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Discontinuities of BFKL amplitudes and the BDS ansatz ☆

V.S. Fadin [a,b], R. Fiore [c,*]

[a] *Budker Institute of Nuclear Physics of SD RAS, 630090 Novosibirsk, Russia*
[b] *Novosibirsk State University, 630090 Novosibirsk, Russia*
[c] *Dipartimento di Fisica, Università della Calabria, and Istituto Nazionale di Fisica Nucleare, Gruppo collegato di Cosenza, Arcavacata di Rende, I-87036 Cosenza, Italy*

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Abstract

We perform an examination of discontinuities of multiple production amplitudes, which are required for further development of the BFKL approach. It turns out that the discontinuities of $2 \to 2 + n$ amplitudes obtained in the BFKL approach contradict to the BDS ansatz for amplitudes with maximal helicity violation in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory with large number of colors starting with $n = 2$. Explicit expressions for the discontinuities of the $2 \to 3$ and $2 \to 4$ amplitudes in the invariant mass of pairs of produced gluons are obtained in the planar $N = 4$ SYM in the next-to-leading logarithmic approximation. These expressions can be used for checking the conjectured duality between the light-like Wilson loops and the MHV amplitudes.

1. Introduction

The BFKL (Balitsky–Fadin–Kuraev–Lipatov) approach [1–4] is based on the multi-Regge form of scattering amplitudes with gluon quantum numbers in all cross-channels. For the ampli-

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* Corresponding author.
E-mail addresses: fadin@inp.nsk.su (V.S. Fadin), roberto.fiore@cs.infn.it (R. Fiore).

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tude $\mathcal{A}_{2\to n+2}$ of the process $A+B\to A'+G_1+\cdots+G_n+B'$ of production of n gluons with momenta $k_1,k_2,\ldots,k_n$ in the multi-Regge kinematics (MRK) this form can be written as

$$\Re\mathcal{A}_{2\to n+2} = 2s\Gamma^{R_1}_{A'A}\left(\prod_{i=1}^{n}\frac{1}{t_i}\left(\frac{s_i}{|\vec{k}_{i-1}||\vec{k}_i|}\right)^{\omega(t_i)}\gamma^{G_i}_{R_iR_{i+1}}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{|k_n||\vec{q}_{n+1}|}\right)^{\omega(t_{n+1})}\Gamma^{R_{n+1}}_{B'B}, \tag{1.1}$$

where $\omega(t)$ is called gluon trajectory (in fact, the trajectory is $1+\omega(t)$), $\Gamma^R_{A'A}$ and $\Gamma^R_{B'B}$ are the particle–particle–Reggeon (PPR) vertices, or the scattering vertices, and $\gamma^{G_i}_{R_iR_{i+1}}$ are the Reggeon–Reggeon–gluon (RRG) vertices, or the production vertices. Moreover,

$$s=(p_A+p_B)^2,\ s_i=(k_{i-1}+k_i)^2,\ \ i=1,\ldots,n+1,\ \ k_0\equiv P_{A'},\ \ k_{n+1}\equiv P_{B'},$$
$$q_1=p_A-p'_A,\ q_{j+1}=q_j-k_j,\ j=1,\ldots,n,\ \ q_{n+1}=p_{B'}-p_B\ , \tag{1.2}$$

the vector sign means transverse to the p_A, p_B plane components. In the MRK

$$s\gg s_i\gg|t_i|\simeq\vec{q}_i^{\,2},\qquad s\simeq\frac{\prod_{i=1}^{n+1}s_i}{\prod_{i=1}^{n}\vec{k}_i^{\,2}}. \tag{1.3}$$

The Reggeon vertices and the gluon trajectory are known in the next-to-leading order (NLO), that means the one-loop approximation for the vertices and the two-loop approximation for the trajectory, in *SYM* as well as in QCD. It is just the accuracy which is required for the derivation of the BFKL equation in the next-to-leading logarithmic approximations (NLLA), taking into account all radiative corrections of the type $\alpha_s(\alpha_s\ln s)^n$. To be precise, note that in this approximation one has to consider not only the amplitudes (1.1), but also amplitudes obtained from them by replacement of one of final particles by a couple of particles with fixed (of order of transverse momenta) invariant mass.

The sign $\Re$ in the Eq. (1.1) means the real part. It is important that this simple factorized form is valid only for the real part of the amplitudes. Fortunately, the imaginary parts are not essential for the derivation of the BFKL equation in the NLLA, because they are suppressed by one power of $\ln s_i$ in comparison with the real ones, and with the NLLA accuracy do not contribute in the unitarity relations. But understanding of properties of the imaginary parts which are associated with the discontinuities in the variables $s_{ij}=(k_i+k_j)^2$ is very important. First, it is necessary for the justification of the BFKL approach, that means a proof of the multi-Regge form of multiple production amplitudes. Second, account of the imaginary parts is indispensable in further development of the BFKL approach. As it was pointed above, they are not essential for derivation of the BFKL equation in the NLLA, but they must be taken into account in the NNLLA.

The idea of the multi-Regge form appeared in Refs. [1,5] from results of fixed order calculations. Later it was proved in the leading logarithmic approximation (LLA) [6] with use of the s-channel unitarity. The proof of the multi-Regge form in the NLLA is based also on the s-channel unitarity [7].

Here it is necessary to recall that as compared with ordinary particles, Reggeons in the Regge–Gribov theory of complex angular momenta possess an additional quantum number, called signature. At large s_i the signature means parity with respect to the substitution $s_i\to -s_i$. The signature of the Reggeized gluon is negative, and the real part of the amplitude presented in Eq. (1.1) coincides with the real part of the amplitude $\mathcal{A}^{\{-\}}_{2\to 2+n}$ with the Reggeized gluons (and, consequently, with the negative signatures) in all t_i channels. Amplitudes with the positive signature in the s_i-channel are suppressed because of the cancellation of leading powers of $\log s_i$, so that with the NLLA accuracy $\Re\mathcal{A}_{2\to 2+n} = \Re\mathcal{A}^{\{-\}}_{2\to 2+n}$.

Compatibility of unitarity with the multi-Regge form leads to the bootstrap relations [8] connecting discontinuities of the amplitudes with products of their real parts and gluon trajectories:

$$\frac{1}{-\pi i}\left(\sum_{l=j+1}^{n+1} \mathrm{disc}_{s_{j,l}} - \sum_{l=0}^{j-1} \mathrm{disc}_{s_{l,j}}\right)\mathcal{A}^{\{-\}}_{2\to n+2} = \big(\omega(t_{j+1}) - \omega(t_j)\big)\,\Re\mathcal{A}_{2\to n+2}\,. \tag{1.4}$$

Here $\Re\mathcal{A}_{2\to n+2}$ is the multi-Regge form (1.1) and the s_{ij}-channel discontinuities must be calculated using this form into the unitarity conditions. Note that for multi-particle amplitudes the discontinuities are not pure imaginary, since a discontinuity in one of the channels can have, in turn, a discontinuity in another channel. But these double discontinuities are sub-sub-leading, so that they are neglected in Eq. (1.4) and in the following.

It turns out [7] that the fulfillment of an infinite set of the relations (1.4) guarantees the multi-Regge form of scattering amplitudes and that all bootstrap relations are fulfilled if several conditions imposed on the Reggeon vertices and the trajectory (bootstrap conditions) hold true. The most complicated condition, which includes the impact factors for Reggeon–gluon transition, was proved recently, both in QCD [9–11] and in its supersymmetric generalizations [12].

The proof that the fulfillment of the bootstrap relations (1.4) is ensured by the bootstrap conditions is based on the form of the discontinuities derived from the unitarity in Ref. [7]. Besides of the Reggeon vertices and the trajectory entering in Eq. (1.1), the discontinuities contain as building blocks the impact factors for particle–particle and Reggeon–particle transitions, the kernel of the BFKL equation and the four-Reggeon gluon production vertex. In fact, the bootstrap conditions are conditions on these building blocks. But since the impact factors for particle–particle and Reggeon–particle transitions, the kernel of the BFKL equation and the four-Reggeon gluon production vertex are expressed in terms of the Reggeon vertices and the trajectory, one can say that the bootstrap conditions are imposed on the Reggeon vertices and the trajectory.

The expressions for discontinuities obtained in Ref. [7] are rather formal, since the impact factors, the kernel and the four-Reggeon gluon production vertex are not given explicitly. In this paper we obtain explicit expressions for the discontinuities of multiple production amplitudes in $N = 4$ SYM with large number of colors (in the planar approximation). Consideration of the discontinuities in this theory is also interesting for two reasons. First, it provides a simple demonstration of imperfection of the BDS (Bern–Dixon–Smirnov) ansatz [13,14] M_{BDS} for multi-particle amplitudes with maximal helicity violation (MHV amplitudes). Second, the discontinuities can be used for the verification of the hypotheses used for the calculation of corrections to this ansatz. It is believed (but not yet proved) that the true amplitudes can be presented as the product of M_{BDS} and the remainder function R, where M^{BDS} contains all infrared divergences and R depends only on the anharmonic ratios of kinematic invariants [15–21]. This property is called dual conformal invariance. Another property is the conjecture (also not yet proved) of correspondence between the MHV amplitudes and expectation values of Wilson loops [19,20,22–25]. All this makes important the direct calculation of the discontinuities.

The paper is organized as follows. In the next Section we introduce the notation, give the general expression for the discontinuities and use it for the calculation of the discontinuity of the amplitude $\mathcal{A}^{\{-\}}_{2\to 2}$. Discontinuities of the amplitude $\mathcal{A}^{\{-\}}_{2\to 3}$ are found in Section 3. Section 4 is devoted to the calculation of discontinuities of the amplitude $\mathcal{A}^{\{-\}}_{2\to 4}$. Discontinuities of amplitudes with a larger number of final particles are considered in Section 5. Conclusions are drawn in Section 6. Appendices A, B and C contain some details of calculations.

2. Definitions, notation and the $\mathcal{A}^{\{-\}}_{2\to 2}$ discontinuity

Let us first present explicit forms of the gluon trajectory and the Reggeon vertices in $N=4$ SYM with the accuracy up to terms vanishing in the limit $\epsilon \to 0$. In the NLO the vertices, as well as the impact factors, are scheme-dependent. We will use the scheme introduced in Ref. [26] and then developed in Ref. [7], which we call standard one. But since usual dimensional regularization is incompatible with supersymmetry, we will use the dimensional reduction instead of the dimensional regularization. The NLO trajectory is given by [27–35]

$$\omega(t) = -2\bar{g}^2\left(\frac{1}{\epsilon}+\ln(-t)\right)+2\bar{g}^4\left[\zeta(2)\left(\frac{1}{\epsilon}+2\ln(-t)\right)-\zeta(3)\right], \tag{2.1}$$

where $\zeta(n)$ is the Riemann zeta-function,

$$\bar{g}^2 = \frac{g^2 N_c \Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}}, \qquad \epsilon = \frac{D-4}{2}, \tag{2.2}$$

$\Gamma(x)$ being the Euler gamma-function, and D is the space–time dimension.

For the gluon polarization vectors in the Reggeon vertices and impact factors we will use the L and R light-cone gauges $(e^L n_2)=0$ and $(e^R n_1)=0$ respectively, with the light-cone vectors n_2 and n_1 such that

$$(n_1 n_2)=1, \;\; (p_A p_B) \simeq (p_A n_2)(p_B n_1) . \tag{2.3}$$

Then,

$$e^L = e^L_\perp - \frac{(e^L_\perp k_\perp)}{kn_2} n_2 , \qquad e^R = e^R_\perp - \frac{(e^R_\perp k_\perp)}{kn_1} n_1 . \tag{2.4}$$

Note that the transverse parts of the polarization vectors in the left and right gauges are different. It is easy to see that the polarization vectors are connected by the gauge transformation:

$$e^L = e^R - 2\frac{(e^R_\perp k_\perp)}{k^2_\perp} k , \qquad e^R = e^L - 2\frac{(e^L_\perp k_\perp)}{k^2_\perp} k . \tag{2.5}$$

For transverse components this means

$$e^L_{\perp\mu} = \Omega_{\mu\nu} e^{R\,\nu}_\perp , \qquad e^R_{\perp\mu} = \Omega_{\mu\nu} e^{L\,\nu}_\perp , \tag{2.6}$$

where

$$\Omega_{\mu\nu} = \Omega_{\nu\mu} = g^\perp_{\mu\nu} - 2\frac{k_{\perp\mu}k_{\perp\nu}}{k^2_\perp} , \qquad \Omega_{\mu\nu}\Omega^{\nu\rho} = g^\rho_\mu . \tag{2.7}$$

Using the results of Refs. [28,36,37] and [38] for the one-loop gluon, quark and scalar corrections correspondingly, for the gluon–gluon–Reggeon vertex we have

$$\Gamma^R_{G'G} = g T^R_{G'G}(\vec{e}\,^{*\prime}\vec{e}\,)\left[1+\bar{g}\,^2(\vec{q}\,^2)^\epsilon\left(-\frac{2}{\epsilon^2}+5\zeta(2)\right)\right]. \tag{2.8}$$

Here q is the Reggeon momentum, $\vec{e}$ and $\vec{e}\,^{*\prime}$ are the polarization vectors of the initial and final gluons G and G' respectively (they have to be taken in the same gauge), $T^R_{G'G}$ is the color group generator in the adjoint representation. For simplicity, here and in the following we use for color indices the same letters as for particles and Reggeons.

