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On L_2-Solvability of Mixed Boundary Value Problems for Elliptic Equations in Plane Non-smooth Domains

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This paper is devoted to an L_2-solvability of mixed boundary value problems (MBVPs) for second order elliptic equations in plane domains with curvilinear polygons as its boundaries. We find a space T' such that the MBVP with data in $L_2(\Omega) \times T'$ is solvable in $L_2(\Omega)$ and calculate the dimension of the kernel of this problem. Moreover we relate our approach to the previous one [P. Grisvard, "Elliptic Boundary Problems in Non-smooth Domains," Pitman, New York, 1985] showing how to overcome difficulties arising there.

0. INTRODUCTION

Throughout this paper we shall be concerned with the problem which can be roughly written in the form: determine $u \in L_2(\Omega)$ such that

$$Au = f \quad \text{in } \Omega, \qquad \gamma_{\mathscr{D}} u = \phi \quad \text{on } \Gamma_{\mathscr{D}}, \qquad B_{\mathscr{N}} u = \psi \quad \text{on } \Gamma_{\mathscr{N}}, \tag{0.1}$$

where $\Omega \subset \mathbb{R}^2$, $\partial\Omega$ is a curvilinear polygon [5], $\partial\Omega = \Gamma_{\mathscr{D}} \cup \Gamma_{\mathscr{N}}$, $\Gamma_{\mathscr{D}} \cap \Gamma_{\mathscr{N}} = \emptyset$, and $\bar{\Gamma}_{\mathscr{D}} \cap \Gamma_{\mathscr{N}} \neq \emptyset$. A is assumed to be an elliptic second order differential operator, $B_{\mathscr{N}}$ is a first order boundary operator and $\gamma_{\mathscr{D}}$ is a trace operator on $\Gamma_{\mathscr{D}}$. Furthermore, $f \in L_2(\Omega)$, ϕ, ψ are certain distributions, defined respectively on $\Gamma_{\mathscr{D}}$ and $\Gamma_{\mathscr{N}}$. In fact, the meaning of the boundary conditions in (0.1) appears to be one of the crucial points of our considerations and will be precised later on.

Mixed boundary value problems (MBVPs), especially in non-smooth domains, have certain properties, which make them essentially different from regular boundary value problems (RBVP). It follows [8, 5], that the solution of MBVP may have singularities, no matter how smooth are the data. In the $H^2(\Omega)$-setting with, say, homogeneous boundary data, this is caused by the fact that the range of A is a proper subspace of $L_2(\Omega)$. A comprehensive theory of two dimensional MBVPs posed in $H^2(\Omega)$ can be found, for example, in [2, 3, 5]. It follows in particular that the defect

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of the range of A in $L_2(\Omega)$ is finite and can be determined explicitly. A theory has also been developed to cover the case of $H^s(\Omega)$ with $s > \frac{1}{2}$ [4, 6], however the case $s < \frac{1}{2}$ is considered rare (see [6], but only for the smooth boundary and [4] for the laplacian in rectilinear polygons), which may be due to the fact that in this case A fails to be injective. However, in our opinion it is worthwhile to have a closer look at such a weak solution for at least two reasons. First, boundary conditions often met in applications, like $\partial u/\partial \nu = \delta_S$, where δ_S is a Dirac's distribution concentrated at $S \in \Gamma_{\mathcal{N}}$ or $\gamma_{\mathcal{D}} u = \phi$, where ϕ is only square-integrable over $\Gamma_{\mathcal{D}}$ (for example, step-function) are outside the scope of even variational setting of MBVPs, being nevertheless of great practical and theoretical importance [1, Chap. 7.1.10]. Second, when we are dealing with either RBVPs or MBVPs in smooth domains, then the operator corresponding to the formally adjoint problem posed in $L_2(\Omega) = H^0(\Omega)$ turns out to be the Hilbert-space adjoint to the original one [7] which makes standard Functional Analysis techniques available. Therefore, it is interesting whether this is also the case for non-smooth domains.

An attempt to construct an $L_2(\Omega)$-setting of a MBVP in polygonal domain, presented in [5], has not been entirely successful since it fails to have the property mentioned above. Therefore, L_2-solutions of formally adjoint homogeneous problem do not necessarily annihilate the range of $A \times \gamma_{\mathcal{D}} \times B_{\mathcal{N}}$, which makes the analysis of it difficult.

It follows that this inconvenience has been caused by an unfortunate method of introducing boundary conditions, which neglects a behaviour of them at the corners of $\partial\Omega$. Here we shall show how to construct the space of boundary data T', preserving basic properties of boundary operators and in such a way that $A \times \gamma_{\mathcal{D}} \times B_{\mathcal{N}} \in \mathcal{L}(L_2(\Omega, A), L_2(\Omega) \times T')$ is surjective, where

$$L_2(\Omega, A) := \{u \in L_2(\Omega); Au \in L_2(\Omega)\}. \tag{0.2}$$

Furthermore, we shall show that in our setting concepts of formally adjoint and Hilbert-space adjoint operators corresponding to MBVPs coincide and calculate the dimension of $\operatorname{Ker} A \times \gamma_{\mathcal{D}} \times B_{\mathcal{N}}$ in $L_2(\Omega, A)$. The paper is completed by a short comparison of our results with those of [5], showing how to simplify an approach to the H^2-theory of MBVPs, given there.

1. Basic Notations and Definitions

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set, whose boundary, $\partial\Omega$, is a curvilinear polygon [5] of a class $C^{k,1}$, $k \geq 1$. Curves, constituting $\partial\Omega$ are denoted by $\bar{\Gamma}_j$, $j \in I = \{1, ..., N\}$, $\bar{\Gamma}_{j+1}$ follows $\bar{\Gamma}_j$ in the anticlockwise direction, $S_j = \bar{\Gamma}_j \cap \bar{\Gamma}_{j+1}$ and by definition $1 := N+1$. We denote by $\mathbf{v}_j$ a $C^{k-1,1}$-vector field, coinciding with the unit outward drawn normal field on Γ_j and τ_j is the corresponding tangent field. For $u \in H^2(\Omega)$ we denote by $\gamma_j u$ (resp. $\gamma_j(\partial u/\partial \nu_j)$) the traces of zero (resp. first) order on Γ_j.

