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Fractional Integration of the H-Function of Several Variables

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Abstract—The main object of the present paper is to derive a number of key formulas for the fractional integration of the multivariable H-function (which is defined by a multiple contour integral of Mellin-Barnes type). Each of the general Eulerian integral formulas (obtained in this paper) are shown to yield interesting new results for various families of generalized hypergeometric functions of several variables. Some of these applications of the key formulas would provide potentially useful generalizations of known results in the theory of fractional calculus.

Keywords—Fractional integration, H-functions of one and more variables, Gamma and Beta functions, Eulerian integrals, Mellin-Barnes contour integrals, Binomial expansion, Appell functions, (Srivastava-Daoust) generalized Lauricella function, Fractional calculus.

1. INTRODUCTION AND PRELIMINARIES

In the theory of Gamma and Beta functions, it is well known that the Eulerian Beta integral

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.1}$$

$$(\Re(\alpha) > 0; \quad \Re(\beta) > 0),$$

can be rewritten (by a simple change of the variable of integration) in its *equivalent* form

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\,dt = (b-a)^{\alpha+\beta-1}\,B(\alpha,\beta), \tag{1.2}$$

$$(\Re(\alpha) > 0; \quad \Re(\beta) > 0; \quad a \neq b).$$

Since

$$(ut+v)^\gamma = (au+v)^\gamma \sum_{\ell=0}^{\infty} \frac{(-\gamma)_\ell}{\ell!} \left\{-\frac{(t-a)u}{au+v}\right\}^\ell, \tag{1.3}$$

$$(|(t-a)u| < |au+v|; \quad t \in [a,b]),$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $(\lambda)_\mu = \Gamma(\lambda+\mu)/\Gamma(\lambda)$, we readily find from (1.2) that (cf., e.g., [1, p. 301, Entry 2.2.6.1])

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^\gamma\,dt = (b-a)^{\alpha+\beta-1}(au+v)^\gamma B(\alpha,\beta)$$

$$\cdot {}_2F_1\left[\alpha,\ -\gamma;\ \alpha+\beta;\ -\frac{(b-a)u}{au+v}\right], \tag{1.4}$$

$$\left(\Re(\alpha)>0;\ \Re(\beta)>0;\ \left|\arg\left(\frac{bu+v}{au+v}\right)\right| \leq \pi-\epsilon\ (0<\epsilon<\pi);\ b\neq a\right),$$

where ${}_pF_q$ denotes, as usual, a generalized hypergeometric function with p numerator and q denominator parameters, and the argument condition emerges from the analytic continuation of the Gaussian hypergeometric function ${}_2F_1$ occurring on the right-hand side of (1.4).

For $\gamma=-\alpha-\beta$, the second member of (1.4) would simplify *considerably*, and if we further set

$$u=\lambda-\mu \qquad \text{and} \qquad v=(1+\mu)b-(1+\lambda)a$$

in terms of the *new* parameters λ and μ, the special case $\gamma=-\alpha-\beta$ of (1.4) would yield (cf., e.g., [2, p. 287, Entry 3.198])

$$\int_a^b \frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{\{b-a+\lambda(t-a)+\mu(b-t)\}^{\lambda+\mu}}\,dt = \frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}}{b-a}B(\alpha,\beta), \tag{1.5}$$

$$(\Re(\alpha)>0;\quad \Re(\beta)>0;\quad b-a+\lambda(t-a)+\mu(b-t)\neq 0\ (t\in[a,b]);\quad a\neq b).$$

Making use of (1.5), Raina and Srivastava [3] addressed the problem of closed-form evaluation of the following general Eulerian integral:

$$\mathcal{I}(z) := \int_a^b \frac{(t-a)^\lambda(b-t)^\mu}{\{f(t)\}^{\lambda+\mu+2}} H_{p,q}^{m,n}\left[z\{g(t)\}^\nu \,\middle|\, \begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right]dt, \tag{1.6}$$

where

$$f(t) := b-a+\rho(t-a)+\sigma(\eta-t), \tag{1.7}$$

$$g(t) := \frac{(t-a)^\gamma(b-t)^\delta\{f(t)\}^{1-\gamma-\delta}}{(b-a)\beta+(\beta\rho+\alpha-\beta)(t-a)+\beta\sigma(b-t)}, \tag{1.8}$$

and $H_{p,q}^{m,n}[z \mid \cdots]$ denotes the familiar H-function of Fox [4, p. 408], defined by (see also, [5, Chapter 2])

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right] := \frac{1}{2\pi i}\int_{\mathcal{L}}\Theta(\zeta)\,z^\zeta\,d\zeta, \tag{1.9}$$

$$\left(i:=\sqrt{-1};\ z\in\mathbb{C}\setminus\{0\};\ z^\zeta=\exp\{\zeta[\log|z|+i\arg(z)]\}\right),$$

where $\log|z|$ represents the *natural* logarithm of $|z|$ and $\arg(z)$ is not necessarily the principal value. Here, for convenience,

$$\Theta(\zeta) := \frac{\prod_{j=1}^{m}\Gamma(b_j-B_j\zeta)\prod_{j=1}^{n}\Gamma(1-a_j+A_j\zeta)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j\zeta)\prod_{j=n+1}^{p}\Gamma(a_j-A_j\zeta)}; \tag{1.10}$$

an empty product is interpreted (as usual) as 1; the integers m, n, p, q satisfy the inequalities

$$0\leq n\leq p \qquad \text{and} \qquad 1\leq m\leq q;$$

the coefficients

$$A_j > 0 \quad (j = 1, \ldots, p) \qquad \text{and} \qquad B_j > 0 \quad (j = 1, \ldots, q)$$

