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# On the Existence of Solutions of Nonlinear Elliptic Boundary Value Problems

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## 1. INTRODUCTION

In this paper we consider nonlinear elliptic boundary value problems of the form

$$\begin{aligned} Lu + f(x, u) &= 0 \qquad \text{in } \Omega, \\ Bu &= g \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $L$ is a uniformly elliptic second-order differential operator, $B$ is a linear first-order boundary operator, and $\Omega$ is an unbounded domain of real $n$-space $\mathbb{R}^n$ with boundary $\partial\Omega$. We are interested in the existence of classical solutions of (1.1).

This problem has been considered by several authors in the case of a bounded domain $\Omega$, and in the case of nonlinear ordinary second-order differential equations on an infinite interval. In particular we mention the work of Nagumo [7], Amann [2], Πeller [5], Bandle [3], Simpson and Cohen [1], Wong [13], and the survey paper of Schmitt [10]. We also refer to the paper by Ogata [9] where bounded solutions of )1.1) are established in exterior domains under assumptions which include that $f(x, u)$ is bounded in $\bar{\Omega} \times R$.

The main purpose of this paper is to extend some of the results on bounded domains to the case when $\Omega$ is unbounded. In particular we show that, under suitable smoothness hypotheses, problem (1.1) has at least one solution if there exist smooth functions $v_0 \leqslant u_0$ on $\bar{\Omega}$ satisfying

$$\begin{aligned} Lu_0 + f(x, u_0) \leqslant 0 \text{ in } \Omega, &\qquad Bu_0 \geqslant g \text{ on } \partial\Omega; \\ Lv_0 + f(x, v_0) \geqslant 0 \textit{ in } \Omega, &\qquad Bv_0 \leqslant g \text{ on } \partial\Omega. \end{aligned}$$

We also present conditions which permits one to conclude the existence of nonnegative solutions, positive solutions, maximal solutions, bounded solutions, and solutions which converge to zero uniformly at $\infty$.

In the following section we introduce notation and formulate our assumptions.



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Section 3 contains the statements and proofs of our main results under the assumption that $\Omega$ is an exterior domain.

In Section 4 we obtain sifficient conditions involving growth and/or integral conditions on $f$ which guarantee the existence of nonnegative solutions of (1.1).

In Section 5 we consider the special case of (1.1) when $B$ is the Dirichlet operator. We show that the results of Section 3 can be established for more general domains.

## 2. Preliminaries

Let $\alpha \in (0, 1)$ be fixed. Denote by $\Omega$ an unbounded domain of real $n$-space $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\bar{\Omega}$. As is usual, we denote by $x = (x_1, ..., x_n)$ the points of $\mathbb{R}^n$ and differentiation with respect to $x_i$ by $D_i$ for $i = 1, 2, ..., n$.

For a bounded domain $M \subset \mathbb{R}^n$, let $C^{m+\alpha}(\bar{M})$, $m = 1, 2, ...$, denote the usual Hölder space. The norm in this space will be denoted by $\| u \|_{m+\alpha, \bar{M}}$.

We consider the second-order linear differential operator

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_i D_j u + \sum_{i=1}^{n} b_i D_i u$$

with real coefficients $a_{ij}$, $b_i$ defined in $\bar{\Omega}$ where we assume that $a_{ij} \in C^{2+\alpha}(\bar{M})$, $b_i \in C^{1+\alpha}(\bar{M})$ for all bounded domains $M \subset \Omega$. The operator $L$ is assumed to be uniformly elliptic on every bounded subdomain of $\Omega$.

Let $B$ denote one of the boundary operators

$$Bu = u,$$

or

$$Bu = \partial u/\partial v + \gamma(x)u, \qquad x \in \partial\Omega.$$

Here $\partial/\partial v$ denotes the outward conormal derivative, and we assume $\gamma > 0$ everywhere on the boundary $\partial\Omega$.

Finally, let $f: \bar{\Omega} \times R \to R$ and $g: \partial\Omega \to R$ be given functions. Then we consider the boundary value problem (1.1) where by a solution $u$ of (1.1) we always mean a function $u$ in $\bar{\Omega}$ such that $u \in C^{2+\alpha}(\bar{M})$ for every bounded domain $M \subset \Omega$ and satisfies (1.1) identically.

The functions $f$, $g$, and $\gamma$ are required to satisfy the following conditions:

*Assumptions* A. (i) $f(x, t) \in C^{\alpha}(\bar{M} \times [a, b])$ for all bounded domains $M \subset \Omega$ and all $-\infty < a < b < \infty$;

(ii) for any given bounded domain $M \subset \Omega$, and for any $-\infty < a < b < \infty$, there exists a positive constant $K$ such that

$$f(x, t_1) - f(x, t_2) \geqslant -K(t_1 - t_2)$$

for all $a \leqslant t_2 \leqslant t_1 \leqslant b$ and for all $x \in \overline{M}$;

(iii) $g \in C^{2+\alpha}(\bar{S})$, $\gamma \in C^{1+\alpha}(\bar{S})$ for any bounded subdomain $S$ of $\partial\Omega$.

## 3. Existence of Solutions in Exterior Domains

In this section we assume that $\Omega$ is an exterior domain with boundary $\partial\Omega$ of class $C^{2+\alpha}$.