By (A, D) or simply A we denote an operator defined on $D \subset L_2(\Omega)$, which on $D \cap C^\infty(\bar{\Omega})$ is defined by strongly elliptic differential expression

$$Au = \sum_{i,k=1}^{2} (a_{ik} u_{,i})_{,k} + \sum_{i=1}^{2} a_i u_{,i} + a_0 u \tag{1.1}$$

where a_{ik}, a_i are real functions, $a_{ik} \in C^{p,1}(\bar{\Omega})$, $p \geqslant 0$, $a_i \in L_\infty(\Omega)$ for $i = 0, 1, 2$ and $k = 1, 2$. We can assume that $a_{ik} = a_{ki}$ for $i, k = 1, 2$.

By $(\partial u/\partial \nu_{A_j'})$, $j \in I$, we denote the conormal derivative, associated with A and $\mathbf{v}_j$.

Further, let $I = \mathscr{D}_\Gamma \cup \mathscr{N}_\Gamma$ and $I = \mathscr{D} \cup \mathscr{N} \cup \mathscr{M}_{12} \cup \mathscr{M}_{21}$, where we say that $j \in \mathscr{D}_\Gamma$ if $\Gamma_j \subset \Gamma_{\mathscr{D}}$ and $j \in \mathscr{N}_\Gamma$ if $\Gamma_j \subset \Gamma_{\mathscr{N}}$, see (0.1) and that $j \in \mathscr{D}$ iff $j, j+1 \in \mathscr{D}_\Gamma$, $j \in \mathscr{N}$ iff $j, j+1 \in \mathscr{N}_\Gamma$ and $j \in \mathscr{M}_{12} (j \in \mathscr{M}_{21})$ iff $j \in \mathscr{D}_\Gamma$, $j+1 \in \mathscr{N}_\Gamma (j \in \mathscr{N}_\Gamma$, $j+1 \in \mathscr{D}_\Gamma)$. Furthermore, $\mathscr{M} := \mathscr{M}_{12} \cup \mathscr{M}_{21}$. We assume that $\mathscr{D}_\Gamma \neq \varnothing$ and therefore

$$\Gamma_{\mathscr{D}} := \bigcup_{j \in \mathscr{D}} \bar{\Gamma}_j \Big\backslash \bigcup_{j \in \mathscr{M}} \{S_j\}, \qquad \Gamma_{\mathscr{N}} := \bigcup_{j \in \mathscr{N}} \bar{\Gamma}_j.$$

We next introduce the first order boundary operators. Let for $j \in I$ and $u \in H^2(\Omega)$

$$B_j u := \gamma_j \left(\frac{\partial u}{\partial \nu_{A_j'}} + b_j \frac{\partial u}{\partial \tau_j} + c_j u \right)$$

and we assume that b_j, c_j are real functions satisfying

$$\begin{gathered} b_j \in C^{k,1}(\bar{\Gamma}_j),\ c_j \in C^{0,\alpha}(\bar{\Gamma}_j) \qquad \text{for} \quad k \geqslant 0, \alpha > \tfrac{1}{2} \\ b_j(S_j) = b_{j+1}(S_j), \qquad c_j(S_j) = c_{j+1}(S_j). \end{gathered} \tag{1.2}$$

By A^* and B_j^* we denote operators, formally adjoint to A and B_j, respectively. For convenience we introduce the following notation

$$\gamma = \gamma_{\mathscr{D}} \times \gamma_{\mathscr{N}} = \{\gamma_j\}_{j \in \mathscr{D}_\Gamma} \times \{\gamma_j\}_{j \in \mathscr{N}_\Gamma}, \ B = B_{\mathscr{D}} \times B_{\mathscr{N}} = \{B_j\}_{j \in \mathscr{D}_\Gamma} \times \{B_j\}_{j \in \mathscr{N}_\Gamma}$$

$$\boldsymbol{\mu}_j := \begin{cases} \mathbf{v}_{A'j} + b_j \boldsymbol{\tau}_j & \text{if} \quad j \in \mathscr{N}_\Gamma \\ \boldsymbol{\tau}_j & \text{if} \quad j \in \mathscr{D}_\Gamma, \end{cases} \tag{1.3}$$

$D^2(\gamma_l, B_m) := \{u \in H^2(\Omega);\ (\gamma_l \times B_m) u = 0\}$ for $l, m = \mathscr{D}$ or $\mathscr{N}$.

Unfortunately we shall need some function spaces besides standard Sobolev spaces. Let Γ_e be a smooth arc and Γ be a connected open arc such that $\Gamma \subset \Gamma_e$, $\bar{\Gamma} = \{S_1\} \cup \Gamma \cup \{S_2\}$ and Γ_{ei}, $i = 1, 2$ be components of the complement of Γ in Γ_e such that $S_i \in \Gamma_{ei}$. We introduce the following notations for $i, j = 1, 2$:

$H^{1/2}(\Gamma, S_i^0)$ – the set of functions belonging to $H^{1/2}(\Gamma)$, which are extendable by zero across S_i continuously in $H^{1/2}(\bar{\Gamma} \cup \bar{\Gamma}_{ei})$.

$$H_0^{3/2}(\Gamma, S_i) := \{u \in H^{3/2}(\Gamma); u(S_i) = 0\}$$

$$H_0^{3/2}(\Gamma, S_i^0) := \{u \in H_0^{3/2}(\Gamma); \partial u/\partial\tau \in H^{1/2}(\Gamma, S_i^0)\}$$

$$H_0^{3/2}(\Gamma, S_i, S_j^0) := H_0^{3/2}(\Gamma, S_i) \cap H_0^{3/2}(\Gamma, S_j^0)$$

and we denote by $\tilde{H}^s(\Gamma)$, $s = \frac{1}{2}, \frac{3}{2}$ the set of functions which are extendable by zero from Γ onto Γ_e, continuously in $H^s(\Gamma_e)$ with suitable topology [5]. Since Sobolev spaces are of local character we can define Hilbert-space topologies in spaces introduced above and properties of them can be deduced from these for spaces defined on $\mathbb{R}_+$ [5]. In particular, these topologies are finer then those of $H^s(\Gamma)$ and $H_0^s(\Gamma)$ and respective injection are dense. As far as functional analytic notions is concerned we use standard notations, see [1, 5]. In particular, we use separate symbols for formal adjoint, *, and for Hilbert space adjoint, prime.