and the *complex* parameters

$$a_j \quad (j = 1, \ldots, p) \qquad \text{and} \qquad b_j \quad (j = 1, \ldots, q)$$

are so constrained that no poles of the integrand in (1.9) coincide, and $\mathcal{L}$ is a suitable contour of the Mellin-Barnes type (in the complex ζ-plane) which separates the poles of one product from those of the other. Furthermore, if we let

$$\Omega := \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0, \tag{1.11}$$

then the integral in (1.9) converges absolutely and defines the H-function, *analytic* in the sector:

$$|\arg(z)| < \frac{1}{2}\Omega\pi, \tag{1.12}$$

the point $z = 0$ being tacitly excluded. In fact, according to Braaksma [6, p. 278], the H-function makes sense and defines an analytic function of z also when *either*

$$\Lambda := \sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j < 0 \qquad \text{and} \qquad 0 < |z| < \infty, \tag{1.13}$$

or

$$\Lambda = 0 \qquad \text{and} \qquad 0 < |z| < R := \prod_{j=1}^{p} A_j^{-A_j} \prod_{j=1}^{q} B_j^{B_j}. \tag{1.14}$$

Recently, Saxena and Nishimoto [7] made use of the integral formula (1.4) in order to evaluate the following Eulerian integrals in terms of an H-function of *two* variables:

$$\mathcal{J}_{\pm\delta}(z) := \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma} \cdot H_{p,q}^{m,n}\left[z(ut+v)^{\pm\delta} \,\middle|\, \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right] dt. \tag{1.15}$$

They also considered a number of interesting special cases of their integral formulas involving (1.15). In each case, however, their result was expressed in terms of an H-function of *two* variables. The present paper has stemmed essentially from our attempt to express the integrals in (1.15), and indeed also those that are contained in (1.15), in terms of special functions of similar or lesser complexity. Thus, in general, we aim at expressing an Eulerian integral of the type (1.15), involving an H-function of r variables, in terms of an H-function of r variables.

By setting $b = x$, each of the Eulerian integrals (considered in the aforementioned works by Raina and Srivastava [3] and Saxena and Nishimoto [7]) can easily be rewritten as a *fractional* integral formula involving the familiar (*fractional*) differintegral operator ${}_aD_x^{\nu}$ defined by (cf., e.g., [8–10])

$${}_aD_x^{\nu}\{f(x)\} := \begin{cases} \frac{1}{\Gamma(-\nu)} \int_a^x (x-t)^{-\nu-1} f(t)\, dt, & (a \in \mathbb{R};\ \Re(\nu) < 0), \\ \frac{d^m}{dx^m}\, {}_aD_x^{\nu-m}\{f(x)\}, & (0 \leq \Re(\nu) < m;\ m \in \mathbb{N} := \{1, 2, 3, \ldots\}), \end{cases} \tag{1.16}$$

provided that the integral exists. In fact, when $a = 0$, the operator

$$D_x^{\nu} := {}_0D_x^{\nu}, \qquad (\nu \in \mathbb{C}), \tag{1.17}$$

corresponds to the classical *Riemann-Liouville fractional derivative* (or *integral*) of order ν (or $-\nu$). Moreover, when $a \to \infty$, equation (1.16) may be identified with the definition of the familiar *Weyl fractional derivative* (or *integral*) of order ν (or $-\nu$) (see also Erdélyi *et al.* [11, Chapter 13]).

The computation of fractional derivatives (and fractional integrals) of special functions of one and more variables is important from the point of view of the usefulness of these results in (for example) the evaluation of series and integrals (cf., e.g., [12,13]), the derivation of generating functions [14, Chapter 5], and the solution of differential and integral equations (cf. [12] and [15, Chapter 3]; (see also [16–18]). Motivated by these and other avenues of applications, Srivastava *et al.* [19,20] obtained several fractional derivative formulas involving the multivariable H-function which was defined by Srivastava and Panda (see [21, p. 271, Equation (4.1) *et seq.*]) and studied systematically by them (see [21–24]; see also [5]). For this multivariable H-function, we shall employ the contracted notations (due essentially to Srivastava and Panda [21]) which are used (among other places) in a subsequent monograph by Srivastava *et al.* [5, p. 251, equation (C.1)]. Thus, following the various conventions and notations explained fairly fully in these earlier works [21–24]; (see also [5,19,20]), let

$$H[z_1,\dots,z_r] \equiv H^{0,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j;\alpha'_j,\dots,\alpha^{(r)}_j)_{1,p} : (c'_j,\gamma'_j)_{1,p_1};\dots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ (b_j;\beta'_j,\dots,\beta^{(r)}_j)_{1,q} : (d'_j,\delta'_j)_{1,q_1};\dots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r} \end{matrix}\right] \tag{1.18}$$

denote the H-function of r complex variables $z_1,\dots,z_r$. Here, for convenience, $(a_j;\alpha'_j,\dots,\alpha^{(r)}_j)_{1,p}$ abbreviates the p-member array

$$\left(a_1;\alpha'_1,\dots,\alpha^{(r)}_1\right),\dots,\left(a_p;\alpha'_p,\dots,\alpha^{(r)}_p\right), \tag{1.19}$$

while $(c^{(k)}_j,\gamma^{(k)}_j)_{1,p_k}$ abbreviates the array of p_k pairs of parameters

$$\left(c^{(k)}_1,\gamma^{(k)}_1\right),\dots,\left(c^{(k)}_{p_k},\gamma^{(k)}_{p_k}\right) \qquad (k=1,\dots,r), \tag{1.20}$$

and so on. Suppose, as usual, that the parameters

$$\begin{cases} a_j, & j=1,\dots,p; \quad c^{(k)}_j, \quad j=1,\dots,p_k; \\ b_j, & j=1,\dots,q; \quad d^{(k)}_j, \quad j=1,\dots,q_k; \end{cases} \qquad (\forall\, k\in\{1,\dots,r\}), \tag{1.21}$$