Let $a > 0$ be chosen such that $\{x \in \mathbb{R}^n : |x| > a\} \subset \Omega$. The following notation will be used:

$$\begin{aligned} \Omega_b &= \{x \in \Omega : |x| < b\}, \\ S_b &= \{x \in \mathbb{R}^n : |x| = b\}, \\ D_{\alpha,b} &= C^{2+\alpha}(\overline{\Omega}_b), \qquad b > 0. \end{aligned}$$

Lemma 3.1. *Let $f$, $g$, and $\gamma$ satisfy the assumptions $A$. If there exist functions $v_0 \leqslant u_0$ on $\overline{\Omega}$ of class $D_{\alpha,b}$ for all $b > 0$ satisfying*

$$Lu_0 + f(x, u_0) \leqslant 0 \text{ in } \Omega, \qquad Bu_0 \geqslant g \text{ on } \partial\Omega,$$

*and*

$$Lv_0 + f(x, v_0) \geqslant 0 \text{ in } \Omega, \qquad Bv_0 \leqslant g \text{ on } \partial\Omega,$$

*then there exists a sequence of functions $u_j$ on $\overline{\Omega}$ with the following properties:*

(1) $v_0 \leqslant u_{j+1} \leqslant u_j \leqslant u_0 \quad$ *in* $\Omega$;

(2) $u_j \in D_{\alpha,a+j}$;

(3) $$\begin{aligned} Lu_j + f(x, u_j) &= 0 && \text{in } \Omega_{a+j}, \\ Bu_j &= g && \text{on } \partial\Omega, \\ u_j &= u_0 && \text{on } S_{a+j} \end{aligned}$$

*for all $j = 1, 2, 3, \ldots$ .*

*Proof.* We first consider the boundary problem

$$\begin{aligned} Lu + f(x, u) &= 0 && \text{in } \Omega_{a+1}, \\ Bu &= g && \text{on } \partial\Omega, \\ u &= u_0 && \text{on } S_{a+1}. \end{aligned} \tag{3.1}$$

Under the hypotheses of Lemma 3.1 a result of Amann [2] implies that problem (3.1) has a solution $U_1$ of class $D_{\alpha,a+1}$ satisfying

$$v_0(x) \leqslant U_1(x) \leqslant u_0(x), \qquad x \in \Omega_{a+1}.$$

Let $u_1$ be the extension of $U_1$ to all of $\bar{\Omega}$ defined as $u_0$ for all $|x| \geqslant a+1$. Then $v_0 \leqslant u_1 \leqslant u_0$ in $\Omega$, and hence $u_1$ satisfies properties (1)–(3) of Lemma 3.1. We use induction to construct the required sequence. Assume $u_i$ satisfies properties (1)–(3) for all $i \leqslant j$. A $u_{j+1}$ satisfying the same properties will be constructed below.

From assumption A(ii) there exists a constant $K > 0$ such that

$$f(x, t_1) - f(x, t_2) \geqslant -K(t_1 - t_2) \tag{3.2}$$

for all $x \in \bar{\Omega}_{a+j+1}$ and for all minimum $v_0(x) \leqslant t_2 \leqslant t_1 \leqslant$ maximum $u_0(x)$, where the minimum and the maximum are taken over $\bar{\Omega}_{a+j+1}$. Let $y_j$ be the unique solution of the boundary value problem

$$\begin{aligned} Ly - Ky &= -f(x, u_j) - Ku_j && \text{in } \Omega_{a+j+1}, \\ By &= g && \text{on } \partial\Omega, \\ y &= u_0 && \text{on } S_{a+j+1}. \end{aligned}$$

It is well known that the above problem has a unique solution $y_j \in D_{\alpha,a+j+1}$. We show next that $y_j$ satisfies the properties

(i) $v_0 \leqslant y_j \leqslant u_0$ in $\Omega_{a+j+1}$;

(ii) $y_j \leqslant u_j$ in $\Omega_{a+j+1}$.

Since $v_0 \leqslant u_j \leqslant u_0$ in $\Omega_{a+j+1}$ by the induction hypothesis, the hypothesis on $u_0$ and (3.2) imply that in $\Omega_{a+j+1}$

$$(L - K)(y_j - u_0) \geqslant -f(x, u_j) - Ku_j + f(x, u_0) + Ku_0 \geqslant 0;$$

and

$$(L - K)(y_j - v_0) \leqslant -f(x, u_j) - Ku_j + f(x, v_0) + Kv_0 \leqslant 0.$$

Furthermore, $B(y_j - u_0) \leqslant 0$ on $\partial\Omega$, $y - u_0 \equiv 0$ on $S_{a+j+1}$, $B(y_j - v_0) \geqslant 0$ on $\partial\Omega$, and $y_j - v_0 \geqslant 0$ on $S_{a+j+1}$. Therefore, the maximum principle for elliptic equations implies that $v_0 \leqslant y_j \leqslant u_0$ on $\bar{\Omega}_{a+j+1}$, which proves (i).

Since $L(y_j - u_j) = 0$ on $\bar{\Omega}_{a+j}$, $B(y_j - u_j) = 0$ on $\partial\Omega$, and $y_j - u_j = y_j - u_0 \leqslant 0$ by (i), we deduce from the maximum principle that $y_j - u_j \leqslant 0$ on $\bar{\Omega}_{a+j}$. But $u_j(x) = u_0(x)$ for $a + j \leqslant |x| \leqslant a + j + 1$. Hence $y_j \leqslant u_j$ on $\bar{\Omega}_{a+j+1}$, which proves (ii).