2. Trace Theorem

In this Section we shall find the space of traces $\gamma_{\mathscr{D}} \times B_{\mathscr{N}}$ of functions, belonging to $L_2(\Omega, A)$. This will be done in three main steps. First we shall construct a Hilbert space T, such that $\gamma_{\mathscr{N}} \times B^*_{\mathscr{D}} \in \mathscr{L}(D^2(\gamma_{\mathscr{D}} \times B^*_{\mathscr{N}}), T)$ and is a surjection. This is not straightforward, since the classical Trace Theorem for Sobolev spaces is not available in polygonal domains, unless the boundary data satisfy certain compatibility conditions at the corners S_j [5, 2, 3]. In the second step, using well-known transposition technique and Green's formula [1, 5] we find the image $(\gamma_{\mathscr{D}} \times B_{\mathscr{N}}) L_2(\Omega, A)$ as a dual of T, T'. Finally, we shall give a representation of T' in terms of Sobolev spaces, defined on smooth parts of $\partial\Omega$ and distributions concentrated at the corners. We start with an algebraic description of T.

Let $\mathscr{G}_j := \bar{\Gamma}_j \cup \bar{\Gamma}_{j+1}$ and let for any functions f_k defined on Γ_k, $k = j, j+1$, $[f_j, f_{j+1}]$ denote the function on $\mathscr{G}_j$, whose restriction to Γ_k are equal respective to f_k. Furthermore, $\boldsymbol{\mu}_j^k := \boldsymbol{\mu}_j(S_k)$.

Theorem 2.1. *Let*

$$\Phi = \{\phi_j\}_{j \in \mathscr{N}_\Gamma} \in \prod_{j \in \mathscr{N}_\Gamma} H^{3/2}(\Gamma_j), \qquad \Psi = \{\psi_j\}_{j \in \mathscr{D}_\Gamma} \in \prod_{j \in \mathscr{D}_\Gamma} H^{1/2}(\Gamma_j). \tag{2.1}$$

There exists $u \in D^2(\gamma_{\mathscr{D}}, B_{\mathscr{N}})$ *satisfying* $(\gamma_{\mathscr{D}} \times B_{\mathscr{N}})u = \{\Phi, \Psi\}$ *iff*

$$\text{(a)} \quad \text{if } j \in \mathscr{D} \text{ then } \psi_k \in H^{1/2}(\Gamma_k, S_j^0) \text{ for } k = j, j+1 \tag{2.2}$$

$$\text{(b)} \quad \text{if } j \in \mathscr{M}_{12} \text{ and } \boldsymbol{\tau}_j^j \parallel \boldsymbol{\mu}_{j+1}^j \text{ then, } \psi_j \in H^{1/2}(\Gamma_j, S_j^0),$$

$$\phi_{j+1} \in H_0^{3/2}(\Gamma_{j+1}, S_j^0) \text{ and } \phi_{j+1} \in H_0^{3/2}(\Gamma_j, S_j),$$

$$\left[k_{j+1}\psi_j, -k_j \frac{\partial \phi_{j+1}}{\partial \tau_{j+1}}\right] \in H^{1/2}(\mathscr{G}_j), \tag{2.3}$$

otherwise, where $k_k = \boldsymbol{\mu}_k^j \cdot \mathbf{v}_k^j$ *for* $k = j, j+1$. *Analogously if* $j \in \mathscr{M}_{21}$,

(c) *if* $j \in \mathscr{N}$ *then*

$$\phi_j(S_j) = \phi_{j+1}(S_j) = k$$

and

$$\frac{\partial \phi_j}{\partial \tau_j} - C_j k \in H^{1/2}(\Gamma_j, S_j^0), \frac{\partial \phi_{j+1}}{\partial \tau_{j+1}} + C_{j+1} k \in H^{1/2}(\Gamma_{j+1}, S_j^0) \tag{2.4}$$

where

$$\text{for } k = j, j+1, \quad C_k = \frac{c}{(a_{11}a_{22} - a_{12}a_{21} + b^2)} \cdot \frac{(\boldsymbol{\mu}_j^j - \boldsymbol{\mu}_{j+1}^j) \cdot \mathbf{v}_k^j}{\mathbf{v}_j^j \cdot \boldsymbol{\tau}_{j+1}^j}$$

and $c = c_j(S_j) = c_{j+1}(S_j)$, $b = b_j(S_j) = b_{j+1}(S_j)$ *and* $a_{sr} = a_{rs}(S_j)$, $s, r = 1, 2$.

Proof. The idea of the proof is analogous to that in [2], so we confine ourselves to the outline of it. The starting point is [5, Theorem 1.5.2.4], asserting that if we are given $f_i, g_i \in H^{3/2-i}(\mathbb{R}_+)$, $i = 1, 2$, then there exists $u \in H^2(\mathbb{R}_+ \times \mathbb{R}_+)$, satisfying $\gamma_1 u = g_0$, $\gamma_2 u = f_0$, $\gamma_1(\partial u/\partial v_1) = g_1$, $\gamma_2(\partial u/\partial v_2) = f_1$, (here $\Gamma_i := \{x_i = 0\}$, $i = 1, 2$) iff

$$f_0(0) = g_0(0), \qquad g_0' - f_1, f_0' - g_1 \in \tilde{H}^{1/2}(\mathbb{R}_+), \tag{2.5}$$

where $f' := df/dt$ for any function f. In a usual way we localize our problem and due to the classical Trace Theorem we only need to prove our theorem in a neighbourhood of a corner S_j with, say, $j = 1$. We focus our attention on the case of acute angle, the obtuse one being analogous, and flattening Γ_1 and Γ_2 (where we use the canonical parameterizations of arcs), we reduce the problem to the following one: determine $u \in H^2(\mathbb{R}_+ \times \mathbb{R}_+)$ satisfying

$$\begin{cases} \gamma_1 u = \phi_1 \\ \gamma_1(\alpha_{11} u_{,1} + \alpha_{12} u_{,2} + c_1 u) = \psi_1 \\ \gamma_2 u = \phi_2 \\ \gamma_2(\alpha_{21} u_{,1} + \alpha_{22} u_{,2} + c_2 u) = \psi_2. \end{cases} \tag{2.6}$$

It can be calculated, that for $k, m = 1, 2$

$$\alpha_{km}(0,0) = \boldsymbol{\mu}_k^1 \cdot \mathbf{v}_m^1 / \boldsymbol{\tau}_m^1 \cdot \mathbf{v}_k \text{ and } \alpha_{kk}(0,0) \neq 0. \tag{2.7}$$

Similarly as in [4, 5] we can use (2.5) to reduce (2.6) to the system of equations on $\mathbb{R}_+$.