are complex numbers, and the associated coefficients

$$\begin{cases} \alpha^{(k)}_j, & j=1,\dots,p; \quad \gamma^{(k)}_j, \quad j=1,\dots,p_k; \\ \beta^{(k)}_j, & j=1,\dots,q; \quad \delta^{(k)}_j, \quad j=1,\dots,q_k; \end{cases} \qquad (\forall\, k\in\{1,\dots,r\}), \tag{1.22}$$

are positive real numbers such that

$$\Lambda_k := \sum_{j=1}^{p}\alpha^{(k)}_j - \sum_{j=1}^{q}\beta^{(k)}_j + \sum_{j=1}^{p_k}\gamma^{(k)}_j - \sum_{j=1}^{q_k}\delta^{(k)}_j \le 0 \tag{1.23}$$

and

$$\Omega_k := \sum_{j=n+1}^{p}\alpha^{(k)}_j - \sum_{j=1}^{q}\beta^{(k)}_j + \sum_{j=1}^{n_k}\gamma^{(k)}_j - \sum_{j=n_k+1}^{p_k}\gamma^{(k)}_j + \sum_{j=1}^{m_k}\delta^{(k)}_j - \sum_{j=m_k+1}^{q_k}\delta^{(k)}_j > 0, \qquad (\forall\, k\in\{1,\dots,r\}), \tag{1.24}$$

where the integers n, p, q, m_k, n_k, p_k, and q_k are constrained by the inequalities $0 \leq n \leq p$, $q \geq 0$, $1 \leq m_k \leq q_k$, and $0 \leq n_k \leq p_k$ $(\forall k \in \{1, \dots, r\})$, and the equality in (1.23) holds true for suitably restricted values of the complex variables $z_1, \dots, z_r$.

Then, it is known that the multiple Mellin-Barnes contour integral (cf., e.g., [5, p. 251, equation (C.1)]) representing the multivariable H-function (1.3) converges absolutely, under the conditions (1.24), when

$$|\arg(z_k)| < \frac{1}{2}\Omega_k \pi, \qquad (\forall\, k \in \{1, \dots, r\}), \tag{1.25}$$

the points $z_k = 0$ $(k = 1, \dots, r)$ and various exceptional parameter values being tacitly excluded. Furthermore, we have (cf. [12, p. 131, equation (1.9)]):

$$H[z_1, \dots, z_r] = \begin{cases} O(|z_1|^{\xi_1} \cdots |z_r|^{\xi_r}), & (\max\{|z_1|, \dots, |z_r|\} \to 0), \\ O(|z_1|^{\eta_1} \cdots |z_r|^{\eta_r}), & (n = 0;\ \min\{|z_1|, \dots, |z_r|\} \to \infty), \end{cases} \tag{1.26}$$

where (with $k = 1, \dots, r$)

$$\begin{aligned} \xi_k &= \min\left\{\frac{\Re(d_j^{(k)})}{\delta_j^{(k)}}\right\}, \qquad (j = 1, \dots, m_k), \\ \eta_k &= \max\left\{\frac{\Re(c_j^{(k)} - 1)}{\gamma_j^{(k)}}\right\}, \qquad (j = 1, \dots, n_k), \end{aligned} \tag{1.27}$$

provided that each of the inequalities in (1.23)–(1.25) holds true.

We remark in passing that, throughout the present work, we shall assume that the convergence (and existence) conditions corresponding appropriately to the ones detailed above are satisfied by each of the various H-functions involved in our results which are presented in the following sections.

2. EULERIAN INTEGRALS OF THE MULTIVARIABLE H-FUNCTION

In this section, we first state one of our main integral formulas associated with the H-function of several variables:

$$\begin{aligned} &\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma} \cdot H\left[z_1(ut+v)^{-\rho_1}, \dots, z_r(ut+v)^{-\rho_r}\right] dt \\ &= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}\, B(\alpha,\beta) \sum_{\ell=0}^{\infty} \frac{(\alpha)_\ell}{\ell!\,(\alpha+\beta)_\ell} \left\{-\frac{(b-a)u}{au+v}\right\}^{\ell} \cdot H^{0,n+1:m_1,n_1;\dots;m_r,n_r}_{p+1,q+1:p_1,q_1;\dots;p_r,q_r} \\ &\left[\begin{matrix} z_1(au+v)^{-\rho_1} \\ \vdots \\ z_r(au+v)^{-\rho_r} \end{matrix} \,\middle|\, \begin{matrix} (1+\gamma-\ell;\rho_1,\dots,\rho_r), & (a_j;\alpha_j',\dots,\alpha_j^{(r)})_{1,p} : \\ (1+\gamma;\rho_1,\dots,\rho_r), & (b_j;\beta_j',\dots,\beta_j^{(r)})_{1,q} : \end{matrix} \right. \\ &\qquad\qquad \left. \begin{matrix} (c_j',\gamma_j')_{1,p_1};\dots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d_j',\delta_j')_{1,q_1};\dots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right] \end{aligned} \tag{2.1}$$

provided (in addition to the appropriate convergence and existence conditions) that

$$\min\{\rho_1, \dots, \rho_r\} > 0; \qquad \left|\frac{(b-a)u}{au+v}\right| < 1; \quad b \neq a;$$

and

$$\min\{\Re(\alpha),\, \Re(\beta)\} > 0.$$

Furthermore, if we employ the notation (cf. equation (1.18))

$$H^*[z_1,\dots,z_r] = H[z_1,\dots,z_r]|_{n=0}, \tag{2.2}$$

we also obtain the following companion of the integral formula (2.1):