We consider now the boundary value problem

$$\begin{aligned} Lu + f(x, u) &= 0 && \text{in } \Omega_{a+j+1}, \\ Bu &= g && \text{on } \partial\Omega, \\ u &= u_0 && \text{on } S_{a+j+1}. \end{aligned} \tag{3.3}$$

Since $Ly_j + f(x, y_j) = f(x, y_j) - f(x, u_j) + K(y_j - u_j) \leqslant 0$, $Bu_j = g$ on $\partial\Omega$, $y_j = u_0$ on $S_{a+j+1}$, and $y_j \geqslant v_0$ in $\Omega_{a+j+1}$ from properties (i) and (ii), we can apply the result of Amann [2] to conclude that (3.3) has a solution $U_{j+1} \in D_{\alpha, a+j+1}$ satisfying $v_0 \leqslant U_{j+1} \leqslant y_j$ in $\Omega_{a+j+1}$. Let $u_{j+1}$ be the extension of $U_{j+1}$ to all of $\bar{\Omega}$ defined as $u_0(x)$ for $|x| \geqslant a + j + 1$. It is now easy to check that $u_{j+1}$ satisfies all the properties (1)–(3) of Lemma 3.1 completing the inductive construction.

*Remark* 1. Without assumption A(ii) on $f$, and with the other hypotheses of Lemma 3.1, it is easy to see from the above argument that there exists a sequence of functions $u_j$ on $\Omega$ satisfying properties (2) and (3) of Lemma 3.1. In fact, assumption A(ii) on $f$ was only used in the argument to construct a monotone sequence.

LEMMA 3.2. *Let the sequence $\{u_j\}$ be as in Lemma 3.1. Then for any given integer $J \geqslant 1$ there exists a positive constant $K$, depending on $n$, $\alpha$, $J$, $u_0$, and $v_0$ but independent of $j$, such that*

$$\| u_j \|_{2+\alpha, \Omega_{a+J}} \leqslant K \tag{3.4}$$

*for all $j \geqslant J$.*

*Proof.* For any given $J$, the functions $u_j$, $j \geqslant J$, are solutions of the boundary problem

$$\begin{aligned} Lu + f(x, u_j(x)) &= 0 && \text{in } \Omega_{a+J}, \\ Bu &= g && \text{on } \partial\Omega, \\ u &= u_j && \text{on } S_{a+J}. \end{aligned} \tag{3.5}$$

Since the sequence $\{u_j\}$ is uniformly bounded on $\bar{\Omega}_{a+J}$, the functions $f_j$ defined by

$$f_j(x) = f(x, u_j(x)), \qquad x \in \bar{\Omega}_{a+J},$$

are uniformly bounded on $\bar{\Omega}_{a+J}$. It then follows from Lemma 3.2 of Amann [2, p. 132] that $u_j \in W_p^1(\Omega_{a+J})$ and

$$\begin{aligned} \| u_j \|_{W_p^1(\Omega_{a+J})} &\leqslant \delta[\| f_j \|_{L_q(\Omega_{a+J})} + \| g \|_{L_p(\partial\Omega)} \\ &\quad + \| u_j \|_{L_p(S_{a+J})} + \| g \|_{L_q(\partial\Omega)} \\ &\quad + \| u_j \|_{L_q(S_{a+J})} \\ &\leqslant K \end{aligned} \tag{3.6}$$

for some positive constant $K$ independent of $j$, where $\delta$ is a positive constant independent of $j$, $p > 1$, and $q = p/(p-1)$.

We apply the Sobolev embedding lemma to (3.6) with $p = n/(1-\alpha)$ to conclude that $u_j \in C^\alpha(\bar{\Omega}_{a+J})$ and

$$\| u_j \|_{\alpha,\bar{\Omega}_{a+J}} \leqslant K_1, \qquad j \geqslant J, \tag{3.7}$$

for some $K_1 > 0$ independent of $j$.

The $L_p$-estimate of Agmon, Douglis, and Nirenberg [1. Theorem 15.2] applied to (3.5) has the form

$$\begin{aligned} \| u_j \|_{W_p{}^2(\Omega_{a+J})} \leqslant \delta_1[\| f_j \|_{L_p(\Omega_{a+J})} \\ + \| g_j \|_{1-1/p} \end{aligned} \tag{3.8}$$

for some $\delta_1 > 0$ independent of $j$, $j \geqslant J$, where

$$\begin{aligned} g_j(x) = g(x), &\qquad x \in \partial\Omega, \\ u_j(x), &\qquad x \in S_{a+J} \end{aligned}$$

and

$$\| g_j \|_{1-1/p} = \inf \| v \|_{W_p{}^1(\Omega_{a+J})}$$

with the infimum being taken with respect to all functions $v \in C^1(\bar{\Omega}_{a+j})$ which equal $g_j$ on $\partial\Omega_{a+J}$. From (3.6) and (3.8) we deduce that there exists a positive constant $K_2$ independent of $j$ such that

$$\| u_j \|_{W_p{}^2(\Omega_{a+J})} \leqslant K_2 \tag{3.9}$$

From (3.9) with $p = n/(1-\alpha)$ and the Sobolev embedding lemma we conclude that there exists a positive constant $K_3$ independent of $j$ such that $u_j \in C^{1+\alpha}(\bar{\Omega}_{a+J})$ and

$$\| u_j \|_{C^{1+\alpha}(\bar{\Omega}_{a+J})} \leqslant K_3 \tag{3.10}$$

for all $j \geqslant J$.

