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Spiral Waves for λ–ω Systems, II*

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1. INTRODUCTION

In a recent paper [1], the author established that the λ–ω equations introduced by Kopell and Howard [2–4], support rotating-spiral wave solutions. The specific equations studied were

$$\frac{\partial}{\partial t}(u + iv) = (\lambda + i\omega)(u + iv) + \Delta_2(u + iv), \tag{1.1}$$

where λ and ω were the following functions of $A = \sqrt{u^2 + v^2}$:

$$\lambda = 1 - A, \qquad \omega = 1 + \omega_1(A - 1), \qquad \text{and} \qquad \omega_1 > 0, \tag{1.2}$$

and Δ_2 was the two-dimensional Laplacian. The rotating spirals were solutions of (1.1) of the form:

$$u + iv = A(r)\exp\left(i\left(\Omega t \pm \Theta - \int_0^r k(s)\,ds\right)\right), \tag{1.3}$$

where

$$\Omega = 1 - \omega_1 k_\infty^2 \tag{1.4}$$

and A and k satisfy

$$\lim_{r\to 0^+} (A, k)(r) = (0,0) \qquad \text{and} \qquad \lim_{r\to\infty} (A, k)(r) = (1 - k_\infty^2, k_\infty), \tag{1.5}$$

with $k_\infty > 0$. These solutions obtain so long as

$$0 < \omega_1 \leq 0(k_\infty) \qquad \text{and} \qquad 0 < k_\infty \ll 1. \tag{1.6}$$

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In this we re-examine the spiral-wave problem when (1.2) *is replaced by*

$$\lambda = 1 - A, \qquad \omega = 1 - \omega_1(A - 1), \qquad \text{and} \qquad \omega_1 > 0. \quad (1.2)^{\#}$$

Over the years there has been some controversy as to the appropriate sign of ω_1; that is, in systems of actual interest (of which (1.1) and (1.2) or $(1.2)^{\#}$ is supposed to be a prototype) which possess an isolated, orbitally stable limit cycle (in our case the solution $A \equiv 1$) is the frequency ω an increasing or decreasing function of the amplitude. It is not our purpose to answer this question here; rather it is to show that (1.1) has spiral solutions of the form (1.3) with A and k satisfying (1.4) independently of whether (1.2) or $(1.2)^{\#}$ holds.[1] There are essential differences in these two cases. When (1.2) holds there is a two-parameter family of solutions (indexed by k_∞ and ω_1) whereas when $(1.2)^{\#}$ holds there is only a one-parameter family.

2. Asymptotic Equations

A. The Equations

It is a simple matter to check that if (1.1) has a solution of the form (1.3), if (1.4) holds, and if λ and ω are given by $(1.2)^{\#}$, then A and k satisfy

$$(DEA)\ (rA_r)r + rA\left(1 - A - k^2 - \frac{1}{r^2}\right) = 0, \qquad r > 0,$$

$$(DEk)\ (rA^2k)_r = \omega_1 rA^2\left(1 - k_\infty^2 - A\right), \qquad r > 0,$$

$$(BC)\ \lim_{r\to 0^+}(A,k)(r) = (0,0) \quad \text{and} \quad \lim_{r\to\infty}(A,k)(r) = \left(1 - k_\infty^2, k_\infty\right),$$

where $k_\infty > 0$ and $\omega_1 > 0$.

Rather than work with A it is convenient to work with the independent variable $\phi \overset{\text{def}}{=} A + k^2 - 1$. Then ϕ and k satisfy

$$(DE\phi)\ (r\phi_r)_r - r(1+\phi)\left(\frac{1}{r^2}+\phi\right) = \left(\left(r(k^2)_r\right)_r - r\left(\phi + \frac{1}{r^2}\right)k^2\right), \qquad r > 0,$$

$$(DEk)\ \left(r(1+\phi-k^2)^2k\right)_r = \omega_1 r(1+\phi-k^2)^2(k^2 - k_\infty^2)$$
$$\qquad - \omega_1 r(1+\phi-k^2)^2\phi, \qquad r > 0,$$

$$(BC)\ \lim_{r\to 0^+}(\phi(r), k(r)) = (-1,0) \quad \text{and} \quad \lim_{r\to\infty}(\phi(r), k(r)) = (0, k_\infty).$$

[1]When $(1.2)^{\#}$ holds, Ω is given by $1 + \omega_1 k_\infty^2$.