The Reggeon—Reggeon–gluon vertex was obtained in the Born approximation in Ref. [5] and looks as

$$\gamma^{G(B)}_{R_1R_2} = gT^{G}_{R_1R_2}e^{*}_{\mu}(k)C^{\mu}(q_2,q_1)\,, \tag{2.9}$$

where

$$\begin{aligned} C_{\mu}(q_2,q_1) &= -q_{1\mu} - q_{2\mu} + p_{1\mu}\left(\frac{q_1^2}{kp_1} + 2\frac{kp_2}{p_1p_2}\right) - p_{2\mu}\left(\frac{q_2^2}{kp_2} + 2\frac{kp_1}{p_1p_2}\right) \\ &= -q_{1\perp\mu} - q_{2\perp\mu} - \frac{p_{1\mu}}{2(kp_1)}\left(k_\perp^2 - 2q_{1\perp}^2\right) + \frac{p_{2\mu}}{2(kp_2)}\left(k_\perp^2 - 2q_{2\perp}^2\right). \end{aligned} \tag{2.10}$$

The vertex is gauge invariant, being $C_{\mu}(q_2,q_1)k_{\mu} = 0$. In the light cone gauges (2.4) we get

$$e^{*}_{\mu}(k)C_{\mu}(q_2,q_1) \;=\; e_\perp^{L*}C_\perp^{L}(q_2,q_1) = e_\perp^{R*}C_\perp^{R}(q_2,q_1)\;, \tag{2.11}$$

where

$$\begin{aligned} C_\perp^{L}(q_2,q_1) &= C_\perp(q_2,q_1) - \frac{n_2C(q_2,q_1)}{kn_2}k_\perp = -2\left(q_{1\perp} - k_\perp\frac{q_{1\perp}^2}{k_\perp^2}\right), \\ C_\perp^{R}(q_2,q_1) &= C_\perp(q_2,q_1) - \frac{n_1C(q_2,q_1)}{kn_1}k_\perp = -2\left(q_{2\perp} + k_\perp\frac{q_{2\perp}^2}{k_\perp^2}\right). \end{aligned} \tag{2.12}$$

It makes sense to note that using the light-cone gauges does not mean loss of generality. One can restore any vertex in a gauge invariant form from its form in one of the gauges (2.4). Let us demonstrate it here for the vertex (2.9), denoting $C(q_2,q_1)$ there as C for brevity. Note that C can be changed by adding terms proportional to k without changing the vertex (2.9), as well as $C_\perp^L$ and $C_\perp^R$ defined in formulas (2.12), and without loss of the gauge invariance. Let us choose these terms in such a way that C goes to C^L subject to the condition $(C^L n_2) = 0$. Then, we have

$$C^L = C_\perp^L + \frac{n_1C^L}{n_1n_2}n_2\,. \tag{2.13}$$

On the other hand, from $kC^L = 0$ we have

$$C_\perp^L k_\perp + \frac{(n_1C^L)(n_2k)}{n_1n_2} = 0\,, \tag{2.14}$$

so that

$$C^L = C_\perp^L - \frac{C_\perp^L k_\perp}{kn_2}n_2\,. \tag{2.15}$$

As it has been said, using in Eq. (2.9) C^L instead of C does not change the vertex leaving it gauge invariant. Thus, we obtain the gauge-invariant form of the vertex from its form in the light-cone gauge. Using the relations (2.12) one can see that C^L is equal to $C - k$, where C is the original form given by Eq. (2.10).

One-loop gluon corrections to the vertex were calculated in Refs. [36,39–41]. In the last paper they were obtained at arbitrary $D = 4 + 2\epsilon$ dimension. With the same accuracy, the quark and scalar corrections were obtained in Refs. [42] and [35] respectively. In the $N = 4$ SYM, with the accuracy resulting when the terms singular at small $\vec{k}$ are given at arbitrary D, but the other terms in the limit $\epsilon \to 0$, we have in the dimensional reduction

$$\gamma^{G}_{R_1 R_2}(q_1, q_2) = \gamma^{G(B)}_{R_1 R_2}(q_1, q_2)\left(1 - \bar{g}^2\left[\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2} - \frac{\pi^2}{2} + \frac{1}{2}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)\right]\right). \quad (2.16)$$

A general representation for the discontinuities was derived in Ref. [7] (it is presented also in Ref. [43]). The discontinuity of $\mathcal{A}^{\{-\}}_{2\to n+2}$ in the $s_{i,j}$-channel is represented as

$$-4i(2\pi)^{D-2}\delta(q_{i\perp} - q_{(j+1)\perp} - \sum_{l=i}^{l=j} k_{l\perp})\,\mathrm{disc}_{s_{i,j}}\,\mathcal{A}_{2\to n+2}$$
$$= \frac{\Gamma^{R_1}_{A'A}}{t_1}\left(\frac{s_1}{|\vec{q}_1||\vec{k}_1|}\right)^{\omega(t_1)}\left(\prod_{l=2}^{i}\frac{\gamma^{G_{l-1}}_{R_{l-1}R_l}}{t_l}\left(\frac{s_l}{|\vec{k}_{l-1}||\vec{k}_l|}\right)^{\omega(t_l)}\right)$$
$$\times\langle G_i R_i|\left(\prod_{l=i+1}^{j-1}\left(\frac{s_l}{|\vec{k}_{l-1}||\vec{k}_l|}\right)^{\hat{\mathcal{K}}}\hat{\mathcal{G}}_l\right)\left(\frac{s_j}{|\vec{k}_{j-1}||\vec{k}_j|}\right)^{\hat{\mathcal{K}}}|G_j R_{j+1}\rangle$$
$$\times\left(\prod_{l=j+1}^{n}\left(\frac{s_l}{|\vec{k}_{l-1}||\vec{k}_l|}\right)^{\omega(t_l)}\frac{\gamma^{G_l}_{R_l R_{l+1}}}{t_l}\right)\left(\frac{s_{n+1}}{|\vec{k}_n||\vec{q}_{n+1}|}\right)^{\omega(t_{n+1})}\frac{\Gamma^{R_{n+1}}_{B'B}}{t_{(n+1)}}. \quad (2.17)$$

Here the bra- and ket-states $\langle G_i R_i|$ and $|G_j R_{j+1}\rangle$ denote the impact factors for the Reggeon–gluon transitions, $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{G}_l}$ are the operators of the BFKL kernel and the gluon production, which acts in the space of states $|\mathcal{G}_1\mathcal{G}_2\rangle$ of two t-channel Reggeons with the orthonormality property

$$\langle\mathcal{G}_1'\mathcal{G}_2'|\mathcal{G}_1\mathcal{G}_2\rangle = \vec{r}_1^{\,2}\vec{r}_2^{\,2}\delta(\vec{r}_1 - \vec{r}_1^{\,\prime})\delta(\vec{r}_2 - \vec{r}_2^{\,\prime})\delta_{c_1c_1'}\delta_{c_2c_2'}, \quad (2.18)$$

where $\vec{r}_i$ and $\vec{r}_i^{\,\prime}$ are the Reggeon transverse momenta and c_i and c_i' are their color indices. The operators are specified by their matrix elements and the states are defined by their projections on the two-Reggeon states.

If $i = 0$ we must omit all factors to the left of $\langle G_0 R_0|$ and replace $\langle G_0 R_0|$ by the impact factors of $A \to A'$ transition $\langle A'A|$ and $k_0 - q_0$ by $p_{A'} - p_A$; in the case $j = n+1$ we must omit all factors to the right of $|G_{n+1}R_{n+2}\rangle$ and perform the substitutions $|GJ_{n+1}R_{n+2}\rangle \to |B'B\rangle$, $k_{n+1} + q_{n+2} \to p_{B'} - p_B$.

For the discontinuity $\mathrm{disc}_s\,\mathcal{A}_{2\to 2} = \mathcal{A}_{2\to 2}(s + i0) - \mathcal{A}_{2\to 2}(s - i0)$ we have

$$-4i(2\pi)^{D-2}\delta(\vec{q} - \vec{q}_B)\,\mathrm{disc}_s\,\mathcal{A}_{2\to 2} = 2s\langle A'A|e^{\hat{\mathcal{K}}\ln\left(\frac{s}{\vec{q}^{\,2}}\right)}|B'B\rangle, \quad (2.19)$$

where $\vec{q} = p_A - p_{A'}$, $\vec{q}_B = p_{B'} - p_B$.

We have to pay attention here on the fundamental difference between the sense of the representation (2.19) used here and that of the formally quite similar representation of the discontinuities of amplitudes with the Pomeron exchange. The BFKL Pomeron means the positive signature and the color singlet in the t-channel, while the amplitudes considered in this paper are the amplitudes with the negative signature and the adjoint representation of the color group in the t-channel. The gluon Reggeization makes the discontinuities (2.19) much simpler than the discontinuities of amplitudes with the Pomeron exchange. Indeed, the bootstrap conditions of the gluon Reggeization [7] tells us that

$$\langle A'A| = \Gamma^R_{A'A} g \langle R_\omega(\vec{q})| \,, \qquad |B'B\rangle = g|R_\omega(\vec{q}_B)\rangle \Gamma^R_{B'B} \,, \tag{2.20}$$

$$\hat{\mathcal{K}}|R_\omega(\vec{q})\rangle = \omega(t)|R_\omega(\vec{q})\rangle \,, \tag{2.21}$$

$$\frac{g^2\vec{q}^{\,2}}{2(2\pi)^{D-1}}\langle R'_\omega(\vec{q}^{\,\prime})|R_\omega(\vec{q})\rangle = \delta_{R'R}\delta(\vec{q}-\vec{q}^{\,\prime})\omega(t) \,, \tag{2.22}$$

where $\Gamma^R_{A'A}$ and $\Gamma^R_{B'B}$ are the scattering Reggeon vertices entering in the form (1.1), $|R_\omega(\vec{q})\rangle$ is the process independent eigenstate of the kernel $\hat{\mathcal{K}}$ with eigenvalue $\omega(t)$ and normalization (2.22). It is transformed according to the adjoint representation of the color group. In the right side of Eq. (2.22) R' and R are the color indices of the eigenstates; in Eq. (2.20) summation over the color indices R is assumed. Note that the bra and ket vectors are related by the left–right substitution, where $A \leftrightarrow B$, $A' \leftrightarrow B'$, $n_1 \leftrightarrow n_2$, that means, in particular, replacement of the left and right gauges.

Fulfillment of the bootstrap conditions (2.20)–(2.22) was proved in the NLO both in QCD [44,45] and SYM [38]. Using these conditions we have from the representation (2.19)

$$\text{disc}_s\,\mathcal{A}_{2\to 2} = -i\pi\left(\frac{2s}{t}\right)\omega(t)\Gamma^R_{A'A}\left(\frac{s}{\vec{q}^{\,2}}\right)^{\omega(t)}\Gamma^R_{B'B} \,. \tag{2.23}$$

It is easy to see that the result (2.23) with account of the form (1.1) at $n=0$ is in agreement with the bootstrap relation (1.4).

Finally, let us present the eigenstate $|R_\omega(\vec{q})\rangle$. From Refs. [44] (see also Ref. [9]) and [12] we obtain with the accuracy up to terms vanishing in the limit $\epsilon \to 0$

$$\langle \mathcal{G}_1\mathcal{G}_2|R_\omega(\vec{q})\rangle = \delta(\vec{q}-\vec{r}_1-\vec{r}_2)T^R_{\mathcal{G}_1\mathcal{G}_2}\left(1+\bar{g}^2\left[-\zeta(2)-\frac{1}{2}\ln\left(\frac{\vec{r}_1^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{q}^{\,2}}\right)\right]\right) \,, \tag{2.24}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the momenta of the Reggeons $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively.