The Schauder-type inequality [1, Theorem 9.1] applied to (3.5) has the form

$$\begin{aligned} \| u_j \|_{C^{2+\alpha}(\bar{\Omega}_{a+J})} \leqslant \delta_2(\| f_j \|_{\alpha,\bar{\Omega}_{a+J}} \\ + \| g_j \|_{1+\alpha,\partial\Omega_{a+J}}) \end{aligned} \tag{3.11}$$

for some $\delta_2 > 0$ independent of $j$. The conclusion of the lemma follows from (3.10) and (3.11). We combine Lemmas 3.1 and 3.2 to prove the following main result.

THEOREM 3.3. *Under the hypotheses of Lemma* 3.1, *the boundary value problem* (1.1) *has a solution* $\hat{u}$ *satisfying*

$$v_0(x) \leqslant \hat{u}(x) \leqslant u_0(x) \qquad \text{in } \Omega.$$

*Proof.* Let $\{u_j\}$ be the sequence constructed in Lemma 3.1. For each integer $i = 1, 2,\ldots$ it follows from Lemma 3.2 that there exists a positive constant $K_0$, independent of $j$, such that $\| u_j \|_{2+\alpha, \Omega_{a+i}} \leqslant K_0$ for all $j \geqslant i$. The compactness of the injection $C^{2+\alpha}(\bar{\Omega}_{a+1}) \to C^2(\bar{\Omega}_{a+1})$ then implies that $\{u_j : j \geqslant 1\}$ has a subsequence $\{u_j^1\}$ which converges in the $C^2(\bar{\Omega}_{a+1})$ norm to a function $u^1$ on $\bar{\Omega}_{a+1}$. Define $u_j^0 = u_j$ for convenience and define $\{u_j^i\}$ inductively ot be subsequence of $\{u_j^{i-1}\}$ which converges in the $C^2(\bar{\Omega}_{a+i})$ norm to a function $u^i$ on $\bar{\Omega}_{a+i}$, $i = 1, 2,\ldots$. Define $\hat{u}$ in $\Omega$ by $\hat{u}(x) = u^i(x)$ if $x \in \Omega_{+ai}$; this definition is consistent since $\Omega_{a+i} \subset \Omega_{a+i+1}$ and $u^{i+1} = u^i$ on $\bar{\Omega}_{a+i}$ obviously for each $i = 1, 2,\ldots$.

We shall show that $\hat{u}$ is the required solution. For any bounded domain $\bar{M} \subset \Omega$, $\bar{M} \subset \bar{\Omega}_{a+i}$ for some integer $i$, and hence the diagonal sequence $\{u_j^j(x)\}$ converges in the $C^2(\bar{M})$ norm to $u^i = \hat{u}$ on $\bar{M}$. In particular $u_j^j$ and $Lu_j$ converge uniformly to $\bar{M}$ to $u$ and $Lu$, respectively. Since $Lu_j = -f(x, u_j)$ in $M$ by Lemma 3.1, it follows that $\hat{u}$ is a solution of (1.1) of class $C^2(\bar{M})$, and hence of class $C^{2+\alpha}(\bar{M})$ by a standard regularity arguments based on Schauder estimates. Since $v_0(x) \leqslant u_j^j(x) \leqslant u_0(x)$ for each $j = 1, 2,\ldots$, the function $\hat{u}$ also satisfies $v_0(x) \leqslant \hat{u}(x) \leqslant u_0(x)$ in $\Omega$.

*Remark* 2. Without assumption A(ii) on $f$ we can still construct a sequence of functions $\{u_j\}$ satisfying properties (2) and (3) of Lemma 3.1. (See Remark 1.) We can then use Lemma 3.2 and the Ascoli–Arzela theorems to construct sequences $\{u_j^k\}$, $k = 1, 2,\ldots$, satisfying the following properties

(i) $\{u_j^{k+1}\} \subset \{u_j^k\} \subset \{u_j\}$, $k = 1, 2,\ldots$;

(ii) for each $k = 1, 2,\ldots$, $\{u_j^k\}$ converges uniformly on $\bar{\Omega}_{a+k}$ to a function $u^k \in D_{\alpha, a+k}$ satisfying $Lu^k + f(x, u^k) = 0$ in $\Omega_{a+k}$ and $Bu^k = g$ on $\Omega\partial$.

If we define a function $u$ on $\bar{\Omega}$ by

$$u(x) = u^k(x) \qquad \text{for} \quad x \in \bar{\Omega}_{a+k},$$

then it is easy to see that the diagonal sequence $\{u_j^j\}$ converges to $u$ and that $u$ is a solution of problem (1.1).

COROLLARY 3.4. *Assume* $f$, $g$, *and* $\gamma$ *satisfy assumptions* $A$(i), A(iii). *Furthermore, assume that* $f(x, 0) \geqslant 0$ *in* $\Omega$, *and* $g(x) \geqslant 0$ *on* $\partial\Omega$.