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\cdot H^*\left[z_1(ut+v)^{\rho_1},\dots,z_r(ut+v)^{\rho_r}\right]dt$$
$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}\,B(\alpha,\beta)\sum_{\ell=0}^{\infty}\frac{(\alpha)_\ell}{\ell!\,(\alpha+\beta)_\ell}\left\{-\frac{(b-a)u}{au+v}\right\}^{\ell}\cdot H^{0,1:n_1,m_1;\dots;n_r,m_r}_{q+1,\,p+1:\,q_1,\,p_1;\dots;\,q_r,\,p_r}$$
$$\left[\begin{matrix} z_1^{-1}(au+v)^{-\rho_1}\\ \vdots \\ z_r^{-1}(au+v)^{-\rho_r}\end{matrix}\ \middle|\ \begin{matrix}(1+\gamma-\ell;\rho_1,\dots,\rho_r),(1-b_j;\beta_j',\dots,\beta_j^{(r)})_{1,q}:\\ (1+\gamma;\rho_1,\dots,\rho_r),(1-a_j;\alpha_j',\dots,\alpha_j^{(r)})_{1,p}:\end{matrix}\right.$$
$$\left.\begin{matrix}(1-d_j',\delta_j')_{1,q_1};\dots;(1-d_j^{(r)},\delta_j^{(r)})_{1,q_r}\\ (1-c_j',\gamma_j')_{1,p_1};\dots;(1-c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\end{matrix}\right] \tag{2.3}$$

provided (in addition to the appropriate convergence and existence conditions) that

$$\min\{\rho_1,\dots,\rho_r\}>0;\quad \left|\frac{(b-a)u}{au+v}\right|<1;\qquad b\neq a;$$

and

$$\min\{\Re(\alpha),\,\Re(\beta)\}>0.$$

DERIVATION OF THE INTEGRAL FORMULA 2.1. *For a simple and direct proof of the integral formula (2.1), we first replace the multivariable H-function occurring on the left-hand side by its Mellin-Barnes contour integral [5, p. 251, equation (C.1)], collect the powers of $(ut+v)$, and apply the binomial expansion (1.3) with, of course, γ replaced by*

$$\gamma-\sum_{k=1}^{r}\rho_k\,\xi_k,$$

where $\xi_1,\dots,\xi_r$ denote the variables of the aforementioned Mellin-Barnes contour integral. We then make use of the Eulerian integral (1.2) and interpret the resulting Mellin-Barnes contour integral as an H-function of the r variables:

$$\frac{z_1}{(au+v)^{\rho_1}},\dots,\frac{z_r}{(au+v)^{\rho_r}}.$$

We are thus led finally to the integral formula (2.1).

The (*sufficient*) conditions of validity of the integral formula (2.1), which we stated already with (2.1), would follow by appealing to the principle of analytic continuation.

DERIVATION OF THE INTEGRAL FORMULA 2.3. *Our proof of the integral formula (2.3) is much akin to that of (2.1), which we have outlined above. Indeed, in the proof of (2.3), we apply the binomial expansion (1.3) with γ replaced by*

$$\gamma+\sum_{k=1}^{r}\rho_k\,\xi_k,$$

and then set

$$\xi_k=-\zeta_k,\qquad (k=1,\dots,r),$$

with a view to interpreting the resulting Mellin-Barnes contour integral as an H-function of the r variables:

$$\frac{1}{z_1(au+v)^{\rho_1}},\dots,\frac{1}{z_r(au+v)^{\rho_r}}.$$

The details may be omitted.

Each of the integral formulas (2.1) and (2.3) can be put in a much more general setting. As a matter of fact, if we employ the binomial expansions (1.3) *and*

$$(yt+z)^{\delta} = (by+z)^{\delta} \sum_{m=0}^{\infty} \frac{(-\delta)_m}{m!} \left\{ \frac{(b-t)y}{by+z} \right\}^m , \tag{2.4}$$

$$(|\,(b-t)y\,| < |\,by+z\,|; \quad t \in [a,b]),$$

simultaneously, we shall similarly obtain the following (*symmetrical*) generalizations of the integral formulas (2.1) and (2.3):

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta} \\
&\quad \cdot H\left[z_1(ut+v)^{-\rho_1}(yt+z)^{-\sigma_1}, \ldots, z_r(ut+v)^{-\rho_r}(yt+z)^{-\sigma_r}\right] dt \\
&= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta} \cdot \sum_{\ell,m=0}^{\infty} \frac{B(\alpha+\ell,\beta+m)}{\ell!\; m!} \left\{-\frac{(b-a)u}{au+v}\right\}^{\ell} \left\{\frac{(b-a)y}{by+z}\right\}^m \\
&\quad \cdot H^{0,n+2:m_1,n_1;\ldots;m_r,n_r}_{p+2,q+2:\,p_1,\,q_1;\ldots;\,p_r,\,q_r} \left[\begin{matrix} z_1(au+v)^{-\rho_1}(by+z)^{-\sigma_1} \\ \vdots \\ z_r(au+v)^{-\rho_r}(by+z)^{-\sigma_r} \end{matrix} \right| \\
&\quad (1+\gamma-\ell;\rho_1,\ldots,\rho_r), (1+\delta-m;\,\sigma_1,\ldots,\sigma_r),\, (a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,p} : \\
&\qquad (1+\gamma;\,\rho_1,\ldots,\rho_r),\, (1+\delta;\sigma_1,\ldots,\sigma_r),\, (b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,q} : \\
&\qquad\qquad \left. \begin{matrix} (c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d_j',\delta_j')_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix} \right],
\end{aligned} \tag{2.5}$$

provided (in addition to the appropriate convergence and existence conditions) that