(a) Let $1 \in \mathscr{D}$, then $\phi_1 = \phi_2 = 0$ and we can assume that $B_{\mathscr{D}}^* = \{\partial/\partial v_j\}_{j \in \mathscr{D}_\Gamma}$ thus $\alpha_{12} = \alpha_{21} = 0$. Then by (2.5) we reduce (2.6) to

$$\begin{cases} \alpha_{11} g_1 = \psi_1 \\ \alpha_{22} f_1 = \psi_2 \end{cases} \quad \text{and} \quad g_1, f_1 \in \tilde{H}^{1/2}(\mathbb{R}_+).$$

Hence, (2.7) implies $\psi_1, \psi_2 \in \tilde{H}^{1/2}(\mathbb{R}_+)$.

(b) Let $1 \in \mathscr{M}_{12}$, then $\phi_1 = \psi_2 = 0$, hence $\phi_2(0) = 0$. Thus, by (2.5) we obtain from (2.6)

$$\begin{cases} \alpha_{11} g_1 = \psi_1, \\ \alpha_{21} \phi_2' + c_2 \phi_2 \in \tilde{H}^{1/2}(\mathbb{R}_+), \end{cases} \quad \text{where } \phi_2' - g_1 \in \tilde{H}^{1/2}(\mathbb{R}_+).$$

Thus, $\alpha_{21}\phi_2' \in \tilde{H}^{1/2}(\mathbb{R}_+)$ and if $\boldsymbol{\mu}_2^1 \cdot \mathbf{v}_1^1 \neq 0$, then by (2.7), $\phi_2 \in \tilde{H}^{3/2}(\mathbb{R}_+)$ and accordingly $\psi_1 \in \tilde{H}^{1/2}(\mathbb{R}_+)$. On the other hand, if $\boldsymbol{\mu}_2^1 \cdot \mathbf{v}_1^1 = 0$, then ϕ_2' can be arbitrary, but then $\alpha_{11}(0,0)\,\phi_2' - \psi_1 \in \tilde{H}^{1/2}(\mathbb{R}_+)$ and having transformed it back to Γ_1 and Γ_2 we obtain (2.3).

(c) Let $j \in \mathscr{N}$, then in the analogous way we reduce (2.6) to

$$\begin{cases} \alpha_{11}\phi_2' + \alpha_{12}\phi_1' + c_1\phi_1 \in \tilde{H}^{1/2}(\mathbb{R}_+) \\ \alpha_{21}\phi_2' + \alpha_{22}\phi_1' + c_2\phi_2 \in \tilde{H}^{1/2}(\mathbb{R}_+). \end{cases}$$

Eliminating ϕ_2 we obtain

$$D\phi_1' + [c_1\alpha_{11} - c_2\alpha_{21}]\,\phi_1 \in \tilde{H}^{1/2}(\mathbb{R}_+),$$

where $D = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$ and $D(0,0) \neq 0$, since $\mathbf{v}_1^1 \nparallel \mathbf{v}_2^1$. Therefore, ϕ_1' is extendable by a constant accross zero. After transformation to Γ_1 this condition can be written as

$$\frac{\partial \phi_1}{\partial \tau_1} - \frac{c_1\phi_1(\mathbf{v}_1^1 \cdot \boldsymbol{\tau}_2^1)(\boldsymbol{\mu}_1^1 - \boldsymbol{\mu}_2^1)\cdot \mathbf{v}_1^1}{G(\boldsymbol{\mu}_1^1, \boldsymbol{\mu}_2^1, \mathbf{v}_1^1, \mathbf{v}_2^1)} \in H^{1/2}(\Gamma_1, S_1^0), \tag{2.8}$$

where G is the Gramm's determinant of given vectors. After some calculations involving (2.7), (2.8) yields (2.4). Similarly we can obtain the condition for ϕ_2. The proof is then complete. ∎

Let T denote the set of functions, satisfying (2.1)–(2.4). In the next step we introduce the topology, in which T becomes a Hilbert space. We have

LEMMA 2.1. *Let T_j, $j \in I$, consist of all pairs of functions defined on Γ_j and Γ_{j+1}, satisfying condition* (2.1) *and the respective one of* (2.2)–(2.4). *Then the norm $\|\cdot\|_j$ turns T_j into a Hilbert space, where*

(a) *if $j \in \mathscr{D}$ then $\|\cdot\|_j$ is induced from $H^{1/2}(\Gamma_j, S_j^0) \times H^{1/2}(\Gamma_{j+1}, S_j^0)$*

(b) *if $j \in \mathscr{M}_{12}$ and $\boldsymbol{\tau}_j^j \nparallel \boldsymbol{\mu}_{j+1}^j$ then $\|\cdot\|_j$ is induced from $H^{1/2}(\Gamma_j, S_j^0) \times H_0^{3/2}(\Gamma_{j+1}, S_j^0)$*

(c) *if $j \in \mathscr{M}_{12}$ and $\boldsymbol{\tau}_j^j \parallel \boldsymbol{\mu}_{j+1}^j$ then, (see* (2.3)*)*

$$\|\{\psi_j, \phi_{j+1}\}\|_j := \left\| \left[k_{j+1}\psi_j, -k_j \frac{\partial \varphi_{j+1}}{\partial \tau_{j+1}} \right] \right\|_{H^{1/2}(\mathscr{G}_j)}.$$

The case $j \in \mathscr{M}_{21}$ is analogous to (b) *and* (c), *respectively.*

(d) *if $j \in \mathscr{N}$ then*

$$\begin{aligned} &\|\{\phi_j, \phi_{j+1}\}\|_j^2 \\ &\quad = \|[\phi_j, \phi_{j+1}]\|_{H^1(\mathscr{G}_j)}^2 + \left\| \frac{\partial \phi_j}{\partial \tau_j} - C_j \phi_j(0) \right\|_{H^{1/2}(\Gamma, S^0)}^2 \\ &\quad + \left\| \frac{\partial \phi_{j+1}}{\partial \tau_{j+1}} + C_{j+1}\phi_{j+1}(0) \right\|_{H^{1/2}(\Gamma_{j+1}, S_j^0)}^2. \end{aligned}$$