*Then, a necessary and sufficient condition for the existence of a nonnegative* *solution of* (1.1) *is the existence of a nonnegative function* $u_0$ *in* $\bar{\Omega}$ *of class* $D_{\alpha,a+j}$ *for all* $j = 1, 2,\ldots$, *satisfying*

$$\begin{aligned} Lu + f(x, u_0) &\leqslant 0 \qquad \textit{in } \Omega, \\ Bu_0 &\geqslant g \qquad \textit{on } \partial\Omega. \end{aligned}$$

The proof follows easily from Theorem 3.3 and Remark 2 by taking $v_p \equiv 0$ on $\bar{\Omega}$.

COROLLARY 3.5. *Assume* $f$, $g$, *and* $\gamma$ *satisfy assumptions* A. *Furthermore, assume that* $f(x, 0) \geqslant 0$ *in* $\Omega$, *and* $g \geqslant 0$ *on* $\partial\Omega$ *with the strict inequality holding for least one point* $x \in \partial G$.

*Then, a necessary and sufficient condition for the existence of a solution* $u$ *of* (1.1) *satisfying* $u > 0$ *in* $\Omega$ *is the existence of a nonnegative function* $u_0$ *in* $\bar{\Omega}$ *of class* $D_{\alpha,a+j}$ *for all* $j = 1, 2,\ldots$, *satisfying*

$$\begin{aligned} Lu_0 + f(x, u_0) &\leqslant 0 \qquad \textit{in } \Omega, \\ Bu_0 &\geqslant g \qquad \textit{on } \partial\Omega. \end{aligned}$$

*Proof.* By Corollary 3.4 is nonnegative solution $u$ of (1.1) exists. We show that $u$ is positive in $\Omega$. Let $J$ be an arbitrary integer. In view of assumption A(ii), we can select a constant $K > 0$ such that

$$f(x, u) - f(x, 0) \geqslant -Ku \qquad \text{in } \Omega_{a+J}. \tag{3.12}$$

Since $f(x, 0) \geqslant 0$ in $\bar{\Omega}$ by hypothesis, (3.12) implies that $Lu - Ku \leqslant 0$ in $\Omega_{a+J}$. We also have $Bu = g \geqslant 0$ on $\partial\Omega$ with the strict inequality for at least one point on $\partial\Omega$ by hypothesis. The maximum principle then implies that $u > 0$ in $\Omega_{a+J}$, and since $J$ is arbitrary, $u > 0$ in $\Omega$.

For the following corollaries the operator $L$ is required to be in the divergence form. In particular, let $L_1$ denote the operator defined by

$$L_1 u \equiv \sum_{i,j=1}^{n} D_i(p_{ij}(x)\, D_j u),$$

where $p_{ij}$ are real function on $\bar{\Omega}$ of class $C^{2+\alpha}(\bar{M})$ for all bounded domains $M \subset \Omega$, and the matrix $(p_{ij}(x))$ is assumed to be positive definite on every bounded subdomain of $\Omega$. Consider the boundary value problem

$$\begin{aligned} L_1 u + f(x, u) &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.13}$$

COROLLARY 3.6. *Assume* $f(x, 0) \geqslant 0$ *in* $\Omega$, $f(x, u)$ *is monotonic decreasing in* $u$ *for all* $u > 0$ *and* $x \in \Omega$, $g > 0$ *on* $\partial\Omega$, *and assumptions* A *hold.*

*If there exists a positive function $u_0$ on $\bar{\Omega}$ of class $D_{\alpha,a+j}$ for all $j = 1, 2, \ldots$, satisfying $L_1 u_0 + f(x, u_0) \leqslant 0$ in $\Omega$ and $u_0 \geqslant g$ on $\partial\Omega$, then there exists a maximal positive solution $\hat{u} \leqslant u_0$ of* (3.13) *in the sense that, for every nonnegative solution $u \leqslant u_0$ of* (3.13), *the inequality $u \leqslant \hat{u}$ holds. Moreover, if this condition is satisfied and $u_0(x)$ converges to* 0 *uniformly as $x \to \infty$,* (3.13) *has a unique solution $\hat{u}(x)$ which converges to* 0 *uniformly as $|x| \to \infty$.*

*Proof.* Let $\{u_j\}$ be the sequence constructed in Lemma 3.1 with $v_0 \equiv 0$. Then the pointwise limit $\hat{u}(x) = \text{limit}_{j\to\infty} u_j(x)$ is a nonnegative solution of (3.13) by Lemma 3.2, satisfying $\hat{u} \leqslant u_0$ in $\Omega$. From Corollary 3.5 and the hypothesis $g > 0$ on $\partial\Omega$, we conclude that $\hat{u} > 0$ on $\bar{\Omega}$.

We show next that $\hat{u}$ is maximal. Let $0 \leqslant u \leqslant u_0$ by any solution of (3.13). Let $J$ be an arbitrary positive integer. Then the functions $u$ and $u_J$, are solutions of the equation $L_1 u + f(x, u) = 0$ in $\Omega_{a+J}$. Furthermore, $u_J > 0$ in $\bar{\Omega}_{a+j}$ as can be proved by a similar argument to the one used in Corollary 3.5. Also $u_J \geqslant u$ on $\partial\Omega_{a+j}$. We then apply a result of Bushard [4] to conclude that $u_J \geqslant u$ on $\bar{\Omega}_{a+J}$. Since $u_J(x) = u_0(x)$ for $|x| \geqslant J$, $u_J \geqslant u$ on $\bar{\Omega}$ and $\hat{u}(x) = \lim_{J\to\infty} u_J(x) \geqslant u(x)$, $x \in \bar{\Omega}$. This proves that $\hat{u}$ is maximal. Finally, if $u_0(x)$ converges to zero uniformly as $|x| \to \infty$, we show that (3.13) has a unique solution $\hat{u}$ satisfying this property. That such a $\hat{u}$ exists follows from the first part of the proof. To show uniqueness, let $u$ be any other positive zolution satisfying this property. Let $\epsilon > 0$ be arbitrary, and choose an integer $J$ such that $\hat{u}(x) < u(x) + \epsilon$ for all $|x| \geqslant J + a$. The monotonicity hypothesis of $f$ implies that the function $w = u + \epsilon$ satisfies