$$\min_{1\le k\le r}\{\rho_k,\sigma_k\} > 0; \quad \max\left\{\left|\frac{(b-a)u}{au+v}\right|, \left|\frac{(b-a)y}{by+z}\right|\right\} < 1; \qquad b \neq a;$$

and

$$\min\{\Re(\alpha),\, \Re(\beta)\} > 0;$$

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta} \\
&\quad \cdot H^*\left[z_1(ut+v)^{\rho_1}(yt+z)^{\sigma_1}, \ldots, z_r(ut+v)^{\rho_r}(yt+z)^{\sigma_r}\right] dt \\
&= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta} \cdot \sum_{\ell,m=0}^{\infty} \frac{B(\alpha+\ell,\beta+m)}{\ell!\; m!} \left\{-\frac{(b-a)u}{au+v}\right\}^{\ell} \left\{\frac{(b-a)y}{by+z}\right\}^m \\
&\quad \cdot H^{0,2:n_1,m_1;\ldots;n_r,m_r}_{q+2,\,p+2:q_1,\,p_1;\ldots;\,q_r,\,p_r} \left[\begin{matrix} z_1^{-1}(au+v)^{-\rho_1}(by+z)^{-\sigma_1} \\ \vdots \\ z_r^{-1}(au+v)^{-\rho_r}(by+z)^{-\sigma_r} \end{matrix} \right| \\
&\quad (1+\gamma-\ell;\rho_1,\ldots,\rho_r), (1+\delta-m;\,\sigma_1,\ldots,\sigma_r),\, (1-b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,q} : \\
&\qquad (1+\gamma;\rho_1,\ldots,\rho_r),\, (1+\delta;\,\sigma_1,\ldots,\sigma_r),\, (1-a_j;\,\alpha_j',\ldots,\alpha_j^{(r)})_{1,p} : \\
&\qquad\qquad \left. \begin{matrix} (1-d_j',\delta_j')_{1,q_1};\ldots;(1-d_j^{(r)},\delta_j^{(r)})_{1,q_r} \\ (1-c_j',\gamma_j')_{1,p_1};\ldots;(1-c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \end{matrix} \right],
\end{aligned} \tag{2.6}$$

provided (in addition to the appropriate convergence and existence conditions) that

$$\min_{1\le k\le r}\{\rho_k,\sigma_k\} > 0; \quad \max\left\{\left|\frac{(b-a)u}{au+v}\right|, \left|\frac{(b-a)y}{by+z}\right|\right\} < 1; \quad b \neq a;$$

and

$$\min\{\Re(\alpha),\ \Re(\beta)\} > 0,$$

$H^*[z_1, \ldots, z_r]$ being given by (2.2).

3. APPLICATIONS INVOLVING SIMPLER SPECIAL FUNCTIONS AND FRACTIONAL INTEGRATION

We begin by remarking that, by making use of (1.4) and noting that (cf., e.g., [25, p. 288])

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}\ {}_2F_1[\alpha,\beta;\gamma;z] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+\zeta)\Gamma(\beta+\zeta)}{\Gamma(\gamma+\zeta)}\cdot\Gamma(-\zeta)(-z)^{\zeta}\,d\zeta, \tag{3.1}$$

$$\left(i := \sqrt{-1};\quad |\arg(-z)| \leq \pi - \epsilon\ (0 < \epsilon < \pi);\qquad \gamma \neq 0, -1, -2, \ldots\right),$$

where the path of integration is indented, if necessary, in order to separate the poles at

$$\zeta = 0, 1, 2, \ldots,$$

from the poles at

$$\zeta = -\alpha - s \quad\text{and}\quad \zeta = -\beta - s,$$
$$(s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

we can easily evaluate each of the Eulerian integrals (2.1) and (2.3) in terms of an H-function of $r+1$ variables, the *additional* variable being

$$z_{r+1} = \frac{(b-a)u}{au+v}, \qquad (b \neq 0).$$

Thus, in their special case when $r = 1$, the integral formulas (2.1) and (2.3) can be shown to correspond to the main results of Saxena and Nishimoto [7, p. 69, equations (4.1) and (4.4)].

In terms of the Appell function F_3 defined by (cf., e.g., [26, p. 14])

$$F_3[\alpha,\alpha',\beta,\beta';\gamma;x,y] = \sum_{\ell,m=0}^{\infty} \frac{(\alpha)_\ell(\alpha')_m(\beta)_\ell(\beta')_m}{(\gamma)_{\ell+m}}\,\frac{x^\ell}{\ell!}\,\frac{y^m}{m!}, \tag{3.2}$$

$$(\max\{|x|,\ |y|\} < 1),$$

it is not difficult to deduce from (1.2), and the expansions (1.3) and (2.4), that

$$\begin{aligned}\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta}\,dt &= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)\\ &\cdot F_3\left[\alpha,\beta,-\gamma,-\delta;\ \alpha+\beta;\ -\frac{(b-a)u}{au+v},\ \frac{(b-a)y}{by+z}\right],\end{aligned} \tag{3.3}$$

where, for convergence,

$$\max\left\{\left|\frac{(b-a)u}{au+v}\right|,\ \left|\frac{(b-a)y}{by+z}\right|\right\} < 1; \qquad b \neq a;$$

and

$$\min\{\Re(\alpha),\ \Re(\beta)\} > 0.$$

Furthermore, since [26, p. 25, equation (34)]

$$F_3[\alpha,\gamma-\alpha,\beta,\beta';\ \gamma;\ x,y] = (1-y)^{-\beta'}\,F_1\left[\alpha,\beta,\beta';\ \gamma;\ x,\ \frac{y}{y-1}\right], \tag{3.4}$$

where F_1 denotes another Appell function defined by (cf., e.g., [26, p. 14])