It is necessary to note here that the accuracy of Eq. (2.24) does not provide preservation of nonvanishing in the limit $\epsilon \to 0$ terms of the $\bar{g}^2$ order in the product

$$\langle R'_\omega(\vec{q}^{\,\prime})|R_\omega(\vec{q})\rangle = \sum_{\mathcal{G}_1\mathcal{G}_2}\int \langle R'_\omega(\vec{q}^{\,\prime})|\mathcal{G}_1\mathcal{G}_2\rangle \frac{d\vec{r}_1 d\vec{r}_2}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\delta(\vec{q}-\vec{r}_1-\vec{r}_2)\langle \mathcal{G}_1\mathcal{G}_2|R_\omega(\vec{q})\rangle$$

(the summation here is performed over color states of the Reggeons $\mathcal{G}_1$ and $\mathcal{G}_2$) because of the infrared divergency of the integration measure. To provide the preservation one has to keep in $\langle \mathcal{G}_1\mathcal{G}_2|R_\omega(\vec{q})\rangle$ terms of the order $\mathcal{O}(\bar{g}^2\epsilon)$.

3. Discontinuities of the 2 → 3 amplitude

3.1. Discontinuities in the s_1 and s_2 channels

For the s_1-channel discontinuity we obtain from the general form (2.17)

$$-4i(2\pi)^{D-2}\delta(\vec{q}_1-\vec{k}-\vec{q}_2)\,\text{disc}_{s_1}\mathcal{A}_{2\to 3} = 2s\langle A'A|e^{\hat{\mathcal{K}}\ln\left(\frac{s_1}{|\vec{q}_1||k_1|}\right)}|GR_2\rangle\frac{1}{t_2}\left(\frac{s_2}{|k_1||\vec{q}_2|}\right)^{\omega(t_2)}\Gamma^{R_2}_{B'B} \,, \tag{3.1}$$

where $q_1 = p_A - p_{A'}$, k and q_2 are the momenta of the gluon G and the Reggeon R_2, $|GR_2\rangle$ is the impact factor for the Reggeon–gluon transition. The bootstrap conditions (2.20) and (2.21) give us

$$-4i(2\pi)^{D-2}\delta(\vec{q}_1 - \vec{k} - \vec{q}_2)\,\mathrm{disc}_{s_1}\mathcal{A}_{2\to 3}$$
$$= 2s\Gamma^R_{A'A}\left(\frac{s_1}{|k_1||\vec{q}_1|}\right)^{\omega(t_1)}\frac{1}{t_2}\left(\frac{s_2}{|k_1||\vec{q}_2|}\right)^{\omega(t_2)}\Gamma^{R_2}_{B'B}\,g\langle R_\omega(\vec{q}_1)|GR_2\rangle\,. \tag{3.2}$$

The impact factors for Reggeon–gluon transitions were calculated in Refs. [10,12] in the special scheme (the so-called bootstrap scheme) which simplifies the proof of the most complicated bootstrap condition

$$\langle GR_1| - g\vec{q}_1^{\,2}\langle R_\omega(\vec{q}_1)|\hat{\mathcal{G}} = g\gamma^G_{R_1R}\langle R_\omega(\vec{q}_1 - \vec{k})|\,, \tag{3.3}$$

where $\hat{\mathcal{G}}$ is the gluon production operator, k is the gluon momentum. The Reggeon R in the condition (3.3) has the momentum $q_1 - k$ and the same color indices as the eigenstate $\langle R_\omega(\vec{q}_1 - \vec{k})|$; summation over them is assumed. The eigenfunction $\langle R_\omega(\vec{q}_1)|\mathcal{G}_1\mathcal{G}_2\rangle$ in the bootstrap scheme also was obtained in Ref. [12]. We could calculate the matrix element $\langle R_\omega(\vec{q}_1)|GR_1\rangle$ in Eq. (3.2) just in this scheme. It turns, however, that it is much more convenient, especially in the further calculations, to use the scheme which we call conformal. It is associated with the modified kernel $\hat{\mathcal{K}}_m$, introduced in Ref. [46], which is obtained from the usual BFKL kernel in the adjoint representation, by subtraction of the gluon trajectory depending on the total t-channel momentum. One of advantages of this kernel is its infrared safety, which permits to consider this kernel at physical transverse dimension $D-2=2$. But the most important advantage is its behavior under Möbius transformations in the two-dimensional transverse momentum space. It is not difficult to see that in the leading order $\mathcal{K}_m$ is Möbius invariant. But in the NLO in the standard scheme, in which the kernel was initially calculated [47,48], it is not Möbius invariant. The existence of the scheme where the modified kernel is Möbius invariant (Möbius scheme) was conjectured in Ref. [49] and then proved in Ref. [50], where the transformation from the standard $\mathcal{K}_m$ to the conformal (Möbius invariant) kernel $\mathcal{K}_c$ was found. It reads

$$\hat{\mathcal{K}}_c = \hat{\mathcal{K}}_m - \frac{1}{4}\left[\hat{\mathcal{K}}^B\left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}^B\right]\right]\,, \tag{3.4}$$

where $\hat{\mathcal{K}}^B$ is the LO kernel. Note that since $\hat{\mathcal{K}}$ and $\hat{\mathcal{K}}_m$ differ only for the trajectory depending on the total t-channel momentum, which is a C-number, in all commutators $\hat{\mathcal{K}}$ can be replaced by $\hat{\mathcal{K}}_m$ and vice versa. We will use the following representations for the kernel:

$$\langle\mathcal{G}'_1\mathcal{G}'_2|\hat{\mathcal{K}}|\mathcal{G}_1\mathcal{G}_2\rangle = \delta(\vec{r}_1 + \vec{r}_2 - \vec{r}_1^{\,\prime} - \vec{r}_2^{\,\prime})\,\vec{q}^{\,2}\sum_R(\mathcal{P}_R)^{\mathcal{G}'_1\mathcal{G}'_2}_{\mathcal{G}_1\mathcal{G}_2}\,K^R(\vec{r}_1, \vec{r}_2;\,\vec{l})\,. \tag{3.5}$$

Here $\vec{r}_i$ and $\vec{r}_i^{\,\prime}$ are the Reggeon momenta, $\vec{q} = \vec{r}_1 + \vec{r}_2$, $\vec{l} = \vec{r}_1 - \vec{r}_1^{\,\prime}$, $\mathcal{P}_R$ is the projection operator on the representation R of the color group, and

$$K^R(\vec{r}_1, \vec{r}_2;\,\vec{l}) = K^R_r(\vec{r}_1, \vec{r}_2;\,\vec{l}) + \frac{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}{\vec{q}^{\,2}}\left(\omega(-\vec{r}_1^{\,2})\delta(\vec{r}_1 - \vec{r}_1^{\,\prime}) + \omega(-\vec{r}_2^{\,2})\delta(\vec{r}_2 - \vec{r}_2^{\,\prime})\right)\,, \tag{3.6}$$

where K^R_r is called the real part of the kernel.

In general, the kernel $K^R_r(\vec{r}_1, \vec{r}_2;\,\vec{l})$ depends on R. But at large N_c only the antisymmetric and symmetric adjoint representations do survive in the decomposition (3.5), with

$$(\mathcal{P}_{Aa})^{\mathcal{G}_1'\mathcal{G}_2'}_{\mathcal{G}_1\mathcal{G}_2} = \frac{1}{N_c} f_{i\mathcal{G}_1\mathcal{G}_2} f_{i\mathcal{G}_1'\mathcal{G}_2'} , \qquad (\mathcal{P}_{As})^{\mathcal{G}_1'\mathcal{G}_2'}_{\mathcal{G}_1\mathcal{G}_2} = \frac{1}{N_c} d_{i\mathcal{G}_1\mathcal{G}_2} d_{i\mathcal{G}_1'\mathcal{G}_2'} , \tag{3.7}$$

and the same kernel $K_r^R(\vec{r}_1, \vec{r}_2; \vec{l})$. Therefore in the following we will omit the index of representation R.

In the LO the real part of the kernel is given by

$$K_r^B(\vec{r}_1, \vec{r}_2; \vec{l}) = \frac{g^2 N_c}{2(2\pi)^{D-1}} \left(\frac{\vec{r}_1^{\,2} \vec{r}_2^{\,\prime\,2} + \vec{r}_2^{\,2} \vec{r}_1^{\,\prime\,2}}{\vec{q}^{\,2} \vec{l}^{\,2}} - 1 \right) , \tag{3.8}$$

whereas the gluon trajectory has the representation

$$\omega^B(t) = \frac{g^2 N_c t}{2(2\pi)^{D-1}} \int \frac{d\vec{l}}{\vec{l}^{\,2} (\vec{q} - \vec{l})^2} , \qquad t = -\vec{q}^{\,2} . \tag{3.9}$$

The difference with usual denotation is in the factor $\vec{q}^{\,2}$ in the representation (3.5). Its extraction is necessary to make the modified kernel

$$K_m(\vec{r}_1, \vec{r}_2; \vec{l}) = K(\vec{r}_1, \vec{r}_2; \vec{l}) - \frac{\vec{r}_1^{\,2} \vec{r}_2^{\,2}}{\vec{q}^{\,2}} \delta(\vec{r}_1 - \vec{r}_1^{\,\prime}) \delta(\vec{r}_2 - \vec{r}_2^{\,\prime}) \, \omega(t) \tag{3.10}$$

explicitly invariant at $D = 4$ with respect to the Möbius transformations

$$z_i \to \frac{a z_i + b}{c z_i + d_i} , \tag{3.11}$$

where a, b, c and d are complex numbers, $z_i = x_i + i y_i$, x_i and y_i are the Cartesian components of the "dual" transverse momenta $\vec{p}_i$ such that

$$\vec{r}_1 = \vec{p}_1 - \vec{p}_2, \qquad \vec{r}_2 = \vec{p}_4 - \vec{p}_1, \qquad \vec{r}_1^{\,\prime} = \vec{p}_3 - \vec{p}_2, \qquad \vec{r}_2^{\,\prime} = \vec{p}_4 - \vec{p}_3 . \tag{3.12}$$

The need for the factor $\vec{q}^{\,2}$ is clear from another point of view: the kernel could be explicitly Möbius invariant only when the corresponding normalization condition is Möbius invariant. The condition (2.18) is not invariant; to make it invariant one needs to multiply both sides on $1/\vec{q}^{\,2}$ and include $1/\vec{q}^{\,2}$ in the left-hand side in definition of the two-Reggeon states. This can be seen from the invariance of the corresponding measure,

$$d\vec{r}_1^{\,\prime} d\vec{r}_2^{\,\prime} \frac{\vec{q}^{\,2}}{\vec{r}_1^{\,\prime\,2} \vec{r}_2^{\,\prime\,2}} \delta(\vec{r}_1 + \vec{r}_2 - \vec{r}_1^{\,\prime} - \vec{r}_2^{\,\prime}) = \frac{d^2 z}{|z|^2} , \tag{3.13}$$

where $\vec{q} = \vec{r}_1 + \vec{r}_2$ and $z = r_2'^{+} r_1^{+} / (r_1'^{+} r_2^{+})$ is invariant. Here and in the following we use the chiral components $r^+ = x + iy$ and $r^- = x - iy$ for the two-dimensional vectors $\vec{r} = (x, y)$. Vice versa, the two conjugate complex numbers z and z^* are confronted with the vector $\vec{z}$ through the components $(z + z^*)/2$ and $(z - z^*)/(2i)$. At the same time, $d\vec{r} = dxdy = dr^+ dr^-/2$, $\delta(\vec{r}) = 2\delta(r^+)\delta(r^-)$ and we define $\delta^2(z)$ in such a way that $\delta^2(z) = \delta(z^+)\delta(z^-)/2 = \delta(\vec{z})$.