$$L_1 w + f(x, w) \leqslant L_1 u + f(x, u) = 0$$

in $\Omega_{a+J}$. Since $w > 0$ on $\partial\Omega_{a+J}$, and $\hat{u} \leqslant u + \epsilon$ on $\delta\Omega_{a+J}$, Bushard's result [4] implies that $\hat{u} \leqslant u + \epsilon$ on $\bar{\Omega}_{a+j}$, and consequently on $\bar{\Omega}$. Since $\epsilon$ is arbitrary, $\hat{u} \leqslant u$ on $\bar{\Omega}$. The inequality $u \leqslant \hat{u}$ on $\bar{\Omega}$ can be proved similarly, completing the proof of Corollary 3.6.

## 4. Criteria for the Existence of Nonnegative Solutions

In this section we derive sufficient conditions on the coefficient $f$ and the boundary data $g$ which giarantee the existence of a nonnegative solution of the boundary value problem

$$\begin{aligned} \Delta u + f(x, u) &= 0, \qquad |x| > 1, \\ u(x) &= g(x), \qquad |x| = 1, \end{aligned} \tag{4.1}$$

where $f$ and $g$ are assumed to satisfy assumptions A.

We note that simplicity of presentation and comparison to known results have been considered in formulating our criteria. In fact analogs of the results below can be obtained for the more general problem (1.1) by the introduction of more notation and complexities.

COROLLARY 4.1. *Problem* (4.1) *has a solution u satisfying*

$$0 \leqslant u(x) \leqslant C\,|\,x\,|^{2-n+\epsilon}, \qquad |\,x\,| > 1$$

*if the following conditions hold*:

(1) $f(x, 0) \geqslant 0, \qquad |\,x\,| \geqslant 1;$

(2) $g(x) \geqslant 0, \qquad |\,x\,| = 1;$

(3) $\sup_{|x|=r} f(x, C\,|\,x\,|^{2-n+\epsilon}) \leqslant C\epsilon(\epsilon - 2 - \epsilon)\, r^{\epsilon-n}$

*for all* $|\,x\,| \geqslant 1$, *where* $C = \text{maximum}_{|x|=1}\, g(x)$, *and* $0 < \epsilon < n - 1$. *Moreover,* $u(x) > 0$ *for* $|\,x\,| > 1$ *if* $g \not\equiv 0$ *on* $|\,x\,| = 1$.

*Proof.* Let $u_0(x) = C\,|\,x\,|^{2-n+\epsilon}$. Then condition (3) implies that $Lu_0 + f(x, u_0) \leqslant 0$, $|\,x\,| > 1$, and $u_0 \geqslant g$, $|\,x\,| = 1$. The conclusions of Corollary 4.1 then follow from Corollary 3.5.

The above corollary applies, in particular, to the problem

$$\Delta u + p(x)\, u^{\gamma} = 0, \qquad |\,x\,| > 1,$$
$$u(x) = g, \qquad |\,x\,| = 1,$$

where $\gamma > 0$ and $p$ is of class $C^{\alpha}(N_b)$ for any $N_b = \{x\colon 1 \leqslant |\,x\,| \leqslant b\}$, $0 < b < \infty$. In this case condition (3) of Corollary 4.1 becomes

$$\sup_{|x|=r} p(x) \leqslant C^{1-\gamma}\epsilon(n - 2 - \epsilon)\, r^{b}, \tag{3}^1$$

where $b = -n + (n-1)\gamma - (\gamma - 1)\epsilon$.

This is quite sharp in the case $p(x) \geqslant 0$, $|\,x\,| \geqslant 1$, and $\gamma > 1$ is the quotient of two odd integers, in view of a result of the author and Swanson [8] which asserts that all solutions of $\Delta u + p(x)\, u^{\gamma} = 0$ change sign in $G_a = \{x\colon |\,x\,| > a\}$ for any $a > 1$ if

$$\int_1^{\infty} r^d p_M(r)\, dr = \infty, \qquad d = n - 1 - \gamma(n - 2),$$

where $p_M$ is the spherical mean of $p(x)$ on a sphere of radius $r$.

In the linear case $\gamma = 1$, we note that $b = -2$, and if $\epsilon = \frac{1}{2}(n - 2)$, then (3) becomes

$$\sup_{|x|=r} p(x) \leqslant \frac{(n-2)^2}{4}\, r^{-2}, \qquad r > 1.$$

This is quite sharp in view of the known [6] Hille–Kneser criterion.