Proof. Statements (a) and (b) are obvious. Let us consider (c). Since $T_j \subset H^{1/2}(\Gamma_j) \times H_0^{3/2}(\Gamma_{j+1}, S_j)$ and $\partial/\partial\tau_{j+1}$: $H_0^{3/2}(\Gamma_{j+1}, S_j) \to H^{1/2}(\Gamma_{j+1})$ is invertible, then the mapping $T_j \ni \{\psi_j, \phi_{j+1}\} \to [k_{j+1}\psi_j, k_j(\partial\phi_{j+1}/\partial\tau_{j+1})] \in H^{1/2}(\mathscr{G}_j)$ establishes an isomorphism, which ends the proof. The statement (d) can be proved similarly by introducing the mapping

$$\{\phi_j, \phi_{j+1}\} \to \left\{ \frac{\partial \phi_j}{\partial \tau_j} - C_j\phi_j(0), \frac{\partial \phi_{j+1}}{\partial \tau_{j+1}} + C_{j+1}\phi_{j+1}(0) \right\} \in \prod_{k=1}^{j+1} H^{1/2}(\Gamma_k, S_j^0).$$

However, this mapping is not injective and does not incorporate condition $\phi_j(0) = \phi_{j+1}(0)$, whence the necessity of adding the term $\| \ \|_{H^1(\mathscr{G}_j)}$. Thus the lemma has been proved. ∎

Having done this, we can introduce a suitable topology over the whole boundary. We take a finite covering of $\partial\Omega$ $\mathscr{U} := \{U_j\}_{j=1}^{2N}$, such that

$S_j \in U_m$ iff $j = m$ and $U_{N+j} \cap \Gamma_m \neq \emptyset$ iff $j = m, j = 1, ..., N$. Subsequently, we introduce a partition of unity $\mathscr{A} := \{\alpha_j\}_{j=1}^{2N}$ subordinated to $\mathscr{U}$ and define

$$\|\{\Phi, \Psi\}\|_T^2 := \sum_{j=1}^{N} \|\alpha_j\{\Phi, \Psi\}\|_j^2 + \sum_{j \in \mathscr{N}_\Gamma} \|\alpha_{N+j}\phi_j\|_{H^{3/2}}^2(\Gamma_j) + \sum_{j \in \mathscr{D}_\Gamma} \|\alpha_{N+j}\psi_j\|_{H^{1/2}}^2(\Gamma_j). \tag{2.8}$$

As in the case of locally defined topology in Sobolev space on, say, manifolds, (2.8) can be proved to introduce a Hilbert space topology independent of a choice of $\mathscr{U}$ and $\mathscr{A}$.

LEMMA 2.3. *The topology in T, defined by* (2.8), *is stronger than that induced from* $\prod_{j \in \mathscr{D}_\Gamma} H^{1/2}(\Gamma_j) \times \prod_{j \in \mathscr{N}_\Gamma} H^{3/2}(\Gamma_j)$.

Proof. Let $\mathscr{H}_j := H^{1/2+k}(\Gamma_j) \times H^{1/2+l}(\Gamma_{j+1})$, where $k = l = 0$ if $j \in \mathscr{D}$, $k = l = 1$ if $j \in \mathscr{N}$, and $k = 0, l = 1$ (resp. $k = 1, l = 0$) if $j \in \mathscr{M}_{12}$ (resp. $\mathscr{M}_{21}$). It is enough to prove that $T_j \subset \mathscr{H}_j$ with finer topology. For $j \in \mathscr{D}$ and $j \in \mathscr{M}_{12}$ (resp. $j \in \mathscr{M}_{21}$) with $\boldsymbol{\tau}_j^j \nparallel \boldsymbol{\mu}_{j+1}^j$ (resp. $\boldsymbol{\mu}_j^j \nparallel \boldsymbol{\tau}_{j+1}^j$) the assertion is obvious. Let $j \in \mathscr{M}_{12}$ and $\boldsymbol{\tau}_j^j \parallel \boldsymbol{\mu}_{j+1}^j$, then by Lemma 2.1 (c) and continuity of the operator of restriction we have

$$\|\{\psi_j, \phi_{j+1}\}\|_j^2 \leqslant C\left(\|\psi_j\|_{H^{1/2}(\Gamma_j)}^2 + \left\|\frac{\partial \phi_{j+1}}{\partial \tau_{j+1}}\right\|_{H^{1/2} \subset \Gamma_{j+1})}^2\right) \leqslant C' \|\{\psi_j, \phi_{j+1}\}\|_{H^{1/2}(\Gamma_j) \times H^{3/2}(\Gamma_{j+1})}$$

and the assertion holds. We can proceed similarly in the remaining cases, hence the lemma holds. ∎

THEOREM 2.4. *Operator* $\gamma_{\mathscr{N}} \times B_{\mathscr{D}}^*$ *is continuous from* $D^2(\gamma_{\mathscr{D}}, B_{\mathscr{N}}^*)$ *onto T.*

Proof. The surjectivity of $\gamma_{\mathscr{N}} \times B_{\mathscr{D}}^*$ follows from the very definition of T. By (2.8) and the classical Trace Theorem it is enough to prove the statement for each T_j, $j \in I$. Let us fix some j and consider a sequence (u_n) such that $u_n \to u$ in $D^2(\gamma_{\mathscr{D}}, B_{\mathscr{N}}^*)$ and $(\gamma_{\mathscr{D}} \times B_{\mathscr{N}}^*) u_n \to v$ in T_j as n tends to infinity. However, Lemma 2.3. implies then that $(\gamma_{\mathscr{D}} \times B_{\mathscr{N}}^*) u_n \to v$ in $\mathscr{H}_j$, thus $v = (\gamma_{\mathscr{D}} \times B_{\mathscr{N}}^*) u$ by classical Trace Theorem. Therefore, the application of the Closed Graph Theorem ends the proof. ∎

Theorem 2.4 makes the standard transposition technique [1, 5] available. However, before we formulate the main result of this section we note that $(\gamma_{\mathscr{D}} \times B_{\mathscr{N}}^*) \in \mathscr{L}(D^2(\gamma_{\mathscr{N}}, B_{\mathscr{D}}^*) T)$ is of global character, due to the compatibility conditions at S_j which define T, whereas we would rather have the parts corresponding to $\gamma_{\mathscr{D}}$ and $B_{\mathscr{N}}^*$ to be separated. Since we are to apply the transposition technique, this separation should be done in terms of decomposition of T into orthogonal subspaces, which in general does not coincide with restrictions of data to particular subdomains. To this end we denote by $T_{\mathcal{N}}$ the subspace of $\prod_{j\in\mathcal{N}_\Gamma} H^{1/2}(\Gamma_j)$ defined by conditions:

(i) if $j\in\mathcal{N}$, then $\{\phi_j, \phi_{j+1}\}\in T_j$.