The transformation (3.4) gives [50] $K_c(\vec{r}_1, \vec{r}_2; \vec{l}) = K_c(z)$, where $z = r_1^+ r_2'^{+} / (r_2^+ r_1'^{+})$ and

$$K_c(z) = K_c^B(z) \left(1 - \frac{g^2 N_c}{8\pi^2} \zeta(2) \right) + \delta^{(2)}(1 - z) \left(\frac{g^2 N_c}{8\pi^2} \right)^2 3\zeta(3) + \frac{1}{8\pi} \left(\frac{g^2 N_c}{8\pi^2} \right)^2$$
$$\times \left[\left(\frac{1}{2} - \frac{1 + |z|^2}{|1 - z|^2} \right) \ln^2 |z|^2 - \frac{1 - |z|^2}{2|1 - z|^2} \ln |z|^2 \ln \frac{|1 - z|^4}{|z|^2} \right.$$

$$+\left(\frac{1}{1-z}-\frac{1}{1-z^*}\right)(z-z^*)\int\limits_0^1\frac{dx}{|x-z|^2}\,\ln\frac{|z|^2}{x^2}\Bigg]. \tag{3.14}$$

Here

$$K_c^B(z)=\frac{g^2N_c}{32\pi^3}\left(\frac{z+z^*}{|1-z|^2}-\delta^{(2)}(1-z)\int\frac{d\vec{l}}{|l|^2}\frac{l+l^*}{|1-l|^2}\right)\,, \tag{3.15}$$

with the properties

$$K_c(z)=K_c(z^*)=K_c(1/z),\;\; K_c(0)=0\,. \tag{3.16}$$

The transformation (3.4) has to be accompanied by the corresponding transformation of the impact factors and the eigenstate $\langle R_\omega|$. The eigenstate $\langle R_\omega|_c$ which corresponds to the kernel $\hat{\mathcal{K}}_c$ (3.4) is

$$\langle R_\omega(\vec{q})|_c=\langle R_\omega(\vec{q})|-\frac{1}{4}\langle R_\omega(\vec{q})|^B\left[\ln\left(\hat{\vec{r}}_1^{\,2}\hat{\vec{r}}_2^{\,2}\right),\hat{\mathcal{K}}_r^B\right]\,, \tag{3.17}$$

where $\hat{\mathcal{K}}_r^B$ is the real part of the LO kernel (3.8), $\hat{\vec{r}}_1$ and $\hat{\vec{r}}_2$ are the Reggeon momentum operators. Using (see Appendix A for details)

$$\langle R_\omega(\vec{q})|^B\left[\ln\left(\hat{\vec{r}}_1^{\,2}\hat{\vec{r}}_2^{\,2}\right),\hat{\mathcal{K}}_r^B\right]|\mathcal{G}_1\mathcal{G}_2\rangle$$
$$=-2\bar{g}^2\delta(\vec{q}-\vec{r}_1-\vec{r}_2)T^R_{\mathcal{G}_1\mathcal{G}_2}\ln\left(\frac{\vec{r}_1^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{q}^{\,2}}\right)\,, \tag{3.18}$$

we obtain from Eq. (2.24)

$$\langle R_\omega(\vec{q})|\mathcal{G}_1\mathcal{G}_2\rangle_c=\delta(\vec{q}-\vec{r}_1-\vec{r}_2)T^R_{\mathcal{G}_1\mathcal{G}_2}\left(1-\bar{g}^2\zeta(2)\right)\,. \tag{3.19}$$

Now turn to the impact factor $|GR_2\rangle$. The impact factor corresponding to the kernel $\hat{\mathcal{K}}_c$ (3.4) is obtained from $|GR_2\rangle$ in the standard scheme by the transformation

$$|GR_2\rangle\to|GR_2\rangle+\frac{1}{4}\left[\ln\left(\hat{\vec{r}}_1^{\,2}\hat{\vec{r}}_2^{\,2}\right),\hat{\mathcal{K}}^B\right]|GR_2\rangle^B\,. \tag{3.20}$$

It was found, however, in Ref. [51] that impact factors for Reggeon–gluon transitions acquire the most simple form in the scheme in which not only the kernel, but also the energy evolution parameter is conformal invariant. Transition to this scheme, which is called conformal scheme, means the additional transformation for the impact factor $|GR_2\rangle$,

$$|GR_2\rangle\to|GR_2\rangle-\frac{1}{2}\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)\hat{\mathcal{K}}_m^B|GR_2\rangle^B\,. \tag{3.21}$$

Together with the transformation (3.20) it gives

$$|GR_2\rangle\to|GR_2\rangle_c=|GR_2\rangle-\frac{1}{4}\left[\ln\left(\hat{\vec{r}}_1^{\,2}\hat{\vec{r}}_2^{\,2}\right),\hat{\mathcal{K}}_r^B\right]|GR_2\rangle^B$$
$$-\frac{1}{2}\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)\hat{\mathcal{K}}_m^B|GR_2\rangle^B\,. \tag{3.22}$$

Note that the transformation (3.21) does not affect the matrix element $\langle R_\omega(\vec{q}_1)|GR_2\rangle$ because $\langle R_\omega|$ is the eigenstate of $\mathcal{K}_m$ with the eigenvalue equal to 0.

For amplitudes with the negative signature, the impact factors are antisymmetric with respect to the $\mathcal{G}_1 \leftrightarrow \mathcal{G}_2$ exchange. In fact, putting

$$\langle GR_1| = \langle GR_1|_s - \langle GR_1|_u \ , \tag{3.23}$$

we have

$$\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle_u = \langle GR_1|\mathcal{G}_2\mathcal{G}_1\rangle_s \ . \tag{3.24}$$

As it follows from Ref. [51], in the conformal scheme, the impact factors $\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle_s$ of gluons with the polarization vectors

$$\vec{e}_\lambda^{\,L} = \frac{1}{\sqrt{2}}\left(\vec{e}_x + i\lambda\vec{e}_y\right) \ , \qquad \vec{e}_\lambda^{\,L\,*} = \frac{1}{\sqrt{2}}\left(\vec{e}_x - i\lambda\vec{e}_y\right) \tag{3.25}$$

for helicities $\lambda = \pm 1$ have the form

$$\begin{aligned} &\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle_s \\ &\quad = \langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle_s^B \left[1 + \bar{g}^2\left(I_\lambda(z) - \frac{1}{2}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right) - \frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2} + 2\zeta(2)\right)\right]. \end{aligned} \tag{3.26}$$

Here $z = -q_1^+ r_2^+/(k^+ r_1^+)$,

$$\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle_s^B = -g^2\delta(\vec{q}_1 - \vec{r}_1 - \vec{r}_2 - \vec{k})\left(T^{R_1}T^G\right)_{\mathcal{G}_1\mathcal{G}_2} \vec{e}_\lambda^{\,L\,*}\vec{C}^{\,L}(\vec{r}_1,\vec{q}_1) \ , \tag{3.27}$$

$$\vec{C}^{\,L}(\vec{r}_1,\vec{q}_1) = -2\left(\vec{q}_1 - (\vec{q}_1 - \vec{r}_1)\frac{\vec{q}_1^{\,2}}{(\vec{q}_1 - \vec{r}_1)^2}\right) \ , \tag{3.28}$$

and $I_{+1}(z) = I(z)$, $I_{-1}(z) = I^*(z) = I(z^*)$, where

$$\begin{aligned} I(z) = \frac{1-z}{8}\Bigg(&\ln\left(\frac{|1-z|^2}{|z|^2}\right)\ln\left(\frac{|1-z|^4}{|z|^6}\right) \\ &- 6\,\mathrm{Li}_2(z) + 6\,\mathrm{Li}_2(z^*) - 3\ln|z|^2\ln\frac{1-z}{1-z^*}\Bigg) \\ &- \frac{1}{2}\ln|1-z|^2\ln\frac{|1-z|^2}{|z|^2} - \frac{3}{8}\ln^2|z|^2 \ , \end{aligned} \tag{3.29}$$

$$\mathrm{Li}_2(z) = -\int_0^1 \frac{dx}{x}\ln(1 - xz) \ . \tag{3.30}$$

Note that $I(0) = 0$, $I(1/z) = I(z)/z$. In the two-dimensional transverse momentum space, with the polarization vectors (3.25) we have

$$\vec{e}_+^{\,L\,*}\vec{C}^{\,L}(\vec{r}_1,\vec{q}_1) = \sqrt{2}\frac{q_1^- r_1^+}{(q_1 - r_1)^+} \ , \qquad \vec{e}_-^{\,L\,*}\vec{C}^{\,L}(\vec{r}_1,\vec{q}_1) = \frac{q_1^+ r_1^-}{(q_1 - r_1)^-} \ . \tag{3.31}$$

The set of diagrams for the process $A + B \to A' + G + B'$ is evidently invariant with respect to rotating around the gluon line and the exchange $A \leftrightarrow B$. It means that the impact factor $\langle\mathcal{G}_1\mathcal{G}_2|GR_2\rangle$ can be obtained from $\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle$ by the replacement

$$n_1 \leftrightarrow n_2, \qquad \vec{q}_1 \to -\vec{q}_2,$$
$$\vec{r}_{1,2} \to -\vec{r}_{1,2}\,, \qquad \left(T^{R_1}T^G\right)_{\mathcal{G}_1\mathcal{G}_2} \to \left(T^{R_2}T^G\right)_{\mathcal{G}_1\mathcal{G}_2}\,. \tag{3.32}$$

The replacement $n_1 \leftrightarrow n_2$ means also $\vec{e}_\lambda^{\,L} \leftrightarrow \vec{e}_\lambda^{\,R}$. With account of Eqs. (2.6) and (2.7) we have

$$\vec{e}_\lambda^{\,R} = -\left(\frac{k^+}{k^-}\right)^\lambda \vec{e}_{-\lambda}^{\,L} = -\left(\frac{k^+}{k^-}\right)^\lambda \left(\vec{e}_x - i\lambda\vec{e}_y\right)\,. \tag{3.33}$$

Using the substitutions (3.32) and formulas (3.33) we obtain

$$\langle\mathcal{G}_1\mathcal{G}_2|GR_2\rangle_s$$
$$= \langle\mathcal{G}_1\mathcal{G}_2|GR_2\rangle_s^B\left[1+\bar{g}^2\left(I_\lambda^*(z) - \frac{1}{2}\ln^2\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right) - \frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2} + 2\zeta(2)\right)\right], \tag{3.34}$$

where $z = q_2^+r_2^+/(k^+r_1^+)$,

$$\langle\mathcal{G}_1\mathcal{G}_2|GR_2\rangle_s^B = g^2\delta(\vec{r}_1+\vec{r}_2-\vec{k}-\vec{q}_2)\left(T^{R_2}T^G\right)_{\mathcal{G}_1\mathcal{G}_2}\vec{e}_\lambda^{\,R\,*}\vec{C}^{\,R}(\vec{q}_2,\vec{r}_1) \tag{3.35}$$

and

$$\vec{C}^{\,R}(\vec{q}_2,\vec{r}_1) = -2\left(\vec{q}_2 + (\vec{r}_1-\vec{q}_2)\frac{\vec{q}_2^{\,2}}{(\vec{r}_1-\vec{q}_2)^2}\right)\,. \tag{3.36}$$

Using now Eqs. (3.19) and (3.34)–(3.36) we arrive to

$$\langle R_\omega(\vec{q}_1)|GR_2\rangle_s$$
$$= \frac{g^2N_c}{2\vec{q}_1^{\,2}}\delta(\vec{q}_1-\vec{k}-\vec{q}_2)T^G_{R_1R_2}\vec{e}_\lambda^{\,R\,*}\int d\vec{r}_1 d\vec{r}_2\frac{\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\delta(\vec{r}_1+\vec{r}_2-\vec{k}-\vec{q}_2)$$
$$\times\vec{C}^{\,R}(\vec{q}_2,\vec{r}_1)\left[1+\bar{g}^2\left(I_{-\lambda}(z) - \frac{1}{2}\ln^2\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right) - \frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2} + \zeta(2)\right)\right], \tag{3.37}$$

where $\lambda = \pm 1$ is the gluon helicity, $I_{+1}(z) = I(z)$, $I_{-1}(z) = I(z^*)$, $I(z)$ is defined in Eq. (3.29), and $z = q_2^+r_2^+/(k^+r_1^+)$. The integral with $I_{-\lambda}(z)$ in Eq. (3.37) is not singular and can be calculated in two-dimensional transverse momentum space. Using the measure (3.13) and formulas

$$\vec{e}_+^{\,R\,*}\vec{C}^{\,R}(\vec{q}_2,\vec{r}_1) = \sqrt{2}\frac{k^-}{k^+}\frac{q_2^+r_1^-}{(r_1-q_2)^-} = \sqrt{2}\frac{q_2^+q_1^-}{k^+(1-z^*)}\,,$$
$$\vec{e}_-^{\,R\,*}\vec{C}^{\,R}(\vec{q}_2,\vec{r}_1) = \sqrt{2}\frac{k^+}{k^-}\frac{q_2^-r_1^+}{(r_1-q_2)^+} = \sqrt{2}\frac{q_2^-q_1^+}{k^-(1-z)}\,, \tag{3.38}$$

we obtain that the contribution of the term with $I_{-\lambda}(z)$ in Eq. (3.37) is equal zero. Indeed, for the positive helicity it is proportional to

$$\int\frac{d^2z}{|z|^2(1-z^*)}I(z^*) = 0\,. \tag{3.39}$$

The result (3.39) follows from the fact that in the expansion of the integrand in powers of $(z^*)^n$ at $|z| < 1$ and in powers of $(1/z^*)^n$ at $|z| > 1$ there are only terms with $n > 0$ (remind that, as pointed out previously, $I(z) = 0$, $\;I(z) = zI(1/z)$). For the negative helicity the result is obtained by complex conjugation. It means that the term with $I_{-\lambda}(z)$ in Eq. (3.37) can be omitted. The remaining integral (details of the calculation are given in Appendix B) is

$$\int d\vec{r}_1 d\vec{r}_2 \frac{\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\delta(\vec{r}_1+\vec{r}_2-\vec{k}-\vec{q}_2)\left(\vec{q}_2+(\vec{r}_1-\vec{q}_2)\frac{\vec{q}_2^{\,2}}{(\vec{r}_1-\vec{q}_2)^2}\right)$$
$$=\pi^{1+\epsilon}\Gamma(1-\epsilon)\left(\vec{q}_2+\vec{k}\frac{\vec{q}_2^{\,2}}{\vec{k}^{\,2}}\right)\left(\frac{1}{\epsilon}+\ln\left(\frac{\vec{q}_1^{\,2}\vec{k}^{\,2}}{\vec{q}_2^{\,2}}\right)\right). \tag{3.40}$$