COROLLARY 4.2. *Problem* (4.1) *has a maximal positive solution* $\hat{u}$ *satisfying*

$$0 < \hat{u}(x) \leqslant C \mid x \mid^{2-n+\epsilon}, \qquad \mid x \mid \geqslant 1$$

*if the conditions* (1), (2), (3) *of Corollary* 4.1 *hold and*

(4) $g(x) > 0$ *for* $\mid x \mid = 1$;

(5) $f(x, u)$ *is monotonic decreasing in* $u$, $u > 0$, $x \in \Omega$, *where* $C = \max_{|x|=1} g(x)$.

*Moreover, if these conditions hold and* $n \geqslant 3$, (4.1) *has a unique solution* $\hat{u}(x)$ *converging to zer uniformly as* $\mid x \mid \to \infty$.

*Proof.* The proof follows from Corollary 3.6 by taking $u_0(x) = C \mid x \mid^{2-n+\epsilon}$, where $C = \sup_{|x|=1} g(x)$, and $0 < \epsilon < n - 2$ satisfying condition (3).

COROLLARY 3.4. *For any* $\epsilon$ *satisfying* $\epsilon(2 - n + \epsilon) \leqslant 0$, *problem* (4.1) *has a solution* $u$ *satisfying*

$$0 \leqslant u(x) \leqslant c \mid x \mid^{2-n+\epsilon}$$

*if*

(1) $f(x, u) \leqslant 0, f(x, 0) = 0$ *for all* $\mid x \mid \geqslant 1$ *and all* $u > 0$;

(2) $g(x) \geqslant 0$, $\mid x \mid = 1$, *where* $c = \sup_{|x|=1} g(x)$.

*Moreover, if* $f(x, u)$ *is monotonic decreasing in* $u$ *for all* $u > 0$ *and* $\mid x \mid > 1$, *and* $g(x) > 0$ *for* $\mid x \mid = 1$, (4.1) *has a maximal solution* $\hat{u}$ *satisfying*

$$0 < \hat{u}(x) \leqslant c \mid x \mid^{2-n+\epsilon}$$

*for any* $\epsilon$ *satisfying* $\epsilon(2 - n + \epsilon) \leqslant 0$.

*Proof.* Let $u_0(x) = c \mid x \mid^{2-n+\epsilon}$, where $c = \sup_{|x|=1} g(x)$, and $\epsilon(2 - n + \epsilon) \leqslant 0$. Then (1) implies that $\Delta u_0 + f(x, u_0) \leqslant 0$ for $\mid x \mid > 1$, and $u_0(x) \geqslant g(x)$ for $\mid x \mid = 1$. The conclusion of Corollary 4.3 then follows from Corollary 3.6.

Other criteria can be obtained by applying known one-dimensional criteria. As an example we use a result of Wong [13, Lemma 1.5] to obtain a sufficient condition which guarantees that the boundary value problem

$$\begin{aligned} n = 2, \qquad \Delta u &= H(x, u), \qquad \mid x \mid > 1, \\ u(x) &= g(x), \qquad \mid x \mid = 1, \end{aligned} \tag{4.3}$$

has a nonnegative solution $u(x)$ converging uniformly to zero as $\mid x \mid \to \infty$. Here $H$ and $g$ are required to satisfy the following conditions

(a) $-H$ and $g$ satisfy conditions $A$;

(b) $H(x, t) \geqslant tG(\mid x \mid, t)$ for all $\mid x \mid \geqslant 1$ and for all $t \geqslant 0$, where $G$ is continuous and positive for all $|x| \geqslant 1$ and for all $t > 0$, $G(|x|, 0) \equiv 0$, and $G$ is monotonic increasing in $t$ for all $|x| \geqslant 1$ and all $t > 0$;

(c) $H(x, 0) \equiv 0.\ \ |x| \geqslant 1$;

(d) $g(x) \geqslant 0,\ |x| = 1$.

Consider the ordinary differential equation

$$\frac{d}{dr}\left(r\frac{d\rho}{dr}\right) = r\rho G(r, \rho). \tag{4.4}$$

COROLLARY 4.3. *Assume $H$ and $g$ satisfy conditions $a$, $b$, $c$, $d$. Then* (4.3) has *a unique solution $u$ which converges to zero uniformly as $|x| \to \infty$ if for each $K > 0$*

$$\int_1^\infty r \log rG(r, K)\, dr = \infty. \tag{4.5}$$

*Proof.* Liouville's change of variables $r = e^s$, $h(s) = \rho(e^s)$ transforms (4.4) into

$$h'' = e^{2s}h(s)\, G(e^s, h(s)). \tag{4.6}$$

A result of Wong [13, Lemma 1.5] implies that (4.6) has a positive solution converging to zero as $s \to \infty$ of for some $a > 0$,

$$\int_a^\infty se^{2s}h(s)\, G(e^s, K)\, ds = \infty \tag{4.7}$$

for each $K > 0$. Since (4.7) is equivalent to (4.5), it follows that (4.5) is a sufficient condition for (4.4) to have a positive solution $\rho_0(r)$ on $[1, \infty]$ converging to zero as $r \to \infty$. Let $c \geqslant 1$ be chosen such that the function $u_0(|x|) = c\rho_0(|x|)$ satisfies $u_0(1) \geqslant \sup_{|x|=1} g(x)$. From the hypothesis (b) we obtain

