp\left(-\frac{(1+j\tau)t^2}{2\sigma^2}\right)\cdot H_n\left(\frac{t}{\sigma}\right) \tag{2}$$

where $H_n(t)$ is the Hermite function of order $n$, $H_n(t)=(-1)^n \exp(t^2)\frac{d^n}{dt^n}(\exp(-t^2))$. The values of $\sigma,\tau$ and $\alpha$ are as follow

$$\sigma^2 = \frac{2|b|}{\sqrt{4-(a+d)^2}}, \tau = \frac{\mathrm{sgn}(b)(a-d)}{\sqrt{4-(a+d)^2}}, \alpha = \arccos\left(\frac{a+d}{2}\right)\arcsin\left(\frac{\mathrm{sgn}(b)}{2}\sqrt{4-(a+d)^2}\right) \tag{3}$$

And the corresponding eigenvalues are $\lambda_n=\sqrt{e^{-j\alpha}}e^{-j\alpha n}$. The eigenfunctions of the LCT in (2) have the orthogonality property $\int_{-\infty}^{\infty}\phi_m^{(\sigma,\tau)}\tilde{\phi}_n^{(\sigma,\tau)}dt=\delta_{m,n}$. In fact, in most of the case, (2) are the only possible eigenfunctions of the LCT when $|a+d|<2$ except for some difference of constant phase.

### 3. The discrete linear canonical transform

The eigenfunctions of the LCT in (2) have the orthogonality property. If the eigenfunctions are divided by their norm, we can easily get a set of normal orthogonal functions. Let us assume that their norm is one. In order to find the LCT of the input signal $x(t)$ ,we rewrite $x(t)$ as

$$x(t)=\sum_{n=0}^{\infty}X_n\phi_n^{\sigma,\tau}(t) \tag{4}$$

where $X_n=\int\phi_n^{\sigma,\tau}(t)\,x(t)dt$. Let $\mathcal{L}^{a,b,c,d}$ act on the two sides of the (4) and we have

$$\mathcal{L}^{a,b,c,d}[x(t)]=\mathcal{L}^{a,b,c,d}[\sum_{n=0}^{\infty}\int\phi_m^{\sigma,\tau}(t)x(t)dt\phi_n^{\sigma,\tau}(t)]=\int\sum_{n=0}^{\infty}\sqrt{e^{-j\alpha}}\phi_n^{\sigma,\tau}(u)\phi_n^{\sigma,\tau}(t)x(t)dt \tag{5}$$

In comparison with the definition of the LCT in (1),we can get the expression of the kernel of the LCT

$$K(t,u)=\sum_{n=0}^{\infty}\sqrt{e^{-j\alpha}}e^{-j\alpha n}\phi_n^{\sigma,\tau}(t)\phi_n^{\sigma,\tau}(u) \tag{6}$$

According to (6), the expression of the LCT in (1) can be rewritten as,

$$\mathcal{L}^{a,b,c,d}[x(t)]=\int_{-\infty}^{\infty}K(t,u)x(t)dt=\sum_{n=0}^{\infty}\phi_n^{\sigma,\tau}(u)(e^{-j\alpha(\frac{1}{2}+n)})\int_{-\infty}^{\infty}x(t)\phi_n^{\sigma,\tau}(t)dt \tag{7}$$

Providing the signal $x(t)$ and it's LCT $X(u)$ are respectively sampl at a rate $1/T_S=\sqrt{N/2\pi}$ and $1/U_s=\sqrt{N/2\pi}$ , where $N$ is the number of the sample of $x(t)$ . The sampling interval is $[-\sqrt{N\pi/2},\sqrt{N\pi/2}]\times[-\sqrt{N\pi/2},\sqrt{N\pi/2}]$. Then we get

$$\mathcal{L}^{a,b,c,d}[x(t)]=X(u)=\sum_{n=0}^{\infty}\phi_n^{\sigma,\tau}(u)e^{-j\alpha(\frac{1}{2}+n)}\int_{-\sqrt{N\pi/2}}^{\sqrt{N\pi/2}}x(t)\phi_n^{\sigma,\tau}(t)dt|_{T_S=\sqrt{2\pi/N}}$$

$$=\sum_{n=0}^{\infty}\phi_n^{\sigma,\tau}(u)\exp(-j\alpha(\frac{1}{2}+n))\sum_{k=-(N-1)/2}^{(N-1)/2}x(kT_s)\phi_n^{\sigma,\tau}(kT_s)T_s$$

$$=\sum_{n=0}^{\infty}\exp(-j\alpha(\frac{1}{2}+n))\phi_n^{\sigma,\tau}(u)T_S\Phi_n^TX_N|_{u=\left[-\sqrt{N\pi/2},\sqrt{N\pi/2}\right]}^{U_S=\sqrt{2\pi/N}}$$

$$=\sum_{n=0}^{N-1}\exp(-j\alpha(\frac{1}{2}+n))T_S\Phi_n\Phi_n^TX_N+\sum_{n=N}^{\infty}\exp(-j\alpha(\frac{1}{2}+n))T_S\Phi_n\Phi_n^TX_N$$

$$\approx T_s(\sum_{n=0}^{N-1}\exp(-j\alpha(\frac{1}{2}+n))\Phi_n\Phi_n^T)X_N=T_s\Phi_ND^{\alpha}\Phi_N^TX_N \tag{8}$$