$$F_1[\alpha, \beta, \beta'; \gamma; x, y] = \sum_{\ell, m=0}^{\infty} \frac{(\alpha)_{\ell+m} (\beta)_\ell (\beta')_m}{(\gamma)_{\ell+m}} \frac{x^\ell}{\ell!} \frac{y^m}{m!}, \tag{3.5}$$

$$(\max\{|x|, \ |y|\} < 1),$$

the integral formula (3.3) can be rewritten in the (*equivalent*) form

$$\begin{aligned} \int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} (ut+v)^\gamma (yt+z)^\delta \, dt = (b-a)^{\alpha+\beta-1} (au+v)^\gamma (ay+z)^\delta B(\alpha, \beta) \\ \cdot F_1 \left[\alpha, \ -\gamma, \ -\delta; \ \alpha+\beta; \ -\frac{(b-a)u}{au+v}, -\frac{(b-a)y}{ay+z} \right], \end{aligned} \tag{3.6}$$

provided that

$$\max\left\{ \left| \frac{(b-a)u}{au+v} \right|, \left| \frac{(b-a)y}{ay+z} \right| \right\} 1; \qquad b \neq a;$$

and

$$\min\{\Re(\alpha), \ \Re(\beta)\} > 0.$$

The last integral formula (3.6) can indeed be proven *directly* by appealing to (1.2), (1.3), and an obvious companion of the expansion (1.3) for $(yt+z)^\delta$. Both (3.3) and (3.6) would reduce, in the *special* case $\delta = 0$, to the known result (1.4). More interestingly, since [26, p. 40]

$$\begin{aligned} &\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')}{\Gamma(\gamma)} F_1[\alpha, \beta, \beta'; \gamma; x, y] \\ &= -\frac{1}{4\pi^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+\xi+\eta)\Gamma(\beta+\xi)\Gamma(\beta'+\eta)}{\Gamma(\gamma+\xi+\eta)} \cdot \Gamma(-\xi)\Gamma(-\eta)(-x)^\xi (-y)^\eta \, d\xi \, d\eta, \end{aligned} \tag{3.7}$$

$$\left(i := \sqrt{-1}; \ \max\{|\arg(-x)|, \ |\arg(-y)|\} < \pi; \ \gamma \neq 0, -1, -2, \dots\right),$$

and

$$\begin{aligned} &\frac{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}{\Gamma(\gamma)} F_3[\alpha, \ \alpha', \ \beta, \ \beta'; \ \gamma; \ x, y] \\ &= -\frac{1}{4\pi^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+\xi)\Gamma(\alpha'+\eta)\Gamma(\beta+\xi)\Gamma(\beta'+\eta)}{\Gamma(\gamma+\xi+\eta)} \cdot \Gamma(-\xi)\Gamma(-\eta)(-x)^\xi (-y)^\eta \, d\xi \, d\eta, \\ &\left(i := \sqrt{-1}; \ \max\{|\arg(-x)|, \ |\arg(-y)|\} < 1; \ \gamma \neq 0, -1, -2, \dots\right), \end{aligned} \tag{3.8}$$

the second member of each of the integral formulas (3.3) and (3.6) can be expressed as a *double* Mellin-Barnes contour integral. Thus, by employing the integral formula (3.3) or (3.6), and then appealing to (3.7) or (3.8), we can evaluate the integals in (2.5) and (2.6) in terms of H-functions of $r+2$ variables, the *additional* variables being

$$z_{r+1} = \frac{(b-a)u}{au+v} \quad \text{and} \quad z_{r+2} = -\frac{(b-a)y}{by+z},$$

or

$$z_{r+1} = \frac{(b-a)u}{au+v} \quad \text{and} \quad z_{r+2} = \frac{(b-a)y}{ay+z}.$$

Each of our integral formulas (2.1) and (2.3), and indeed also (2.5) and (2.6), possesses manifold generality. First of all, by specializing the various parameters and variables involved, these formulas (and indeed also their numerous variations obtained by letting any desired number of exponents

$$\rho_1, \ldots, \rho_r \quad \text{and} \quad \sigma_1, \ldots, \sigma_r$$

decrease to zero in such a manner that each side of the resulting equations exists) can be suitably applied to derive the corresponding results involving a remarkably wide variety of potentially useful functions (or products of several such functions), which are expressible in terms of the E, F, G, and H functions of one, two or more variables. For example, if $n = p = q = 0$, the multivariable H-function occurring on the left-hand side of each of our formulas (2.1) and (2.5), and also in (2.3) and (2.6) when $p = q = 0$, would reduce rather immediately to the product of r different H-functions of Fox [4]. Thus, the table listing various special cases of the H-function (given, amongst other places, in [5, pp. 18,19]) can be employed with a view to deriving Eulerian integral formulas involving any of these simpler special functions desired.

Next we turn to the applications of our main results (2.1) and (2.6) to the (Srivastava-Daoust) generalized Lauricella function of several variables (see, for details, [27, p. 454 *et seq.*]). Indeed, by appealing to the known relationship (cf. [21, p. 272, equation (4.7)]; [5, p. 253, equation (C.9)]), it is not difficult to derive the following integral formulas as special cases of (2.1) and (2.5):