(ii) if $j\in\mathcal{M}_{12}$ $(\mathcal{M}_{21})$ and $\tau_j^j \nparallel \mu_{j+1}^j$ $(\tau_{j+1}^j \nparallel \mu_j^j)$ then $\alpha_j\phi_{j+1}$ $(\alpha_j\phi_j)$ belongs to $H_0^{3/2}(\Gamma_{j+1}, S_j^0)$ $(H_0^{3/2}(\Gamma_j, S_j^0))$.

(iii) if $j\in\mathcal{M}_{12}$ $(\mathcal{M}_{21})$ and $\tau_j^j \parallel \mu_{j+1}^j$ $(\tau_{j+1}^j \parallel \mu_j^j)$ then $\alpha_j\phi_{j+1}$ $(\alpha_j\phi_j)$ belongs to $H_0^{3/2}(\Gamma_{j+1}, S_j)$ $(H_0^{3/2}(\Gamma_j, S_j))$.

By $T_{\mathcal{D}}$ we denote the subspace of $\prod_{j\in\mathcal{D}_\Gamma}(\Gamma_j)$ defined by

(iv) if $j\in\mathcal{D}$, then $\{\psi_j, \psi_{j+1}\}\in T_j$

(v) if $j\in\mathcal{M}_{12}$ $(\mathcal{M}_{21})$ then $\alpha_j\psi_j$ $(\alpha_j\psi_{j+1})$ belongs to $H^{1/2}(\Gamma_j, S_j^0)$ $(H^{1/2}(\Gamma_{j+1}, S_j^0))$.

LEMMA 2.5.

$$T\cong T_{\mathcal{D}}\oplus T_{\mathcal{N}} \tag{2.9}$$

(*where* $\cong$ *denotes equal up to the isometry*) *and*

$$\gamma_{\mathcal{N}}(D^2(\gamma_{\mathcal{D}}, B^*_{\mathcal{N}})) = T_{\mathcal{N}}, \qquad P_{\mathcal{D}}B^*_{\mathcal{D}}(D^2(\gamma_{\mathcal{D}}, B^*_{\mathcal{N}})) = T_{\mathcal{D}},$$

where P_i *denotes orthogonal projector onto* T_i *for* $i=\mathcal{N}, \mathcal{D}$.

Proof. As previously, we can confine ourselves to T_j, $j\in\mathcal{M}_{12}$, and $\tau_j^j\parallel\mu_{j+1}^j$. In this case T_j is isomorphic to $H^{1/2}(\mathcal{G}_j)\cong H^{1/2}(\Gamma_j, S_j^0)\oplus H^{1/2}(\Gamma_{j+1})$ and thus $T_j\cong H^{1/2}(\Gamma_j, S_j^0)\oplus H_0^{3/2}(\Gamma_{j+1}, S_j)$. Therefore $P_{\mathcal{N}}(\gamma_{\mathcal{N}}\times B^*_{\mathcal{D}})$ can be identified with $\gamma_{\mathcal{N}}$ itself, whereas $P_{\mathcal{D}}B^*_{\mathcal{D}}$ equals $B^*_{\mathcal{D}}$ only on $\operatorname{Ker}\gamma_{\mathcal{N}}$. ∎

Remark 2.6. Decomposition (2.9) certainly is not unique. We have selected it in that form so as to preserve original values of the trace operator. However, this is not of great importance, since (2.9) is only an auxiliary step in the construction of trace operators on $L_2(\Omega, A)$ below.

THEOREM 2.7. *There exist unique operators* $\tilde{\gamma}_{\mathcal{D}}\in\mathcal{L}(L_2(\Omega, A), T'_{\mathcal{D}})$ *and* $\tilde{B}_{\mathcal{N}}\in\mathcal{L}(L_2(\Omega, A), T'_{\mathcal{N}})$ *such that the Green's formula*

$$\int_\Omega Auv\,dx - \int_\Omega uA^*v = \langle \tilde{B}_{\mathcal{N}}u, \gamma_{\mathcal{N}}v\rangle_{T_{\mathcal{N}}} - \langle\tilde{\gamma}_{\mathcal{D}}u, P_{\mathcal{D}}B^*_{\mathcal{D}}v\rangle_{T_{\mathcal{D}}} \tag{2.10}$$

holds for every $u \in L_2(\Omega, A)$ *and* $v \in D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}})$. *Operators* $\tilde{\gamma}_{\mathscr{D}}$ *and* $\tilde{B}_{\mathscr{N}}$ *are extensions onto* $L_2(\Omega, A)$ *of operators* $\gamma_{\mathscr{D}}$ *and* $B_{\mathscr{N}}$, *respectively, which were originally defined on, say,* $H^2(\Omega)$.

Proof. The operator $\gamma_{\mathscr{N}} \times B^*_{\mathscr{D}}$, thanks to Theorem 2.4 satisfies assumptions required to assert that there exists a unique operator $\delta \in \mathscr{L}(L_2(\Omega, A), T')$ such that the following formula

$$\int Auv\,dx - \int uA^*v\,dx = \langle \delta u, (\gamma_{\mathscr{N}} \times B^*_{\mathscr{D}})\,v \rangle_T$$

holds for $u \in L_2(\Omega, A)$ and $v \in D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}})$ [1, 5]. However ((2.9)),

$$\langle \delta u, (\gamma_{\mathscr{N}} \times B^*_{\mathscr{D}})\,v \rangle_{T' \times T} = \langle \tilde{B}_{\mathscr{N}} u, \gamma_{\mathscr{N}} v \rangle_{T_{\mathscr{N}}} - \langle \tilde{\gamma}_{\mathscr{D}} u, P_{\mathscr{D}} B^*_{\mathscr{D}} v \rangle_{T_{\mathscr{D}}},$$

where $\tilde{B}_{\mathscr{N}} := P'_{\mathscr{N}}\delta$ and $\tilde{\gamma}_{\mathscr{D}} := -P'_{\mathscr{D}}\delta$, hence (2.10).