For $\mathrm{disc}_{s_1}\mathcal{A}^{\{-\}}_{2\to 3}/\Re\mathcal{A}_{2\to 3}$ from Eqs. (3.2) and (1.1) we get

$$-4i(2\pi)^{D-2}\delta(\vec{q}_1-\vec{k}-\vec{q}_2)\frac{\mathrm{disc}_{s_1}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}}$$
$$=\frac{g\,t_1\langle R_\omega(\vec{q}_1)|GR_2\rangle}{\gamma^G_{R_1R_2}}=2\frac{g\,t_1\langle R_\omega(\vec{q}_1)|GR_2\rangle_s}{\gamma^G_{R_1R_2}}. \tag{3.41}$$

Here the last equality comes from antisymmetry of $\langle R_\omega|\mathcal{G}_1\mathcal{G}_2\rangle$ with respect to $\mathcal{G}_1\leftrightarrow\mathcal{G}_2$ exchange. Then, using Eqs. (2.16), (2.9), (3.37) and the equalities

$$\vec{e}_+^{\,R\,*}\vec{C}^{\,R}(\vec{q}_2,\vec{q}_1)=\vec{e}_+^{\,L\,*}\vec{C}^{\,L}(\vec{q}_2,\vec{q}_1)=\sqrt{2}\frac{q_2^+q_1^-}{k^+},$$
$$\vec{e}_-^{\,R\,*}\vec{C}^{\,R}(\vec{q}_2,\vec{q}_1)=\vec{e}_-^{\,L\,*}\vec{C}^{\,L}(\vec{q}_2,\vec{q}_1)=\sqrt{2}\frac{q_2^-q_1^+}{k^-}, \tag{3.42}$$

which means

$$\gamma^{G(B)}_{R_1R_2}\Big|_{\lambda=+1}=-gT^G_{R_1R_2}\sqrt{2}\frac{q_2^+q_1^-}{k^+},\quad \gamma^{G(B)}_{R_1R_2}\Big|_{\lambda=-1}=-gT^G_{R_1R_2}\sqrt{2}\frac{q_2^-q_1^+}{k^-}, \tag{3.43}$$

we obtain

$$\frac{\mathrm{disc}_{s_1}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}}=\pi i\bar{g}^2\left(\frac{1}{\epsilon}+\ln\left(\frac{\vec{q}_1^{\,2}\vec{k}^{\,2}}{\vec{q}_2^{\,2}}\right)\right)\left(1-2\bar{g}^2\zeta(2)\right). \tag{3.44}$$

Here, it is necessary to make the note analogous to that given at the end of Section 2. The accuracy of Eq. (3.19) for $\langle R_\omega(\vec{q})|\mathcal{G}_1\mathcal{G}_2\rangle_c$ and Eq. (3.34) for $\langle\mathcal{G}_1\mathcal{G}_2|GR_2\rangle_s$ does not provide preservation of nonvanishing in the limit $\epsilon\to 0$ corrections of the $\bar{g}^2$ order in the integral (3.37) (and therefore in the discontinuity $\mathrm{disc}_{s_1}\mathcal{A}^{\{-\}}_{2\to 3}$) because of the infrared divergency of the integration measure in Eq. (3.37). To provide the preservation one has to find $\langle R_\omega(\vec{q})|\mathcal{G}_1\mathcal{G}_2\rangle_c$ and $\langle\mathcal{G}_1\mathcal{G}_2|GR_2\rangle_s$ with higher accuracy. This issue requires special consideration. It applies also to other discontinuities discussed below.

Evidently, the s_2-channel discontinuity can be obtained by the replacement (3.32) and is given by the relations

$$-4i(2\pi)^{D-2}\delta(\vec{q}_1-\vec{k}-\vec{q}_2)\frac{\mathrm{disc}_{s_2}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}}=\frac{g\,t_2\langle GR_1|R_\omega(\vec{q}_2)\rangle}{\gamma^G_{R_1R_2}}, \tag{3.45}$$

$$\frac{g\, t_2\langle GR_1|R_\omega(\vec{q}_2)\rangle}{\gamma^G_{R_1R_2}}$$
$$= \delta(\vec{q}_1-\vec{k}-\vec{q}_2) g^2\pi^{1+\epsilon}\Gamma(1-\epsilon)\left(1-2\bar{g}^2\zeta(2)\right)\left(\frac{1}{\epsilon}+\ln\left(\frac{\vec{q}_1^{\,2}\vec{k}^{\,2}}{\vec{q}_2^{\,2}}\right)\right), \quad (3.46)$$

$$\frac{\mathrm{disc}_{s_2}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} = \pi i \bar{g}^2\left(\frac{1}{\epsilon}+\ln\left(\frac{\vec{q}_2^{\,2}\vec{k}^{\,2}}{\vec{q}_1^{\,2}}\right)\right)\left(1-2\bar{g}^2\zeta(2)\right). \quad (3.47)$$

3.2. Discontinuity in the s channel

According to the representation (2.17), for the s-channel discontinuity we have

$$-4i(2\pi)^{D-2}\delta(\vec{q}_1-\vec{k}-\vec{q}_2)\,\mathrm{disc}_s\mathcal{A}^{\{-\}}_{2\to 3}$$
$$= 2s\langle A'A|e^{\hat{\mathcal{K}}\ln\left(\frac{s_1}{|\vec{q}_1||\vec{k}|}\right)}\hat{\mathcal{G}}e^{\hat{\mathcal{K}}\ln\left(\frac{s_1}{|\vec{k}||\vec{q}_2|}\right)}|B'B\rangle, \quad (3.48)$$

where $\hat{\mathcal{G}}$ is the gluon production operator. Using the bootstrap conditions (2.20) and (2.21), we obtain

$$-4i(2\pi)^{D-2}\delta(\vec{q}_1-\vec{k}-\vec{q}_2)\frac{\mathrm{disc}_s\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} = \frac{g^2\vec{q}_1^{\,2}\vec{q}_2^{\,2}\langle R_\omega(\vec{q}_1)|\hat{\mathcal{G}}|R_\omega(\vec{q}_2)\rangle}{\gamma^G_{R_1R_2}}. \quad (3.49)$$

Then, due to the bootstrap condition (3.3), we have

$$g\vec{q}_1^{\,2}\langle R_\omega(\vec{q}_1)|\hat{\mathcal{G}}|R_\omega(\vec{q}_2)\rangle = \langle GR_1|R_\omega(\vec{q}_2)\rangle - g\gamma^G_{R_1R}\langle R_\omega(\vec{q}_1-\vec{k})|R_\omega(\vec{q}_2)\rangle. \quad (3.50)$$

Both matrix elements here are known: the first comes from the calculation of the s_2-channel discontinuity, see Eq. (3.46), and the second from the bootstrap condition (2.22). Thus, we obtain

$$\frac{\mathrm{disc}_s\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} = -\pi i \bar{g}^2\left(\frac{1}{\epsilon}+\ln\left(\frac{\vec{q}_2^{\,2}\vec{k}^{\,2}}{\vec{q}_1^{\,2}}\right)\right)\left(1-2\bar{g}^2\zeta(2)\right) - \pi i\omega(t_2)$$
$$= \pi i \bar{g}^2\left[\frac{1}{\epsilon}+\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}}\right)+2\bar{g}^2\left(\zeta(3)-\zeta(2)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}}\right)\right)\right]. \quad (3.51)$$

In fact, it was not needed at all to calculate neither the s_2-channel, nor the s-channel discontinuities, because they can be expressed in terms of s-channel discontinuities from the bootstrap relations (1.4). Indeed, for the amplitude $\mathcal{A}^{\{-\}}_{2\to 3}$ there are three relations:

$$\frac{\mathrm{disc}_{s_1}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} + \frac{\mathrm{disc}_{s}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} = -i\pi\omega(t_1),$$
$$\frac{\mathrm{disc}_{s_2}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} + \frac{\mathrm{disc}_{s}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} = -i\pi\omega(t_2),$$
$$\frac{\mathrm{disc}_{s_1}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} - \frac{\mathrm{disc}_{s_2}\mathcal{A}^{\{-\}}_{2\to 3}}{\Re\mathcal{A}_{2\to 3}} = -i\pi\left(\omega(t_1)-\omega(t_2)\right). \quad (3.52)$$

However, they are not independent: the third of them is the difference of the first two. Therefore, there are two relationships between the discontinuities, so that only one of them is independent. It is easy to see that the discontinuities calculated above satisfy the relations (3.52). The fulfillment of the third of them, with account of

$$\omega(t_1)-\omega(t_2)=2\bar{g}^2\ln\left(\frac{\vec{q}_2^{\;2}}{\vec{q}_1^{\;2}}\right)\left(1-2\bar{g}^2\zeta(2)\right)\,, \tag{3.53}$$

follows from Eqs. (3.44) and (3.47) and fulfillment of the second follows from Eqs. (3.47) and (3.51).

4. Discontinuities of the $2\to 4$ amplitude

4.1. Discontinuities in the s_1 and s_3 channels

From the representation (2.17), for the s_1-channel discontinuity we have

$$\begin{aligned}&-4i(2\pi)^{D-2}\delta(\vec{q}_1-\vec{k}_1-\vec{q}_2)\,\mathrm{disc}_{s_1}\mathcal{A}_{2\to 4}^{\{-\}}\\&=2s\langle A'A|e^{\hat{\mathcal{K}}\ln\left(\frac{s_1}{|\vec{q}_1|\,|k|}\right)}|G_1R_2\rangle\frac{1}{t_2}\left(\frac{s_2}{|\vec{k}_1|\,|\vec{k}_2|}\right)^{\omega(t_2)}\gamma_{R_2R_3}^{G_2}\frac{1}{t_3}\left(\frac{s_3}{|k_2|\,|\vec{q}_3|}\right)^{\omega(t_3)}\Gamma_{B'B}^{R_3}\,,\end{aligned} \tag{4.1}$$

therefore, using the bootstrap relations (2.20) and (2.21) and the representation (1.1) of the MRK amplitude, we obtain

$$-4i(2\pi)^{D-2}\delta(\vec{q}_1-\vec{k}_1-\vec{q}_2)\,\frac{\mathrm{disc}_{s_1}\mathcal{A}_{2\to 4}^{\{-\}}}{\Re\mathcal{A}_{2\to 4}}=\frac{g^2t_1\,\langle R_\omega(\vec{q}_1)|G_1R_2\rangle}{\gamma_{R_1R_2}^{G_1}}\,. \tag{4.2}$$

The ratio in the right-hand side of Eq. (4.2) is the same as in Eq. (3.41) with the replacement $G\to G_1$, so that using Eq. (3.44) we arrive to

$$\frac{\mathrm{disc}_{s_1}\mathcal{A}_{2\to 4}^{\{-\}}}{\Re\mathcal{A}_{2\to 4}}=\pi i\bar{g}^2\left(\frac{1}{\epsilon}+\ln\left(\frac{\vec{q}_1^{\;2}\vec{k}_1^{\;2}}{\vec{q}_2^{\;2}}\right)\right)\left(1-2\bar{g}^2\zeta(2)\right)\,. \tag{4.3}$$

Obviously, such ratio for the s_3-channel discontinuity can be obtained by the replacement $\vec{k}_1\to\vec{k}_2$, $\vec{q}_1\to-\vec{q}_3$, $\vec{q}_2\to-\vec{q}_2$; it reads

$$\frac{\mathrm{disc}_{s_3}\mathcal{A}_{2\to 4}^{\{-\}}}{\Re\mathcal{A}_{2\to 4}}=\pi i\bar{g}^2\left(\frac{1}{\epsilon}+\ln\left(\frac{\vec{q}_3^{\;2}\vec{k}_2^{\;2}}{\vec{q}_2^{\;2}}\right)\right)\left(1-2\bar{g}^2\zeta(2)\right)\,. \tag{4.4}$$

4.2. Discontinuity in the s_2 channel

For the s_2-channel discontinuity, using the modified kernel $\hat{\mathcal{K}}_m$, $\hat{\mathcal{K}}=\hat{\mathcal{K}}_m+\omega(t_2)$, we have from the representation (2.17)

$$\begin{aligned}&-4i(2\pi)^{D-2}\delta(\vec{q}-\vec{q}_B)\,\mathrm{disc}_{s_2}\mathcal{A}_{2\to 4}^{\{-\}}\\&=2s\Gamma_{A'A}^{R_1}\frac{1}{t_1}\left(\frac{s_1}{|\vec{q}_1|\,|\vec{k}_1|}\right)^{\omega(t_1)}\frac{1}{t_3}\left(\frac{s_3}{|k_2|\,|\vec{q}_3|}\right)^{\omega(t_3)}\Gamma_{B'B}^{R_3}\\&\quad\times\left(\frac{s_2}{|\vec{k}_1|\,|\vec{k}_2|}\right)^{\omega(t_2)}\langle G_1R_1|e^{\hat{\mathcal{K}}_m\ln\left(\frac{s_2}{|\vec{k}_1|\,|\vec{k}_2|}\right)}|G_2R_3)\rangle\,.\end{aligned} \tag{4.5}$$

Here we meet two new important aspects. First, the energy dependence of the s_2 channel discontinuity (4.5) evidently differs from that predicted by the BDS ansatz [14], where this dependence is the same as for the real part of $\mathcal{A}_{2\to 4}$. Instead, according to Eq. (4.5), there is an additional dependence coming from the matrix element with $\hat{\mathcal{K}}_m$. For agreement with the BDS ansatz the impact factors for Reggeon–gluon transitions have to be proportional to the eigenvector of $\hat{\mathcal{K}}_m$ with zero eigenvalue, what is obviously not so. Note that the discrepancy is manifested already in the leading logarithmic approximation.