Therefore $\mathcal{L}^{a,b,c,d}[x(t)]=T_s\Phi_ND^{\alpha}\Phi_N^TX_N$. Where

$$\Phi_n=[\phi_n(-\frac{N-1}{2}),\phi_n(-\frac{N-1}{2}+1),\cdots,\phi_n(\frac{N-1}{2})],n=0,1,\cdots,N-1 \tag{9}$$

It is the N-point sampling column vector of the function $\phi_n^{\sigma,\tau}(t)$. And $\Phi_N$ is the discrete matrix of the function $\phi_n^{\sigma,\tau}(t)$ with the sampling length $N$ . $X_N$ is the N-point column vector of the signal $x(t)$ , $X_N=[x(-\frac{N-1}{2}),x(-\frac{N-1}{2}+1),\cdots,x(\frac{N-1}{2})]$ . $D^{\alpha}=diag(e^{-j\alpha/2},e^{-j3\alpha/2},\cdots,e^{-j(2N-1)\alpha/2})$ . In the above derivation process, there is a rounding errors items $\sum_{n=N}^{\infty}\exp(-j\alpha(\frac{1}{2}+n))T_s\Phi_n\Phi_n^T X_N$ . When $N$ is sufficiently large, this summation item tends to zero and this rounding errors can't affect the calculation precision. Thus we only need to calculate the multiply of $\Phi_N, D^{\alpha}$ and $X_N$ . Then we can get the LCT of the discrete signal $X_N$ . It takes $N^2$ times to calculate $\Phi_N$ . This algorithm use the discretization method to get the discrete matrix of the kernel function of the LCT. It has an approximation results of the continuous LCT. The algorithm can be given straightforwardly.

**Input:** The N-point column vector $X_N$ of the signal $x(t)$ , and the parameters $a,b,c,d$ .

**Output:** The DLCT of the signal $X_N$ with parameters $a,b,c,d$ , approximating the continuous LCT.

*One:* Calculate the values of $\sigma,\tau$ and $\alpha$ in (3).

*Tow:* Calculate the N-point sampling column vector $\Phi_n$ of the function $\phi_n^{\sigma,\tau}(t)$ in (9). Then we can get the matrix $\Phi_N=[\Phi_0^T,\Phi_0^T,\cdots,\Phi_{N-1}^T]$ .

*Three:* Calculate the diagonal matrix $D^{\alpha}$ .

*Four:* Calculate the DLCT of the signal $X_N$ , $X_u=T_s\Phi_N D^{\alpha}\Phi_N^T X_N$ , where $T_s=\sqrt{2\pi/N}$ .

## 4. Simulation results

In order to verify the correctness of the algorithm proposed in this paper, we exhibit two examples in this section. We begin by implementing the DFRFT of an input rectangular signal applying the algorithm in this paper. The sampling vector of the signal is $X_N=[x_{-36},x_{-35},\cdots,x_{36}]$, and the sampling length is N=73. The $x_k$ is equal to 1 in the range $-6 \le k \le 6$ and is equal to 0 everywhere else. We apply the algorithm to calculate the DFRFT of the signal with angle $\theta=0.105$ . The resulting magnitudes about the real part and imaginary part are shown in Fig. 1(a). Then we apply this algorithm to calculate the DLCT of the rectangular signal mentioned above. We set $a=1/2$, $b=6/5$, $c=-5/6$, $d=0$ in the LCT. The magnitudes of the resulting discrete function are shown in Fig.1(b).

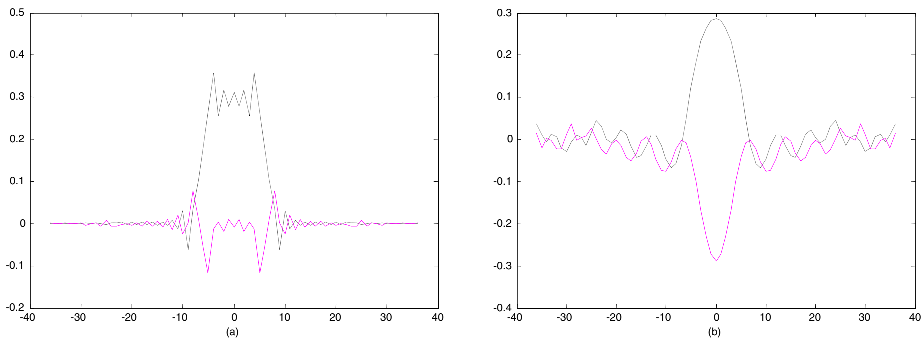


Fig. 1. (a) Magnitude of discrete FRFT; (b) Magnitude of DLCT (the solid line represents the real parts, and the dotted line represents the imaginary parts)

## 5. Conclusions

In this paper, we discuss an approach for the digital computation of the LCT based on the eigenfunctions. With careful consideration of the eigenfunctions of the LCT, we derive an expression of the kernel function of the LCT. According to this expression, the LCT can be expressed in terms of a new definition which, unlike certain earlier definitions, is closely related with the eigenfunctions and eigenvalues of the LCT. Based on the principle of sampling in time and linear canonical transform domains, a new definition of DLCT is put forward. Then we only need to calculate the discrete matrix of the eigenfunction of the LCT and a diagonal matrix to compute the DLCT. This algorithm is significant since it has an approximation result of the continuous LCT. Compared to earlier approaches, this algorithm is more accuracy in computing the continuous LCT. It takes $N^2$ times, where $N$ is the number of samples of the input signal. Therefore, the future work is to accelerate the calculation. There remains much work to be done in improving the efficient and accurate computation of the LCT.

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