Now, if $u \in H^2(\Omega)$, $v \in D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}})$ then the following Green's formula holds [5]:

$$\int_\Omega Auv\,dx - \int_\Omega uA^*v\,dx = \int_{\Gamma_{\mathscr{N}}} (B_{\mathscr{N}} u \cdot \gamma_{\mathscr{N}} v)\,d\sigma - \int_{\Gamma_{\mathscr{D}}} (\gamma_{\mathscr{D}} u \cdot B^*_{\mathscr{D}} v)\,d\sigma. \tag{2.11}$$

Let for a time being $v \in \operatorname{Ker} \gamma_{\mathscr{N}}$. Then $P_{\mathscr{D}} B^*_{\mathscr{D}} v = B^*_{\mathscr{D}} v$ by Lemma 2.5 and $\int (\gamma_{\mathscr{D}} u \cdot B^*_{\mathscr{N}} v)\,d\sigma = \langle \gamma_{\mathscr{D}} u, B^*_{\mathscr{D}} v \rangle_{T_{\mathscr{D}}}$ by density of $T_{\mathscr{D}} \subset L_2(\Omega)$. Thus if $u \in H^2(\Omega)$ then by (2.10) and (2.11) $\langle \tilde{\gamma}_{\mathscr{D}} u - \gamma_{\mathscr{D}} u, B^*_{\mathscr{D}} v \rangle_{T_{\mathscr{D}}} = 0$ for every $v \in \operatorname{Ker} \gamma_{\mathscr{N}}$, hence $\tilde{\gamma}_{\mathscr{D}} u = \gamma_{\mathscr{D}} u$ by surjectivity of $B^*_{\mathscr{D}}$. Now the assertion $\tilde{B}_{\mathscr{N}} u = B_{\mathscr{N}} u$ whenever $u \in H^2(\Omega)$ follows by comparison of (2.10) and (2.11) since again $\gamma_{\mathscr{N}}$ acts onto $T_{\mathscr{N}}$. The proof is then complete. ∎

We shall use the same notation for trace operators and their extensions onto $L_2(\Omega, A)$ in the sequel. This section is completed with a decomposition of $T'_{\mathscr{D}}$ and $T'_{\mathscr{N}}$ into subspaces related to particular Γ_j, $j \in I$. First we prove

LEMMA 2.8. *Let* $j \in \mathscr{N}$ *then* $\alpha_j T_j$ *(see* (2.8)*) admits a decomposition*

$$\alpha_j T_j \cong \tilde{H}^{3/2}(\Gamma_j) \oplus \tilde{H}^{3/2}(\Gamma_{j+1}) \oplus \operatorname{Lin}\{\delta_{S_j}\}$$

where δ_{S_j} *is a Dirac's distribution, concentrated at* S_j *and* Lin *denotes the linear manifold, spanned by elements in brackets.*

Proof. Without losing generality, we assume that $\Gamma_j =]-1, 0[$ and $\Gamma_{j+1} =]0, 1[$. It is clear that T_j admits a decomposition into the direct sum

$$\alpha_j T_j \cong (H_0^{3/2}(\Gamma_j, S_j^0) \times H_0^{3/2}(\Gamma_{j+1}, S_j^0)) \oplus \operatorname{Lin}\{1 + \hat{x}\},$$

where $\hat{x} := [C_j x, -C_{j+1} x]$, see Theorem 2.1. Therefore

$$\alpha_j T_j \ni \Phi = \{\phi_j, \phi_{j+1}\} = \{\phi_{0j}, \phi_{0j+1}\} + \phi(0)(1 + \hat{x}),$$

where $\{\phi_{0j}, \phi_{0j+1}\}$ is an orthogonal projection of Φ onto $H^{3/2}(\Gamma_j, S_j^0) \times H^{3/2}(\Gamma_{j+1}, S_j^0)$. Hence $F \in (\alpha_j T_j)'$ admits a decomposition $F = F_j + F_{j+1} + G$, where $F_k \in H^{3/2}(\Gamma_k, S_j^0)$, $k = j, j+1$ and $G(\Phi) = \Phi(0) \cdot (G(1) + G(\hat{x})) = \beta \delta_{S_j}(\Phi)$. The last expression is meaningful since $\phi_j(S_j) = \phi_{j+1}(S_j)$, (2.4). This ends the proof. ▌

From Lemmas 2.5 and 2.8 and Theorem 2.7 we conclude:

THEOREM 2.9. *Let $u \in L_2(\Omega, A)$, then*

$$\gamma_{\mathscr{D}} u = \{\gamma_j u\}_{j \in \mathscr{D}_\Gamma} \in \prod_{j \in \mathscr{D}_\Gamma} (H^{1/2}(\Gamma_j, S_{j-1}^0, S_j^0) \tag{2.11}$$

$$B_{\mathscr{N}} u = \{B_j u\}_{j \in \mathscr{N}_\Gamma} + \sum_{j \in \mathscr{N}} \beta_j \delta_{S_j}. \tag{2.12}$$

Introducing notation $P(k)$ iff $\mu_k^k \| \mu_{k+1}^k$ and $NP(k)$ otherwise, for $k \in I$ we have, for $j \in \mathscr{N}_\Gamma$

$B_j u \in H^{-3/2}(\Gamma_j)$ iff *$j-1, j \in \mathscr{M}$ and $NP(k)$ for $k = j-1, j$.*

$B_j u \in (H_0^{3/2}(\Gamma_j, S_{j-1}^0, S_j))'$ iff *either $j-1, j \in \mathscr{M}$ and $P(J-1)$, $NP(j)$ or $j-1 \in \mathscr{N}$, $j \in \mathscr{M}_{21}$ and $NP(j)$.*

$B_j u \in (H_0^{3/2}(\Gamma_j, S_{j-1}, S_j^0))'$ iff *either $j-1, j \in \mathscr{M}$ and $NP(j-1)$, $P(j)$ or $j-1 \in \mathscr{M}_{12}$, $j \in \mathscr{N}$ and $NP(j-1)$.*

$B_j u \in (\tilde{H}^{3/2}(\Gamma_j))'$ iff *either $j-1, j \in \mathscr{M}$ and $P(j-1)$, $P(j)$ or $j-1 \in \mathscr{M}_{12}$, $j \in \mathscr{N}$ and $P(j-1)$ or $j-1 \in \mathscr{N}$, $j \in \mathscr{M}_{21}$ and $P(j)$ or else $j-1, j \in \mathscr{N}$.*

3. MIXED BOUNDARY VALUE PROBLEM IN $L_2(\Omega, A)$

THEOREM 3.1. *The operator $A \times \gamma_{\mathscr{D}} \times B_{\mathscr{N}}: L_2(\Omega, A) \to L_2(\Omega, A) \times T'_{\mathscr{D}} \times T'_{\mathscr{N}}$ is surjective and* $\operatorname{Ker}(A \times \gamma_{\mathscr{D}} \times B_{\mathscr{N}}) \cong L_2(\Omega)/A^*(D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}}))$.