Actually, it is well known that the BDS ansatz for n-gluon amplitudes is incomplete at $n \geq 6$. The first indications of the incompleteness were obtained in Ref. [52] in the strong coupling regime using the Maldacena hypothesis [53] about the ADS/CFT duality, and in Ref. [24] using the hypothesis of the scattering amplitude/Wilson loop correspondence. Then the incompleteness was shown by direct two-loop calculations in Ref. [54]. Moreover, disagreement of the BDS ansatz with the BFKL approach is also known [55]. Dignity of the demonstration of the discrepancy presented here is its simplicity.

The second new aspect is seen from the expressions for the impact factors in Eq. (4.5)

$$\langle G_1 R_1|\mathcal{G}_1\mathcal{G}_2\rangle = \langle G_1 R_1|\mathcal{G}_1\mathcal{G}_2\rangle_s - \langle G_2 R_1|\mathcal{G}_2\mathcal{G}_1\rangle_s \,,$$
$$\langle \mathcal{G}_1\mathcal{G}_2|G_2 R_3\rangle = \langle \mathcal{G}_1\mathcal{G}_2|G_2 R_3\rangle_s - \langle \mathcal{G}_2\mathcal{G}_1|G_2 R_3\rangle_s \,, \tag{4.6}$$

where $\langle G_1 R_1|\mathcal{G}_1\mathcal{G}_2\rangle_s$ is given by Eqs. (3.26)–(3.29) and $\langle \mathcal{G}_1\mathcal{G}_2|G_2 R_3\rangle$ by Eqs.(3.34)–(3.36) with the replacement $\vec{k} \to \vec{k}_2$, $\vec{q}_2 \to \vec{q}_3$. The new aspect is the appearance in the discontinuity of the color structure $D^{G_1}_{R_1R_2}D^{G_2}_{R_2R_3}$, where $D^a_{bc} = d_{abc}$, in addition to the structure $T^{G_1}_{R_1R_2}T^{G_2}_{R_2R_3}$ in the real part of the amplitude $\mathcal{A}_{2\to n+2}$ presented in Eq. (1.1). Indeed, using

$$(T^aT^b)_{ij}f_{cij} = i\frac{N_c}{2}T^c_{ab}, \;\; (T^aT^b)_{ij}d_{cij} = \frac{N_c}{2}D^c_{ab} \,, \tag{4.7}$$

we have at large N_c

$$(T^{R_1}T^{G_1})_{ij}(T^{R_3}T^{G_2})_{ij} = \frac{N_c}{4}\left(T^{G_1}T^{G_2} + D^{G_1}D^{G_2}\right)_{R_1R_3} \,,$$
$$(T^{R_1}T^{G_1})_{ij}(T^{R_3}T^{G_2})_{ji} = \frac{N_c}{4}\left(-T^{G_1}T^{G_2} + D^{G_1}D^{G_2}\right)_{R_1R_3} \,. \tag{4.8}$$

Writing explicitly all color factors, we obtain using Eqs. (3.23) and (3.24)

$$\langle G_1 R_1|e^{\hat{\mathcal{K}}_m \ln\left(\frac{s_2}{|\vec{k}_1|\,|\vec{k}_2|}\right)}|G_2 R_3\rangle$$
$$= \frac{N_c}{2}\left(T^{G_1}T^{G_2} + D^{G_1}D^{G_2}\right)_{R_1R_3} \widetilde{\langle G_1 R_1|}_s\, e^{\hat{\mathcal{K}}_m \ln\left(\frac{s_2}{|\vec{k}_1|\,|\vec{k}_2|}\right)}\widetilde{|G_2 R_3\rangle}_s$$
$$+ \frac{N_c}{2}\left(-T^{G_1}T^{G_2} + D^{G_1}D^{G_2}\right)_{R_1R_3} \widetilde{\langle G_1 R_1|}_s\, e^{\hat{\mathcal{K}}_m \ln\left(\frac{s_2}{|\vec{k}_1|\,|\vec{k}_2|}\right)}\widetilde{|G_2 R_3\rangle}_u \,, \tag{4.9}$$

where the tilde sign in the impact factors means rejection of color factors.

It is very convenient to use the conformal representation for calculation of the matrix elements in the right side of Eq. (4.9), but transition to the two-dimensional transverse momentum space in this representation must be done with caution because of the infrared divergency in the first term in the right side of Eq. (4.9). In the leading logarithmic approximation, this problem was considered in details in Ref. [46]. In principle, nothing has changed at the transition to the next-to-leading approximation.

The divergency emerges because of the singularity of the integration measure (3.13) at zero momenta of intermediate Reggeized gluons. As it follows from Eqs. (3.26), (3.27), (3.34) and (3.35), the s-pieces of the impact factors $\langle G_1 R_1|\mathcal{G}_1\mathcal{G}_2\rangle_s$ and $\langle \mathcal{G}_1\mathcal{G}_2|G_2R_3\rangle_s$ vanish at $\vec r_1=0$, but not at $\vec r_2=0$. It means that the first matrix element in the right side of Eq. (4.9) is divergent. Fortunately, the divergence exists only in the zero term of the expansion in powers of the BFKL kernel due to its property (3.16). Therefore, writing

$$\langle \widetilde{G_1R_1}|_s\, e^{\hat{\mathcal{K}}_m \ln\left(\frac{s_2}{|\vec k_1||\vec k_2|}\right)}|\widetilde{G_2R_3}\rangle_s = \langle \widetilde{G_1R_1}|_s \left(e^{\hat{\mathcal{K}}_m \ln\left(\frac{s_2}{|\vec k_1||\vec k_2|}\right)}-1\right)|\widetilde{G_2R_3}\rangle_s + \langle \widetilde{G_1R_1}|_s|\widetilde{G_2R_3}\rangle_s\ , \tag{4.10}$$

we can use for the first term in the right side the conformal representation directly in the two-dimensional space. Using Eqs. (3.26), (3.27), (3.31) and (2.16), (3.43) we have for the positive helicity of the gluon G_1 ($\lambda_1=1$)

$$\frac{\langle \widetilde{G_1R_1}|\widetilde{\mathcal{G}_1\mathcal{G}_2}\rangle_s}{\tilde\gamma_{R_1R_2}} = g\delta(\vec q_1-\vec k_1-\vec r_1-\vec r_2)\frac{1}{1-z_1}[1+\bar g^2(I(z_1)-\zeta(2))]\ , \tag{4.11}$$

where $z_1=-q_1^+r_2^+/(k_1^+r_1^+)$ and the tilde signs means omission of the color factors. Analogously, for the positive helicity of the gluon G_2 ($\lambda_2=1$), we obtain using Eqs. (3.34), (3.35), (3.38) and (2.16), (3.43)

$$\frac{\langle \widetilde{\mathcal{G}_1\mathcal{G}_2}|\widetilde{G_2R_3}\rangle_s}{\tilde\gamma_{R_2R_3}} = -g\delta(\vec q_2-\vec k_2-\vec r_1-\vec r_2)\frac{1}{1-z_2^*}[1+\bar g^2(I(z_2^*)-\zeta(2))]\ , \tag{4.12}$$

where $z_2=q_3^+r_2^+/(k_2^+r_1^+)$. The corresponding results for negative helicities are obtained by complex conjugation of Eqs. (4.11) and (4.12).

In the conformal representation, the energy evolution parameter in Eq. (4.10) is $s_2\vec q_2^{\,2}/(|\vec q_1||\vec q_3||\vec k_1||\vec k_2|)$ (instead of $s_2/(|\vec k_1||\vec k_2|)$, and in the two-dimensional transverse momentum space the kernel takes the form (3.14). It has the representation

$$\langle \widetilde{\mathcal{G}_1\mathcal{G}_2}|\hat K_c|\widetilde{\mathcal{G}'_1\mathcal{G}'_2}\rangle = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\nu\, \omega(\nu,n)\,\langle \widetilde{\mathcal{G}_1\mathcal{G}_2}|\nu,n\rangle\langle\nu,n|\widetilde{\mathcal{G}'_1\mathcal{G}'_2}\rangle\ , \tag{4.13}$$

with the eigenfunctions [50]

$$\langle \widetilde{\mathcal{G}_1\mathcal{G}_2}|\nu,n\rangle = \delta(\vec r_1+\vec r_2-\vec q_2)\frac{1}{\sqrt{2\pi^2}}\left(\frac{r_1^+}{r_2^+}\right)^{\frac{n}{2}+i\nu}\left(\frac{r_1^-}{r_2^-}\right)^{-\frac{n}{2}+i\nu}\ , \tag{4.14}$$

which form an orthonormal set with the integration measure (3.13), the eigenvalues being [49]

$$\omega(\nu,n) = \frac{g^2N_c}{8\pi^2}\left(\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}}-\psi(1+i\nu-\frac{|n|}{2})+\psi(1-i\nu+\frac{|n|}{2}+2\psi(1)\right)$$
$$\times\left(1-\frac{g^2N_c}{8\pi^2}\zeta(2)\right)+\left(\frac{g^2N_c}{8\pi^2}\right)^2$$
$$\times\left(\frac{1}{4}\left(\psi''(1+i\nu+\frac{|n|}{2})+\psi''(1-i\nu+\frac{|n|}{2}\right)\right.$$

$$+\frac{2i\nu\left(\psi'(1-i\nu+\frac{|n|}{2})-\psi'(1+i\nu+\frac{|n|}{2})\right)}{\nu^2+\frac{n^2}{4}}\Bigg)+3\zeta(3)+\frac{1}{4}\frac{|n|\left(\nu^2-\frac{n^2}{4}\right)}{\left(\nu^2+\frac{n^2}{4}\right)^3}\Bigg). \tag{4.15}$$

Here $\psi(x)=(\ln\Gamma(x))'$. Note that $\omega(\nu,n)$ has the important property

$$\omega(0,0)=0\,, \tag{4.16}$$

in accordance with the bootstrap conditions. Using the representation (4.13) and Eqs. (4.11), (4.12), we obtain for positive helicities of both gluons

$$\frac{t_2\langle\widetilde{G_1R_1}|_s\left(e^{\hat{\mathcal{K}}_m\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)}-1\right)|\widetilde{G_2R_3}\rangle_s}{\tilde{\gamma}_{R_1R_2}\tilde{\gamma}_{R_2R_3}}$$
$$=\delta(\vec{q}_1-\vec{k}_1-\vec{k}_2-\vec{q}_3)g^2(1-2\bar{g}^2\zeta(2))$$
$$\times\frac{1}{2}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\nu\left(e^{\omega(\nu,n)\ln\left(\frac{s_2\vec{q}_2^{\,2}}{|\vec{q}_1||\vec{q}_3||\vec{k}_1||\vec{k}_2|}\right)}-1\right)w^{\frac{n}{2}+i\nu}(w^*)^{-\frac{n}{2}+i\nu}$$
$$\times\int\frac{d^2z_1}{\pi|z_1|^2}\frac{1}{1-z_1}\left(1+\bar{g}^2I(z_1)\right)z_1^{\frac{n}{2}+i\nu}(z_1^*)^{-\frac{n}{2}+i\nu}$$
$$\times\int\frac{d^2z_2}{\pi|z_2|^2}\frac{1}{1-z_2^*}\left(1+\bar{g}^2I^*(z_2)\right)(z_2^*)^{\frac{n}{2}-i\nu}z_2^{-\frac{n}{2}-i\nu} \tag{4.17}$$

where $w=k_2^+q_1^+/(k_1^+q_3^+)$.