The advantage to working with ϕ over A is that for any pair $0 < k_\infty$ and $0 < \omega_1$ solutions of $(DE\phi)$ and (DEk) which algebraically meet the boundary conditions at $r = +\infty$ satisfy the asymptotic relationship $\phi \sim -1/r^2$ independently of k_∞ whereas solutions A of (DEA) and (DEk) which algebraically meet the boundary condition at $r = +\infty$ do so at a slower rate; in particular $A \sim 1 - k_\infty^2 - k_\infty(1 - k_\infty^2)^2/\omega_1 r$.

For solutions of $(DE\phi)$, (DEk), and (BC) with k uniformly small, the last observation allows us to replace ϕ by ϕ_0, where ϕ_0 satisfies

$$(DE\phi)_0 \ (r\phi_{0r})_r - r(1+\phi_0)\left(\frac{1}{r^2} + \phi_0\right) = 0, \qquad r > 0,$$

$$(BC\phi)_0 \ \lim_{r\to\infty} \phi_0(r) = -1 \qquad \text{and} \qquad \lim_{r\to\infty} \phi_0(r) = 0,$$

and be assured that not only is ϕ_0 uniformly close to ϕ but that it has the same asymptotic behavior at $r = +\infty$. The approximating equation for k is then obtained by replacing the terms $(1 + \phi - k^2)^2$ in (DEk) with $(1 + \phi_0)^2$; the result is

$$(DEk)_0 \ \left(r(1+\phi_0)^2 k\right)_r = \omega_1 r(1+\phi_0)^2(k^2 - k_\infty^2) - \omega_1 r(1+\phi_0)^2\phi_0$$

$$(BCk)_0 \ \lim_{r\to 0^+} k(r) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} k(r) = k_\infty > 0.$$

B. The Solutions

We start with a few words about the boundary value problem $(DE\phi)_0$ and $(BC\phi)_0$. In [1, Section 2-B] it was shown there exists an increasing function $r \to \mathcal{Q}_0(r)$ satisfying

$$(DE\mathcal{Q})_0 \ (r\mathcal{Q}_{0r})_r + r\mathcal{Q}_0\left(1 - \mathcal{Q}_0 - \frac{1}{r^2}\right) = 0, \qquad r > 0,$$

$$(BC\mathcal{Q})_0 \ \lim_{r\to 0} \mathcal{Q}_0(r) = 0 \qquad \text{and} \qquad \lim_{r\to\infty} \mathcal{Q}_0(r) = 1,$$

and the asymptotic estimate $\mathcal{Q}_0 \sim 1 - 1/r^2$. The solution of $(DE\phi)_0$ and $(BC\phi)_0$ may be expressed in terms of $\mathcal{Q}_0$ by

$$\phi_0(r) = \mathcal{Q}_0(r) - 1. \tag{2.1}$$

In what follows it will be convenient to work with $\mathcal{Q}_0$.

We now turn to the problem (DEk) and (BCk). Instead of working with the wave number $k(r)$ we introduce the unknown

$$\mu(r) \stackrel{\text{def}}{=} \omega_1 k(r), \tag{2.2}$$

and work with the parameters ω_1 and $\mu_\infty \stackrel{\text{def}}{=} \omega_1 k_\infty$. μ satisfies

$$(DE\mu)_0 \; \left(r\mathcal{Q}_0^2\mu\right)_r = r\mathcal{Q}_0^2\left(\mu^2 - \mu_\infty^2\right) + \omega_1^2 r\mathcal{Q}_0^2(r)\left(1 - \mathcal{Q}_0(r)\right), \qquad r > 0,$$

and

$$(BC\mu)_0 \; \lim_{r\to 0^+}\mu(r) = 0 \qquad \text{and} \qquad \lim_{r\to\infty}\mu(r) = \mu_\infty.$$

It is easily checked that μ solves $(DE\mu)_0$ and the boundary condition at $r = \infty$ if and only if

$$\mu(r) = \mu_\infty + \frac{1}{r\mathcal{Q}_0^2(r)}\int_r^\infty \exp\left(-\int_r^s (\mu_\infty + \mu(t))\,dt\right) \times\left(\mu_\infty\left(s\mathcal{Q}_0^2(s)\right)_s - \omega_1^2 s\mathcal{Q}_0^2(s)\left(1 - \mathcal{Q}_0(s)\right)\right)ds \tag{2.3}$$

and μ meets the boundary condition at $r = 0$ if and only if

$$\omega_1^2 = \frac{\int_0^\infty \exp\left(-\int_0^s (\mu_\infty + \mu(t))\,dt\right) s\mathcal{Q}_0^2(s)(\mu_\infty(\mu_\infty + \mu(s)))\,ds}{\int_0^\infty \exp\left(-\int_0^s (\mu_\infty + \mu(t))\,dt\right) s\mathcal{Q}_0^2(s)(1 - \mathcal{Q}_0(s))\,ds}. \tag{2.4}$$