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma} \cdot F^{p:p_1;\ldots;p_r}_{q:q_1;\ldots;q_r}\begin{pmatrix} z_1(ut+v)^{-\rho_1} \\ \vdots \\ z_r(ut+v)^{-\rho_r}\end{pmatrix} dt \\
&= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma} B(\alpha,\beta) F^{p+1:p_1;\ldots;p_r;1}_{q+1:q_1;\ldots;q_r;1} \\
&\left(\begin{matrix} (a_j;\alpha_j',\ldots,\alpha_j^{(r)},0)_{1,p}, & (-\gamma;\rho_1,\ldots,\rho_r,1): \\ (b_j;\beta_j',\ldots,\beta_j^{(r)},0)_{1,q}, & (-\gamma;\rho_1,\ldots,\rho_r,0): \end{matrix}\right. \\
&\qquad \left.\begin{matrix} (c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}; & (\alpha,1); \\ (d_j',\delta_j')_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}; & (\alpha+\beta,1); \end{matrix}\; Z_1,\ldots,Z_r,-\frac{(b-a)u}{au+v}\right),
\end{aligned} \tag{3.9}$$

where, for convenience,

$$Z_k := \frac{z_k}{(au+v)^{\rho_k}}, \qquad (k=1,\ldots,r);$$

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta} \cdot F^{p:p_1;\ldots;p_r}_{q:q_1;\ldots;q_r}\begin{pmatrix} z_1(ut+v)^{-\rho_1}(yt+z)^{-\sigma_1} \\ \vdots \\ z_r(ut+v)^{-\rho_r}(yt+z)^{-\sigma_r}\end{pmatrix} dt \\
&= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta} B(\alpha,\beta)\, F^{p+2:p_1;\ldots;p_r;1;1}_{q+3:q_1;\ldots;q_r;0;0} \\
&\left(\begin{matrix} (a_j;\alpha_j',\ldots,\alpha_j^{(r)},0,0,)_{1,p}, & (-\gamma;\rho_1,\ldots,\rho_r,1,0), \\ (b_j;\beta_j',\ldots,\beta_j^{(r)},0,0)_{1,q}, \quad (-\gamma;\rho_1,\ldots,\rho_r,0,0), & (-\delta;\sigma_1,\ldots,\sigma_r,0,0), \end{matrix}\right. \\
&\qquad \begin{matrix} (-\delta;\sigma_1,\ldots,\sigma_r,0,1): & (c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}; \\ (\alpha+\beta;0,\ldots,0,1,1): & (d_j',\delta_j')_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}; \end{matrix} \\
&\qquad \left.\begin{matrix} (\alpha,1); & (\beta,1); \\ \text{---}; & \text{---}; \end{matrix}\; \Xi_1,\ldots,\Xi_r,-\frac{(b-a)u}{au+v},\frac{(b-a)y}{by+z}\right),
\end{aligned} \tag{3.10}$$

where, for convenience,

$$\Xi_k := \frac{z_k}{(au+v)^{\rho_k}(by+z)^{\sigma_k}}, \qquad (k=1,\ldots,r);$$

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta} \cdot F^{p:p_1;\ldots;p_r}_{q:q_1;\ldots;q_r}\begin{pmatrix} z_1(ut+v)^{-\rho_1}(yt+z)^{-\sigma_1} \\ \vdots \\ z_r(ut+v)^{-\rho_r}(yt+z)^{-\sigma_r}\end{pmatrix} dt \\
&= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(ay+z)^{\delta} B(\alpha,\beta) F^{p+3:p_1;\ldots;p_r;0;0}_{q+3:q_1;\ldots;q_r;0;0}
\end{aligned}$$

$$\begin{pmatrix} (a_j;\alpha'_j,\ldots,\alpha_j^{(r)},0,0)_{1,p}, & (-\gamma;\rho_1,\ldots,\rho_r,1,0), & (-\delta;\sigma_1,\ldots,\sigma_r,0,1), \\ (b_j;\beta'_j,\ldots,\beta_j^{(r)},0,0)_{1,q}, & (-\gamma;\rho_1,\ldots,\rho_r,0,0), & (-\delta;\sigma_1,\ldots,\sigma_r,0,0), \\ (\alpha;0,\ldots,0,1,1): & (c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}; & \text{---};\ \text{---}; \\ (\alpha+\beta;0,\ldots,0,1,1): & (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}; & \text{---};\ \text{---}; \\ \multicolumn{3}{c}{\dfrac{z_1}{(au+v)^{\rho_1}(ay+z)^{\sigma_1}},\ldots,\dfrac{z_r}{(au+v)^{\rho_r}(ay+z)^{\sigma_r}},-\dfrac{(b-a)u}{au+v},-\dfrac{(b-a)y}{ay+z}} \end{pmatrix} \tag{3.11}$$

which is indeed equivalent to the integral formula (3.10).

The conditions of validity of the Eulerian integral formulas (3.9)–(3.11) are obtainable fairly easily from those of their parent formulas (2.1) and (2.5). Thus, in each case, we require that

$$\min\{\Re(\alpha),\ \Re(\beta)\}>0 \quad \text{and} \quad b\neq a,$$

and that all of the multiple hypergeometric series involved are absolutely convergent (cf., e.g., [28–30]).

The special cases of the Eulerian integral formulas (2.3) and (2.6), with the (Srivastava-Daoust) generalized Lauricella function in their integrands, are expressible in terms of multivariable H-functions just as in the parent formulas (2.3) and (2.6), and we skip the details involved.

A *further* special case of the Eulerian integral formula (3.9) when $r=1$ was given by Saxena and Nishimoto [7, p. 71, equation (4.9)] who incidentally expressed their result as an H-function of two variables (instead of a generalized Kampé de Fériet function in two variables). A similar remark would apply also to another Eulerian integral formula given by them [7, p. 72, equation (4.12)].