Proof. Let us denote $D := D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}})$ throughout this proof. Then the operator $A^*: D \to L_2(\Omega)$ is injective and of closed range [3] hence its transpose $(A^*)'$ acts from $L_2(\Omega)$ onto D' and $\operatorname{Ker}(A^*)' \cong L_2(\Omega)/A^*(D)$. If F is defined for $v \in D$ by

$$\langle F, v \rangle = \int_\Omega fv \, dx + \langle \Psi, \gamma_{\mathscr{N}} v \rangle_{T_{\mathscr{N}}} - \langle \Phi, P_{\mathscr{D}} B^*_{\mathscr{D}} v \rangle_{T_{\mathscr{D}}},$$

where $\{f, \Phi, \Psi\} \in L_2(\Omega) \times T'_{\mathscr{D}} \times T'_{\mathscr{N}}$ then $F \in D'$, by Theorem 2.4. Thus, there exists $u \in L_2(\Omega)$ satisfying,

$$\langle (A^*)' u, v \rangle_D = \int_\Omega fv\, dx + \langle \Psi, \gamma_{\mathscr{N}} v \rangle_{T_{\mathscr{N}}} - \langle \Phi, P_{\mathscr{D}} B^*_{\mathscr{D}} v \rangle_{T_{\mathscr{D}}} \tag{3.1}$$

for all $v \in D$. Since $C_0^\infty(\Omega) \subset D$ then (3.1) implies $Au = f$, thus $u \in L_2(\Omega, A)$ and the Green's formula (2.10) is available. Subtracting (3.1) from (2.10) we obtain, that for all $v \in D$,

$$\langle B_{\mathscr{N}} u - \Phi, \gamma_{\mathscr{N}} v \rangle_{T_{\mathscr{N}}} - \langle \gamma_{\mathscr{D}} u - \Phi, P_{\mathscr{D}} B^*_{\mathscr{D}} v \rangle_{T_{\mathscr{D}}} = 0.$$

Hence $B_{\mathscr{N}} u = \Psi$ and $\gamma_{\mathscr{D}} u = \Phi$ by Lemmas 2.4 and 2.5, therefore the surjectivity is proved. Now, let $u \in \operatorname{Ker}(A^*)'$, then by (3.1) $Au = 0$, $u \in L_2(\Omega, A)$ and by Green's formula, which is now available, $(\gamma_{\mathscr{D}} \times B_{\mathscr{N}})\, u = 0$, hence $u \in \operatorname{Ker}(A \times \gamma_{\mathscr{D}} \times B_{\mathscr{N}})$. The opposite inclusion follows from (2.10), thus the proof is complete. ∎

COROLLARY 3.2. *Let $j \in I$ be fixed, and $\lambda^A_{j,m} := (\psi^A_{j+1} - \psi^A_j + m\pi)/\omega^A_j$, where $\psi^A_k \in [0, \pi[$, $k = j, j+1$, is an angle between $\mathbf{\mu}^A_k$ and $\mathbf{\tau}^A_k$ and ω^A_j is an angle between Γ^A_j and Γ^A_{j+1} at S_j. The superscript A indicates that all calculations are to be carried out after having applied a linear transformation, reducing A at S_j to the canonical form, see* [2, 3]. *Then it follows* [2, 3], *that*

$$\dim \operatorname{Ker}(A \times \gamma_{\mathscr{D}} \times B_{\mathscr{N}}) = \sum_{j \in I} \operatorname{card}\{m;\, -1 < \lambda^A_{j,m} < 0\}.$$

We complete this paper by comparing our results with those of Ref. [5]. Instead of $D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}})$, the author there considers spaces $H_j = \{u \in H^2(\Omega);\ (\gamma_m \times B_m)\, u = 0 \text{ unless } j = m\}$ and proves that $\gamma_j \times B_j$ admits an extension, denoted hereafter by $\gamma_j \times B^0_j$, as an operator from $L_2(\Omega, A)$ into $(\tilde{H}^{1/2}(\Gamma_j))' \times (\tilde{H}^{3/2}(\Gamma_j))'$. It is seen that such an extension carries no information about the behaviour of traces at the corners, which is essential for the availability of the global trace theorem. Therefore, $\operatorname{Ker}(A \times \prod_{j \in \mathscr{D}_\Gamma} \gamma_j \times \prod_{j \in \mathscr{N}_\Gamma} B^0_j)$ contains solutions, corresponding to non-homogeneous boundary data, $B_{\mathscr{N}} u = \sum_{j \in \mathscr{N}} \beta_j \delta_{S_j}$, and as such cannot be an annihilator of $A^*(D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}})$. Thus we have

COROLLARY 3.3.

$$\begin{aligned}\dim \operatorname{Ker}\left(A \times \prod_{j \in \mathscr{D}_\Gamma} \gamma_j \times \prod_{j \in \mathscr{N}_\Gamma} B^0_j\right) &= \dim \operatorname{Ker}(A \times \gamma_{\mathscr{D}} \times B_{\mathscr{N}}) + \operatorname{card} \mathscr{N} \\ &= \dim L_2(\Omega)/A^*(D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}})) + \operatorname{card} \mathscr{N}.\end{aligned} \tag{3.2}$$

Note, that since the left hand side can be calculated explicitly, as in [5], (3.2) gives the codimension of $A^*(D^2(\gamma_{\mathscr{D}}, B^*_{\mathscr{N}}))$ in $L_2(\Omega)$. Since under assumption (1.3) we have card $\mathscr{N} = \sum_{j \in \mathscr{N}} \text{card}\{m; \lambda^A_{j,m} = 0\}$, (3.2) coincides with the result given in [5], avoiding, however, complicated estimations appearing there.

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