The second term in Eq. (4.10) must be calculated at $D=4+2\epsilon$. Using Eqs. (3.26)–(3.28) and (3.25) for $\langle\widetilde{G_1R_1}|_s$, Eqs. (3.34)–(3.36) and (3.33) for $|\widetilde{G_2R_3}\rangle_s$, and the results obtained in Appendix C, we have for positive gluon helicities

$$\frac{t_2\langle\widetilde{G_1R_1}|_s|\widetilde{G_2R_3}\rangle_s}{\tilde{\gamma}_{R_1R_2}\tilde{\gamma}_{R_2R_3}}=\delta(\vec{q}_1-\vec{k}_1-\vec{k}_2-\vec{q}_3)g^2$$
$$\times\left(\frac{1}{\epsilon}+\ln\left(\frac{\vec{k}_1^{\,2}\vec{k}_2^{\,2}}{\vec{k}^{\,2}}\right)\right)(1-2\bar{g}^2\zeta(2))\,. \tag{4.18}$$

Calculation of the second term in the right side of Eq. (4.9) is simplified because the infrared divergency is absent in this term, since $\langle\mathcal{G}_1\mathcal{G}_2|G_2R_3\rangle_u=0$ at $\vec{r}_2=0$ according to Eq. (3.24). Therefore, we have for positive helicities of both gluons

$$\frac{t_2\langle\widetilde{G_1R_1}|_s\left(e^{\hat{\mathcal{K}}_m\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)}-1\right)|\widetilde{G_2R_3}\rangle_s}{\tilde{\gamma}_{R_1R_2}\tilde{\gamma}_{R_2R_3}}$$
$$=\delta(\vec{q}_1-\vec{k}_1-\vec{k}_2-\vec{q}_3)g^2(1-2\bar{g}^2\zeta(2))$$
$$\times\frac{1}{2}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\nu\left(e^{\omega(\nu,n)\ln\left(\frac{s_2\vec{q}_2^{\,2}}{|\vec{q}_1||\vec{q}_3||\vec{k}_1||\vec{k}_2|}\right)}-1\right)w^{\frac{n}{2}+i\nu}(w^*)^{-\frac{n}{2}+i\nu}$$

$$\times \int \frac{d^2 z_1}{\pi |z_1|^2} \frac{1}{1-z_1} \left(1+\bar{g}^2 I(z_1)\right) z_1^{\frac{n}{2}+i\nu} (z_1^*)^{-\frac{n}{2}+i\nu}$$
$$\times \int \frac{d^2 z_2}{\pi |z_2|^2} \frac{1}{1-z_2^*} \left(1+\bar{g}^2 I^*(z_2)\right) (z_2^*)^{\frac{n}{2}-i\nu} z_2^{-\frac{n}{2}-i\nu} \tag{4.19}$$

where $w = k_2^+ q_1^+/(k_1^+ q_3^+)$.

4.3. Discontinuities in the s_{02}, s_{13} and s channels

The discontinuities in the s_{02}, s_{13} and s channels can be expressed through the ones calculated above with the help of the bootstrap relations (1.4). There are four such relations, but only three of them are independent. In general, for $\mathcal{A}_{2\to 2+n}$, there are $n+2$ bootstrap relations (1.4), for $j = 0, 1, \dots, n+1$, but their sum is identically zero. Denoting $\mathrm{disc}_{s_{ij}} \mathcal{A}^{\{-\}}_{2\to 4}/\Re\mathcal{A}_{2\to 4} = R^{\{-\}}_{ij}$, we have for $j = 0, 1, 2$ in the relations (1.4)

$$R^{\{-\}}_{01} + R^{\{-\}}_{02} + R^{\{-\}}_{03} = -i\pi\omega(t_1)\,, \quad R^{\{-\}}_{12} + R^{\{-\}}_{13} - R^{\{-\}}_{01} = -i\pi\left(\omega(t_2)-\omega(t_1)\right)\,,$$
$$R^{\{-\}}_{23} - R^{\{-\}}_{12} - R^{\{-\}}_{02} = -i\pi\left(\omega(t_3)-\omega(t_2)\right)\,. \tag{4.20}$$

This result gives

$$\mathrm{disc}_{s_{02}} \mathcal{A}^{\{-\}}_{2\to 4} = \mathrm{disc}_{s_3} \mathcal{A}^{\{-\}}_{2\to 4} - \mathrm{disc}_{s_2} \mathcal{A}^{\{-\}}_{2\to 4} - i\pi\left(\omega(t_2)-\omega(t_3)\right)\Re\mathcal{A}_{2\to 4}\,,$$
$$\mathrm{disc}_{s_{13}} \mathcal{A}^{\{-\}}_{2\to 4} = \mathrm{disc}_{s_1} \mathcal{A}^{\{-\}}_{2\to 4} - \mathrm{disc}_{s_2} \mathcal{A}^{\{-\}}_{2\to 4} - i\pi\left(\omega(t_2)-\omega(t_1)\right)\Re\mathcal{A}_{2\to 4}\,,$$
$$\mathrm{disc}_{s} \mathcal{A}^{\{-\}}_{2\to 4} = \mathrm{disc}_{s_2} \mathcal{A}^{\{-\}}_{2\to 4} - \mathrm{disc}_{s_1} \mathcal{A}^{\{-\}}_{2\to 4} - \mathrm{disc}_{s_3} \mathcal{A}^{\{-\}}_{2\to 4}$$
$$- i\pi\left(\omega(t_3)+\omega(t_1)-\omega(t_2)\right)\Re\mathcal{A}_{2\to 4}\,. \tag{4.21}$$

The same relations can be obtained from the representation of the discontinuities in terms of matrix elements of the evolution operators and the gluon production operators between the impact factor states and use of the bootstrap conditions (2.20)–(2.22) and (3.3).

Thus, all the discontinuities are expressed through the discontinuities in s_1, s_3 and s_2 channels, and the last one evidently disagree with the BDS ansatz. It is necessary to note that in the total imaginary part of $\mathcal{A}^{\{-\}}_{2\to 4}$ in the channel where all s_{ij} are positive, which is defined by the sum of all the discontinuities, the contribution of the s_2-channel discontinuity cancel, so we get

$$\sum_{i=0}^{n}\sum_{j=i+1}^{n+1} \mathrm{disc}_{s_{ij}} \mathcal{A}^{\{-\}}_{2\to 4} = \mathrm{disc}_{s_1} \mathcal{A}^{\{-\}}_{2\to 4} + \mathrm{disc}_{s_3} \mathcal{A}^{\{-\}}_{2\to 4} - i\pi\omega(t_2)\Re\mathcal{A}_{2\to 4}\,. \tag{4.22}$$

5. Discontinuities of amplitudes with larger number of particles

In general, there are $(n+1)(n+2)/2$ s_{ij}-channel discontinuities for the amplitude $\mathcal{A}_{2\to 2+n}$, that means ten discontinuities for $\mathcal{A}_{2\to 5}$. The bootstrap relations (1.4) give $n+1$ connections between them. For $\mathcal{A}_{2\to 5}$ one can choose as independent discontinuities in the channels s_1, s_2, s_3, s_4, s_{13} and, for example, s_{04}. The ratios $\mathrm{disc}_{s_{ij}} \mathcal{A}^{\{-\}}_{2\to 5}/\Re\mathcal{A}_{2\to 5}$ for the first four channels can be obtained from the results for $\mathcal{A}^{\{-\}}_{2\to 4}$ by evident substitutions. But the $\mathrm{disc}_{s_{13}} \mathcal{A}^{\{-\}}_{2\to 5}$ contains the new matrix element

$$\langle G_1 R_1 | e^{\hat{\mathcal{K}}_m \ln\left(\frac{s_2}{|\vec{k}_1|\,|\vec{k}_2|}\right)} \hat{\mathcal{G}}(k_2) e^{\hat{\mathcal{K}}_m \ln\left(\frac{s_3}{|\vec{k}_2|\,|\vec{k}_3|}\right)} | G_3 R_4 \rangle$$

where $\hat{\mathcal{G}}(\vec{k}_2)$ is the gluon production operator and k_2 is the gluon momentum. To calculate it one needs to know its matrix elements $\langle \mathcal{G}_1' \mathcal{G}_2' | \hat{\mathcal{G}}(k_2) | \mathcal{G}_1 \mathcal{G}_2 \rangle$. They are known in the LO, but in the NLO only matrix elements $\langle R_\omega(\vec{q}_2) | \hat{\mathcal{G}}(k_2) | \mathcal{G}_1 \mathcal{G}_2 \rangle$ are known in the "bootstrap scheme" (see, for instance, Refs. [9,11,38]), which was introduced to simplify the proof of validity of the bootstrap conditions. Of course, the matrix elements $\langle \mathcal{G}_1' \mathcal{G}_2' | \hat{\mathcal{G}}(k_2) | \mathcal{G}_1 \mathcal{G}_2 \rangle$ are necessary for calculation of discontinuities of amplitudes with larger number of particles. We intend to discuss this matrix element in subsequent paper.

A few words about the total imaginary part of $\mathcal{A}_{2\to 5}^{\{-\}}$ in the channel where all s_{ij} are positive. With account of the bootstrap conditions it can be greatly simplified, so that its ratio to the real part is expressed through gluon trajectories and the ratios of the type shown in Eqs. (4.3), (4.4).

6. Conclusion

In this paper, using the BFKL approach, we have performed an analysis of the discontinuities of multiple production amplitudes in invariant masses of pairs of produced gluons in the multi-Regge kinematics. We have discovered, in particularly, that the discontinuities of the four gluon production amplitudes contradict the BDS ansatz for MHV amplitudes in planar $\mathcal{N}=4$ super-symmetric Yang–Mills theory. This contradiction is almost obvious and is already apparent in the leading logarithmic approximation. It appears also in amplitudes with more than four produced gluons.

We have obtained explicit expressions of all discontinuities for production of three and four gluons, as well as of some of discontinuities for production of a greater number of gluons in the next-to-leading logarithmic approximation. It turns out that certain discontinuities have a rather complicated form. In particular, their color structure differs from the color structure of the real part of the corresponding amplitude. In the sum of all discontinuities the complicated pieces cancel due to the bootstrap conditions, so that the sum acquires a relatively simple form and the same color structure as the real part. This result can be important for further development of the BFKL approach.

Appendix A

First, consider Eq. (3.18). Using Eq. (2.24) and Eqs. (3.5), (3.6) and (3.8) we have

$$\langle R_\omega |^B \left[\ln\left(\hat{\vec{r}}_1^{\,2} \hat{\vec{r}}_2^{\,2}\right), \hat{\mathcal{K}}_r^B \right] | \mathcal{G}_1 \mathcal{G}_2 \rangle = \frac{\bar{g}^2 \delta(\vec{r}_1 + \vec{r}_2 - \vec{q})}{\Gamma(1-\epsilon)\pi^{1+\epsilon}} T^R_{\mathcal{G}_1\mathcal{G}_2} \int \frac{d\vec{r}_1{}' d\vec{r}_2{}' \delta(\vec{r}_1{}' + \vec{r}_2{}' - \vec{q})}{\vec{r}_1{}'^{\,2} \vec{r}_2{}'^{\,2}}$$
$$\times \left(\frac{\vec{r}_1^{\,2} \vec{r}_2{}'^{\,2} + \vec{r}_2^{\,2} \vec{r}_1{}'^{\,2}}{(\vec{r}_1 - \vec{r}_1{}')^2} - \vec{q}^{\,2} \right) \ln\left(\frac{\vec{r}_1{}'^{\,2} \vec{r}_2{}'^{\,2}}{\vec{r}_1^{\,2} \vec{r}_2^{\,2}} \right). \qquad \text{(A.1)}$$