The relation (2.4) illustrates that the parameters μ_∞ and ω_1 cannot be prescribed independently. The system (2.3) and (2.4) may be solved by iteration; specifically by the scheme

$$\left(\omega_1^{N+1}\right)^2 = \frac{\int_0^\infty \exp\left(-\int_0^s \left(\mu_\infty + \mu^N(t)\right)dt\right) s\mathcal{Q}_0^2(s)\left(\mu_\infty\left(\mu_\infty + \mu^N(s)\right)\right)ds}{\int_0^\infty \exp\left(-\int_0^s \left(\mu_\infty + \mu^N(t)\right)dt\right) s\mathcal{Q}_0^2(s)\left(1 - \mathcal{Q}_0(s)\right)ds} \tag*{$(2.4)_N$}$$

and

$$\mu^{N+1}(r) = \mu_\infty + \frac{1}{r\mathcal{Q}_0^2(r)}\int_r^\infty \exp\left(-\int_r^s \left(\mu_\infty + \mu^N(t)\right)dt\right) \times\left(\mu_\infty\left(s\mathcal{Q}_0^2(s)\right)_s - \left(\omega_1^{N+1}\right)^2 s\mathcal{Q}_0^2(s)\left(1 - \mathcal{Q}_0(s)\right)\right)ds, \tag*{$(2.5)_{N+1}$}$$

where

$$\mu^0(r) \equiv \mu_\infty. \tag{2.5$_0$}$$

There is also a variational formulation to $(DE\mu)_0$ and $(BC\mu)_0$. We introduce the potential η by

$$\frac{\eta_r}{\eta} = -\mu(r). \tag{2.6}$$

Then μ satisfies $(DE\mu)_0$ and $(BC\mu)_0$ if $\eta(r) \neq 0$ solves

$$-\left(r\mathcal{C}_0^2(r)\eta_r\right)_r + \mu_\infty^2 r\mathcal{C}_0^2(r)\eta = \omega_1^2 r\mathcal{C}_0^2(r)\left(1 - \mathcal{C}_0(r)\right)\eta, \tag{2.7}$$

and

$$\lim_{r\to 0^+} \eta_r(r) = 0 \qquad \text{and} \qquad \eta \sim \frac{\exp(-\mu_\infty r)}{r^{1/2}} \qquad \text{as } r \to \infty. \tag{2.8}$$

The positive solution of (2.7) and (2.8) minimizes the Raleigh quotient

$$J(\psi, \mu_\infty^2) \stackrel{\text{def}}{=} \frac{\int_0^\infty r\mathcal{C}_0^2(r)\left(\psi_r^2 + \mu_\infty^2\psi^2(r)\right)dr}{\int_0^\infty r\mathcal{C}_0^2(r)\left(1 - \mathcal{C}_0(r)\right)\psi^2(r)\,dr} \tag{2.9}$$

and the parameter ω_1^2 is given by

$$\omega_1^2 = \inf_\psi J(\psi, \mu_\infty^2) = J(\eta, \mu_\infty^2). \tag{2.10}$$

It is easily checked that $J(\psi, \mu_\infty^2)$ is bounded from below by μ_∞^2 which is equivalent to the constraint that $k_\infty \leq 1$. Upper bounds for ω_1^2 are obtained by evaluating $J(e^{-\mu_\infty \cdot}, \mu_\infty^2)$. As μ_∞ tends to zero this upper bound reduces to

$$\omega_1^2 \leq J\left(e^{-\mu_\infty \cdot}, \mu_\infty^2\right) \sim -\frac{1}{2\log\mu_\infty} \qquad \text{as } \mu_\infty \to 0 \tag{2.12}$$

This in turn yields the lower bound k_∞:

$$k_\infty \geq \frac{\mu_\infty}{\left(J\left(e^{-\mu_\infty \cdot}, \mu_\infty^2\right)\right)^{1/2}} \sim 2^{1/2}\mu_\infty(-\log\mu_\infty)^{1/2} \qquad \text{as } \mu_\infty \to 0^+. \tag{2.13}$$

It should be noted that the upper bound $J(e^{-\mu_\infty \cdot}, \mu_\infty^2)$ is equal to $(\omega_1^1)^2$ (see $(2.4)_1$).

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