$$\begin{aligned} H(x, u_0) &\geqslant c\rho_0(|x|)\, G(|x|, c\rho_0(|x|)) \\ &\geqslant c\rho_0(|x|)\, G(|x|, \rho_0(|x|)) \\ &= \frac{1}{r}\frac{d}{dr}\left(r\frac{du_0}{dr}\right) = \Delta u_0, \qquad r > 1; \end{aligned}$$

and

$$u_0(1) \geqslant g(x), \qquad |x| = 1.$$

Since the function $v_0(x) \equiv 0$ obviously satisfies $\Delta v_0 \geqslant H(x, v_0)$, $|x| > 1$, and $v_0(x) \leqslant g(x)$, $|x| = 1$, Theorem 3.3 implies that (4.3) has a solution $u$ satisfying $0 \leqslant u \leqslant u_0$. Hence $u(x)$ converges to zero uniformly as $|x| \to \infty$. Finally, the uniqueness of $u$ follows by the same argument used in Corollary 3.6, completing the proof of Corollary 4.3.

## 5. Dirichlet Problem in an Unbounded Domain

For the case when $\Omega$ is not an exterior domain, we require that $\Omega$ allows the following decomposition:

There exists a sequence of bounded domains $\Omega_n$, $n = 1, 2,\ldots$, with boundaries $\partial\Omega_n$ of class $C^{2+\alpha}$ such that

(1) $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n = 1, 2,\ldots$, and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$;

(2) $x \in \partial\Omega$ and $|x| \leqslant n$ implies that $x \in \partial\Omega_n$.

Consider the boundary value problem

$$\begin{aligned} Lu + f(x, u) &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

where $L$ is the elliptic operator defined in 2, and $f$, $g$ satisfy the conditions below.

*Assumption* B. (1) $f$ satisfies assumptions A(i), (ii);

(2) $g$ is a real-valued function on $\bar{\Omega}$ of class $C^{2+\alpha}(\bar{M})$ for all bounded domains $M \subset \Omega$.

By replacing the domains $\Omega_{a+j}$ in the proofs of Lemmas 3.1, 3.2 by the domains $\Omega_j$, $j = 1, 2,\ldots$, we obtain

THEOREM 4.1. *Let $f$, $g$, $\Omega$ satisfy the conditions of Section 4. If there exist functions $v_0 \leqslant u_0$ in $\bar{\Omega}$ of class $C^{2+\alpha}(\bar{M})$ for all bounded domains $M \subset \Omega$ satisfying*

$$\begin{aligned} Lu_0 + f(x, u_0) &\leqslant 0 \qquad \text{in } \Omega, \\ u_0 &\geqslant g \qquad \text{on } \bar{\Omega}; \\ Lv_0 + f(x, v_0) &\geqslant 0 \qquad \text{in } \Omega, \\ v_0 &\leqslant g \qquad \text{on } \bar{\Omega}, \end{aligned}$$

*then the boundary value problem* (5.1) *has a solution $\hat{u}$ satisfying*

$$v_0 \leqslant \hat{u} \leqslant u_0 \qquad \text{in } \Omega.$$

*Proof.* The proof is very similar to the proof of Theorem 3.3. In fact, the only modification required in the proofs of Lemmas 3.1 and 3.2 is to replace $\Omega_{a+j}$ by $\Omega_j$, $j = 1, 2,\ldots, n$. The details are left to the reader.

Analogs of the results in Section 3 can be easily written.

## References

1. S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, *Comm. Pure Appl. Math.* 12 (1959), 628–727.

2. H. AMANN, On the existence of positive solutions of nonlinear elliptic boundary value problems, *Indiana Univ. Math. J.* **21** (1971), 125–146.
3. C. BANDLE, Existence theorems, qualitative results and a priori bounds for a class of nonlinear Dirichlet problems, *Arch. Rational Mech. Anal.* **58** (1975), 219–238.
4. L. B. BUSHARD, A comparison result for a class of quasilinear elliptic partial differential equations, *J. Differential Equations* **21** (1976), 439–443.
5. H. B. KELLER, Elliptic boundary value problems suggested by nonlinear diffusion processes, *Arch. Rational Mesh. Anal.* **5** (1969), 363–381.
6. E. HILLE, Non-oscillation theorems, *Trans. Amer. Math. Soc.* **64** (1948), 234–252.
7. M. NAGUMO, On principally linear elliptic equations of the second order, *Osaka J. Math.* **6** (1954), 207–229.
8. E. S. NOUSSAIR AND C. A. SWANSON, Oscillation theory for semilinear Schrödinger equations and inequalities, *Proc. Roy. Soc. Edinburgh Sect. A* **75** (1975/1976), 67–81.
9. A. OGATA, On bounded positive solutions of nonlinear elliptic boundary value problems in an exterior domain, *Funcial. Ekvac.* **17** (1974), 207–222.
10. K. SCHMITT, Boundary value problems for quasilinear second order elliptic equations, *Nonlinear Anal. Theory Methods Appl.* **2**, No. 3 (1978), 263–309.
11. R. B. SIMPSON AND D. S. COHEN, Positive solutions of nonlinear elliptic eigenvalue problems, *J. Math. Mech.* **19** (1970), 895–910.
12. M. H. PROTTER AND H. F. WEINBERGER, "Maximum Principles in Differential Equations," Prentice-Hall, Englewood Cliffs, N. J., 1967.
13. P. K. WONG, Existence and asymptotic behavior of proper solutions of a class of second order nonlinear differential equations, *Pacific J. Math.* **13** (1963), 737–760.