Due to the symmetry under the $\vec{r}_1 \leftrightarrow \vec{r}_2$, $\vec{r}_1{}' \leftrightarrow \vec{r}_2{}'$ exchange, it is sufficient to calculate in Eq. (A.1) the integral with $\ln(\frac{\vec{r}_1{}'^{\,2}}{\vec{r}_1^{\,2}})$ and to add in the answer the term with $\vec{r}_1 \leftrightarrow \vec{r}_2$. The integral is not infrared divergent and can be evaluated at $\epsilon = 0$. It can be done using the decomposition

$$\frac{1}{\vec{r}_1^{\,\prime\,2}\vec{r}_2^{\,\prime\,2}}\left(\frac{\vec{r}_1^{\,2}\vec{r}_2^{\,\prime\,2}+\vec{r}_2^{\,2}\vec{r}_1^{\,\prime\,2}}{(\vec{r}_1-\vec{r}_1^{\,\prime})^2}-\vec{q}^{\,2}\right)=\left(\frac{1}{r_1^{\prime +}}+\frac{1}{r_1^+-r_1^{\prime +}}\right)\left(\frac{1}{r_1^--r_1^{\prime -}}-\frac{1}{q^--r_1^{\prime -}}\right)$$
$$+\left(\frac{1}{r_1^{\prime -}}+\frac{1}{r_1^--r_1^{\prime -}}\right)\left(\frac{1}{r_1^+-r_1^{\prime +}}-\frac{1}{q^+-r_1^{\prime +}}\right), \tag{A.2}$$

and the integral

$$\int\frac{d\vec{l}}{\pi}\left(\frac{1}{(a^+-1^+)}\frac{1}{(b^--1^-)}+\frac{1}{(a^--1^-)}\frac{1}{(b^+-1^+)}\right)\ln\left(\frac{\vec{l}^{\,2}}{\mu^2}\right)\theta(\Lambda^2-\vec{l}^{\,2})$$
$$=\ln\left(\frac{\Lambda^2}{(\vec{a}-\vec{b})^2}\right)\ln\left(\frac{\Lambda^2(\vec{a}-\vec{b})^2}{\mu^4}\right)+\ln\left(\frac{(\vec{a}-\vec{b})^2}{\vec{b}^{\,2}}\right)\ln\left(\frac{(\vec{a}-\vec{b})^2}{\vec{a}^{\,2}}\right). \tag{A.3}$$

The upper integration limit Λ is introduced because the separate terms of the decomposition (A.2) give divergent integrals. In the sum the divergencies cancel and that leads to the result

$$\int\frac{d\vec{r}_1^{\,\prime}d\vec{r}_2^{\,\prime}\delta(\vec{r}_1^{\,\prime}+\vec{r}_2^{\,\prime}-\vec{q})}{\vec{r}_1^{\,\prime\,2}\vec{r}_2^{\,\prime\,2}}\left(\frac{\vec{r}_1^{\,2}\vec{r}_2^{\,\prime\,2}+\vec{r}_2^{\,2}\vec{r}_1^{\,\prime\,2}}{(\vec{r}_1-\vec{r}_1^{\,\prime})^2}-\vec{q}^{\,2}\right)\ln\left(\frac{\vec{r}_1^{\,\prime\,2}\vec{r}_2^{\,\prime\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\right)$$
$$=-2\pi\ln\left(\frac{\vec{r}_1^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{q}^{\,2}}\right). \tag{A.4}$$

Using this result in Eq. (A.1) we come to Eq. (3.18).

Appendix B

Let us consider the integral in Eq. (3.40). The piece of this integral with the term $\vec{q}_2$ is known from the calculation of $\omega^B(t_1)$ and gives (with $\vec{k}=\vec{q}_1-\vec{q}_2$)

$$\vec{q}_2\int d\vec{r}_1d\vec{r}_2\frac{\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\delta(\vec{r}_1+\vec{r}_2-\vec{q}_1)=2\pi^{1+\epsilon}\Gamma(1-\epsilon)\vec{q}_2\left(\frac{1}{\epsilon}+\ln\vec{q}_1^{\,2}\right). \tag{B.1}$$

The integral with the second term can be represented as

$$\int d\vec{r}_1d\vec{r}_2\frac{\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\delta(\vec{r}_1+\vec{r}_2-\vec{q}_1)(\vec{r}_1-\vec{q}_2)\frac{\vec{q}_2^{\,2}}{(\vec{r}_1-\vec{q}_2)^2}$$
$$=\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{2}\frac{\partial}{\partial\vec{q}_2}\int\frac{d\vec{r}_1}{\vec{r}_1^{\,2}(\vec{r}_1-\vec{q}_1)^2}\ln(\vec{r}_1-\vec{q}_2)^2. \tag{B.2}$$

The last integral can be written as sum of two integrals:

$$\int\frac{d\vec{r}_1}{\vec{r}_1^{\,2}(\vec{r}_1-\vec{q}_1)^2}\ln(\vec{r}_1-\vec{q}_2)^2$$
$$=\frac{1}{2}\int\frac{d\vec{l}}{(\vec{q}_2-\vec{l})^2(\vec{k}+\vec{l})^2}\left(\ln\left(\frac{\vec{l}^{\,2}}{\vec{q}_2^{\,2}}\frac{\vec{l}^{\,2}}{\vec{k}^{\,2}}\right)+\ln\left(\vec{q}_2^{\,2}\vec{k}^{\,2}\right)\right). \tag{B.3}$$

Here the second integral is known, whereas in the first one the contributions of the singularities at $(\vec{q}_2 - \vec{l}) = 0$ and $(\vec{k} + \vec{l}) = 0$ cancel and the integral can be calculated at $\epsilon = 0$ using the decomposition

$$\frac{1}{(\vec{q}_2 - \vec{l})^2(\vec{k} + \vec{l})^2} = \frac{1}{\vec{q}_1^{\,2}}\left(\frac{1}{q_2^+ - l^+} + \frac{1}{k^+ + l^+}\right)\left(\frac{1}{q_2^- - l^-} + \frac{1}{k^- + l^-}\right) \tag{B.4}$$

and the integral (A.3). As a result, we have

$$\int \frac{d\vec{r}_1}{\vec{r}_1^{\,2}(\vec{r}_1 - \vec{q}_1)^2}\ln(\vec{r}_1 - \vec{q}_2)^2$$
$$= \pi^{1+\epsilon}\Gamma(1-\epsilon)\frac{1}{\vec{q}_1^{\,2}}\left(\ln\left(\vec{q}_2^{\,2}\vec{k}^{\,2}\right)\left(\frac{1}{\epsilon} + \ln\vec{q}_1^{\,2}\right) + \frac{1}{2}\ln^2\left(\frac{\vec{k}^{\,2}}{\vec{q}_2^{\,2}}\right)\right).$$

Substituting this result in Eq. (B.2) and using Eq. (B.1), we obtain

$$\int d\vec{r}_1 d\vec{r}_2 \frac{\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\delta(\vec{r}_1 + \vec{r}_2 - \vec{q}_1)\left(\vec{q}_2 + (\vec{r}_1 - \vec{q}_2)\frac{\vec{q}_2^{\,2}}{(\vec{r}_1 - \vec{q}_2)^2}\right)$$
$$= \pi^{1+\epsilon}\Gamma(1-\epsilon)\left(\vec{q}_2 + \vec{k}\frac{\vec{q}_2^{\,2}}{\vec{k}^{\,2}}\right)\left(\frac{1}{\epsilon} + \ln\left(\frac{\vec{q}_1^{\,2}\vec{k}^{\,2}}{\vec{q}_2^{\,2}}\right)\right). \tag{B.5}$$

Appendix C

Let us calculate now the chiral components of the tensor

$$J^{ij} = \frac{1}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\int \frac{d\vec{r}}{\vec{r}^{\,2}(\vec{q}_2 - \vec{r})^2}\left(\frac{q_1}{\vec{q}_1^{\,2}} - \frac{(\vec{q}_1 - r)}{(q_1 - \vec{r})^2}\right)^i\left(\frac{q_3}{\vec{q}_3^{\,2}} - \frac{(\vec{q}_3 - r)}{(q_3 - \vec{r})^2}\right)^j. \tag{C.1}$$

Writing

$$\left(\frac{q_1}{\vec{q}_1^{\,2}} - \frac{(\vec{q}_1 - r)}{(q_1 - \vec{r})^2}\right)^i = \left(\frac{q_1}{\vec{q}_1^{\,2}} - \frac{k_1}{\vec{k}_1^{\,2}}\right)^i + \left(\frac{k_1}{\vec{k}_1^{\,2}} - \frac{(\vec{q}_1 - r)}{(q_1 - \vec{r})^2}\right)^i \tag{C.2}$$

we can split the tensor in the sum of two pieces:

$$J^{ij} = J_1^{ij} + J_2^{ij}, \tag{C.3}$$

where

$$J_1^{ij} = \left(\frac{q_1}{\vec{q}_1^{\,2}} - \frac{k_1}{\vec{k}_1^{\,2}}\right)^i \frac{1}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\int \frac{d\vec{r}}{\vec{r}^{\,2}(\vec{q}_2 - \vec{r})^2}\left(\frac{q_3}{\vec{q}_3^{\,2}} - \frac{(\vec{q}_3 - r)}{(q_3 - \vec{r})^2}\right)^j,$$
$$J_2^{ij} = \frac{1}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\int \frac{d\vec{r}}{\vec{r}^{\,2}(\vec{q}_2 - \vec{r})^2}\left(\frac{k_1}{\vec{k}_1^{\,2}} - \frac{(\vec{q}_1 - r)}{(q_1 - \vec{r})^2}\right)^i\left(\frac{q_3}{\vec{q}_3^{\,2}} - \frac{(\vec{q}_3 - r)}{(q_3 - \vec{r})^2}\right)^j. \tag{C.4}$$

The first tensor can be obtained from Eq. (B.5) by the replacement $\vec{q}_1 \to \vec{q}_2$, $\vec{q}_2 \to \vec{q}_3$; we get

$$J_1^{ij} \simeq \left(\frac{q_1}{\vec{q}_1^{\,2}} - \frac{k_1}{\vec{k}_1^{\,2}}\right)^i\left(\frac{q_3}{\vec{q}_3^{\,2}} + \frac{k_2}{\vec{k}_2^{\,2}}\right)^j \frac{1}{\vec{q}_2^{\,2}}\left(\frac{1}{\epsilon} + \ln\left(\frac{\vec{q}_2^{\,2}\vec{k}_2^{\,2}}{\vec{q}_3^{\,2}}\right)\right). \tag{C.5}$$

The tensor J_2^{ij} is infrared finite and can be calculated at $\epsilon = 0$. The calculation of its chiral components can be performed easily using the decomposition of the integrand into a sum of terms of the type $(a^+ - r^+)^{-1}(b^- - r^-)^{-1}$ and the integral

$$\int \frac{d^2 r}{\pi(a^+ - r^+)(b^- - r^-)}\theta(\Lambda^2 - \vec{r}^{\,2}) = \ln\left(\frac{\Lambda^2}{(\vec{a} - \vec{b})^2}\right) . \tag{C.6}$$

It gives

$$J_2^{++} = \left(\frac{k}{\vec{k}^{\,2}} - \frac{k_1}{\vec{k}_1^{\,2}}\right)^+ \left(\frac{q_3}{\vec{q}_3^{\,2}} + \frac{k_2}{\vec{k}_2^{\,2}}\right)^+ \frac{1}{\vec{q}_2^{\,2}} \ln\left(\frac{\vec{q}_3^{\,2}\vec{k}_1^{\,2}}{\vec{k}_2^{\,2}\vec{q}_1^{\,2}}\right),$$

$$J_2^{+-} = \left(\frac{q_1}{\vec{q}_1^{\,2}} - \frac{k_1}{\vec{k}_1^{\,2}}\right)^+ \left(\frac{q_3}{\vec{q}_3^{\,2}} + \frac{k_2}{\vec{k}_2^{\,2}}\right)^- \frac{1}{\vec{q}_2^{\,2}} \ln\left(\frac{\vec{q}_3^{\,2}\vec{k}_1^{\,2}}{\vec{k}^{\,2}\vec{q}_2^{\,2}}\right), \tag{C.7}$$

where $\vec{k} = \vec{k}_1 + \vec{k}_2$ and $J_2^{--} = \left(J_2^{++}\right)^*$, $J_2^{-+} = \left(J_2^{+-}\right)^*$. Therefore, for the $+-$ component we have

$$J^{+-} \simeq \left(\frac{q_1}{\vec{q}_1^{\,2}} - \frac{k_1}{\vec{k}_1^{\,2}}\right)^+ \left(\frac{q_3}{\vec{q}_3^{\,2}} + \frac{k_2}{\vec{k}_2^{\,2}}\right)^- \frac{1}{\vec{q}_2^{\,2}} \left(\frac{1}{\epsilon} + \ln\left(\frac{\vec{k}_1^{\,2}\vec{k}_2^{\,2}}{\vec{k}^{\,2}}\right)\right)$$

$$= -\frac{1}{q_1^- k_1^- q_3^+ k_2^+}\left(\frac{1}{\epsilon} + \ln\left(\frac{\vec{k}_1^{\,2}\vec{k}_2^{\,2}}{\vec{k}^{\,2}}\right)\right). \tag{C.8}$$

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