For $b=x$, each of the Eulerian integral formulas (2.1), (2.3), (2.5), (2.6), (3.9)–(3.11), and indeed, also all such results as (1.4), (1.5), (3.3), and (3.6), can easily be stated as a fractional integral formula involving the operator ${}_aD_x^{-\beta}$ defined by (1.16). Thus, for example, our integral formulas (2.1), (2.3), (2.5), and (2.6) yield the following results which are valid under the conditions stated already (with, of course, $b=x$):

$$\begin{aligned} &{}_aD_x^{-\alpha}\left\{(x-a)^{\beta-1}(ux+v)^{\gamma}H\left[z_1(ux+v)^{-\rho_1},\ldots,z_r(ux+v)^{-\rho_r}\right]\right\} \\ &= (x-a)^{\alpha+\beta-1}(au+v)^{\gamma}\sum_{\ell=0}^{\infty}\frac{\Gamma(\beta+\ell)}{\ell!\,\Gamma(\alpha+\beta+\ell)}\left\{-\frac{(x-a)u}{au+v}\right\}^{\ell} \\ &\cdot H_{p+1,\,q+1:\ p_1,\,q_1;\ldots;\,p_r,\,q_r}^{0,n+1:m_1,n_1;\ldots;m_r,n_r}\begin{bmatrix} z_1(au+v)^{-\rho_1} \\ \vdots \\ z_r(au+v)^{-\rho_r}\end{bmatrix}, \end{aligned} \tag{3.12}$$

where the multivariable H-function parameters are precisely the same as those displayed on the right-hand side of (2.1);

$$\begin{aligned} &{}_aD_x^{-\alpha}\left\{(x-a)^{\beta-1}(ux+v)^{\gamma}\ H^*\left[z_1(ux+v)^{\rho_1},\ldots,z_r(ux+v)^{\rho_r}\right]\right\} \\ &= (x-a)^{\alpha+\beta-1}(au+v)^{\gamma}\sum_{\ell=0}^{\infty}\frac{\Gamma(\beta+\ell)}{\ell!\,\Gamma(\alpha+\beta+\ell)}\left\{-\frac{(x-a)u}{au+v}\right\}^{\ell} \\ &\cdot H_{q+1,p+1:\ q_1,\,p_1;\ldots;\,q_r,\,p_r}^{0,1:n_1,m_1;\ldots;n_r,m_r}\begin{bmatrix} z_1^{-1}(au+v)^{-\rho_1} \\ \vdots \\ z_r^{-1}(au+v)^{-\rho_r}\end{bmatrix}, \end{aligned} \tag{3.13}$$

where the multivariable H-function parameters are precisely the same as those displayed on the right-hand side of (2.3);

$$\begin{aligned} &{}_aD_x^{-\alpha}\Big\{(x-a)^{\beta-1}(ux+v)^{\gamma}(xy+z)^{\delta} \\ &\qquad\cdot H\left[z_1(ux+v)^{-\rho_1}(xy+z)^{-\sigma_1},\ldots,z_r(ux+v)^{-\rho_r}(xy+z)^{-\sigma_r}\right]\Big\} \end{aligned}$$

$$= (x-a)^{\alpha+\beta-1}(au+v)^{\gamma}(xy+z)^{\delta} \cdot \sum_{\ell,m=0}^{\infty} \frac{B(\alpha+m,\beta+\ell)}{\ell!\, m!} \left\{-\frac{(x-a)u}{au+v}\right\}^{\ell} \left\{\frac{(x-a)y}{xy+z}\right\}^{m}$$
$$\cdot H_{p+2,q+2:\,p_1,q_1;\ldots;\,p_r,q_r}^{0,n+2:m_1,n_1;\ldots;m_r,n_r} \begin{bmatrix} z_1(au+v)^{-\rho_1}(xy+z)^{-\sigma_1} \\ \vdots \\ z_r(au+v)^{-\rho_r}(xy+z)^{-\sigma_r} \end{bmatrix}, \tag{3.14}$$

where the multivariable H-function parameters are precisely the same as those displayed on the right-hand side of (2.5);

$$ {}_aD_x^{-\alpha}\Big\{(x-a)^{\beta-1}(ux+v)^{\gamma}(xy+z)^{\delta}$$
$$\cdot H^{*}\left[z_1(ux+v)^{\rho_1}(xy+z)^{\sigma_1},\ldots,z_r(ux+v)^{\rho_r}(xy+z)^{\sigma_r}\right]\Big\}$$
$$= (x-a)^{\alpha+\beta-1}(au+v)^{\gamma}(xy+z)^{\delta} \cdot \sum_{\ell,m=0}^{\infty} \frac{B(\alpha+m,\beta+\ell)}{\ell!\, m!} \left\{-\frac{(x-a)u}{au+v}\right\}^{\ell} \left\{\frac{(x-a)y}{xy+z}\right\}^{m}$$
$$\cdot H_{q+2,p+2:\,q_1,p_1;\ldots;\,q_r,p_r}^{0,2:n_1,m_1;\ldots;n_r,m_r} \begin{bmatrix} z_1^{-1}(au+v)^{-\rho_1}(xy+z)^{-\sigma_1} \\ \vdots \\ z_r^{-1}(au+v)^{-\rho_r}(xy+z)^{-\sigma_r} \end{bmatrix}, \tag{3.15}$$

where the multivariable H-function parameters are precisely the same as those displayed on the right-hand side of (2.6).

The fractional integral formulas (3.12) and (3.13), with their right-hand sides expressed as an H-function of $r+1$ variables, the additional variable being

$$z_{r+1} = \frac{(x-a)u}{au+v},$$

would obviously generalize the *main* results (Theorem 5.1 and Theorem 5.2, respectively) of Saxena and Nishimoto [7]. As a matter of fact, in *each* of these *main* results of Saxena and Nishimoto [7], the factor $\Gamma(\alpha)$ appears *erroneously* on the right-hand side and should be deleted.

For numerous further results involving fractional calculus of special functions in one and more variables, see the works (amongst others) by Srivastava *et al.* [19,20] and Lavoie *et al.* [31].

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