Locally Twisted Cubes are 4-Pancyclic

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Abstract—The locally twisted cube is a newly introduced interconnection network for parallel computing. Ring embedding is an important issue for evaluating the performance of an interconnection network. In this paper, we investigate the problem of embedding rings into a locally twisted cube. Our main contribution is to find that, for each integer \( l \in \{4, 5, \ldots, 2^n\} \), a ring of length \( l \) can be embedded into an \( n \)-dimensional locally twisted cube so that both the dilation and the load factor are one. As a result, a locally twisted cube is Hamiltonian. We conclude that a locally twisted cube is superior to a hypercube in terms of ring embedding capability. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

One of the central issues in evaluating an interconnection network is to study how well other existing networks can be embedded into this network. This problem can be modeled by the following graph embedding problem: given a host graph, which represents the network into which other networks are to be embedded, and a guest graph, which represents the network to be embedded, the problem is to find a mapping from each node of the guest graph to a node of the host graph, and a mapping from each edge of the guest graph to a path in the host graph.
Two common measures of effectiveness of an embedding are the dilation, which measures the slowdown in the new architecture, and the load factor, which gauges the processor utilization [1]. Graph embeddings have two main applications:

1. to transplant parallel algorithms developed for one network to a different one, and
2. to allocate concurrent processes to processors in the network.

Rings (cycles) are common guest graphs with many applications. The problem of ring embedding has been studied for hypercube structure and variations with or without failures [2–11]. A locally twisted cube is a newly introduced interconnection network for parallel processing [12]. One advantage of an n-dimensional locally twisted cube is that the diameter is only about half of the diameter of an n-dimensional cube. In this paper, we deal with the problem of embedding rings into a locally twisted cube. Our main contribution is to find that an n-dimensional locally twisted cube contains a cycle of length l for each l ∈ {4, 5, . . . , 2^n}. This result implies that each such ring can be embedded into the locally twisted cube so that both the dilation and the load factor are one. We conclude that the locally twisted cube is superior to hypercube in terms of ring embedding capability.

2. PRELIMINARIES

In this paper, we use a graph G = (V(G), E(G)) to represent an interconnection network, in which the nodes represent the processors and the edges represent the communication links between the processors. A Hamiltonian cycle (Hamiltonian path) in a graph is a cycle (path) that goes through every node of the graph exactly once. A Hamiltonian graph (Hamiltonian path graph) is a graph that contains Hamiltonian cycles (Hamiltonian paths). For fundamental graph-theoretic terminology, the reader is referred to [13].

Given a host graph and a guest graph, the graph embedding problem is to find a mapping from each node of the guest graph to a node of the host graph, and a mapping from each edge of the guest graph to a path in the host graph. There are two common measures of effectiveness of an embedding. One measure is the dilation, which is defined as the length of a longest path onto which an edge is mapped. The dilation measures the slowdown in the new architecture. The other measure is the load factor, which is defined as the maximum number of nodes mapped onto one node. The load factor gauges the processor utilization [1].

**DEFINITION 2.1.** Let G = (V(G), E(G)) be a graph, and k ≤ |V(G)| be a positive integer.

1. If G contains a cycle of length l for each integer l ∈ {k, k + 1, . . . , |V(G)|}, then G is k-pancyclic.
2. If G contains a path of length k between any two distinct nodes, then G is k-path-connected.
3. If G is (|V(G)| − 1)-path-connected, then G is Hamiltonian-path-connected.

Next, we review the concept of locally twisted cubes.

**DEFINITION 2.2.** Let n ≥ 2. The n-dimensional locally twisted cube, LTQ_n, is defined recursively as follows.

1. LTQ_2 is a graph consisting of four nodes labeled with 00, 01, 10, and 11, respectively, connected by four edges (00, 01), (00, 10), (01, 11), and (10, 11).
2. For n ≥ 3, LTQ_n is built from two disjoint copies of LTQ_{n−1} according to the following steps. Let 0LTQ_{n−1} denote the graph obtained by prefixing the label of each node of one copy of LTQ_{n−1} with 0, let 1LTQ_{n−1} denote the graph obtained by prefixing the label of each node of the other copy of LTQ_{n−1} with 1, and connect each node x = (x_2 x_3 . . . x_n) of 0LTQ_{n−1} with the node 1(x_2+x_n)x_3 . . . x_n of 1LTQ_{n−1} by an edge, where ‘+’ represents the modulo 2 addition.
Locally Twisted Cubes

(a) Ordinary drawing of LTQ₃.
(b) Symmetric drawing of LTQ₃.
(c) LTQ₄.

Figure 1. LTQₙ for n = 3, 4.

Figure 1 shows two examples of locally twisted cubes. Let \( \{0, 1\}^n \) denote the set of all 0-1 binary strings of length \( n \). The locally twisted cubes can also be equivalently defined as follows.

**DEFINITION 2.2'.** Let \( n \geq 2 \). The \( n \)-dimensional locally twisted cube, \( \text{LTQ}_n \), is a graph with \( \{0, 1\}^n \) as the node set. Two nodes \( x = x_1x_2\ldots x_n \) and \( y = y_1y_2\ldots y_n \) of \( \text{LTQ}_n \) are adjacent if and only if one of the following conditions are satisfied.

1. There is an integer \( 1 < k < n - 2 \) such that
   (a) \( x_k = \bar{y}_k \),
   (b) \( x_{k+1} = y_{k+1} + x_n \), and
   (c) all the remaining bits of \( x \) and \( y \) are identical.
   If so, \( y \) is called the \( k \)-dimensional neighbor of \( x \), denoted by \( y = N_k(x) \).

2. There is an integer \( k \in \{n - 1, n\} \) such that \( x \) and \( y \) differ only in the \( k \)-th bit. If so, \( y \) is called the \( k \)-th-dimensional neighbor of \( x \), denoted by \( y = N_k(x) \).

\( \text{LTQ}_n \) is an \( n \)-regular graph, and the labels of any two neighboring nodes of \( \text{LTQ}_n \) differ in at most two successive bits. It is known [12] that the diameter of \( \text{LTQ}_n \) is \( \lceil (n + 3)/2 \rceil \).

**3. PANCYCLICITY OF LOCALLY TWISTED CUBE**

To study the pancyclicity of locally twisted cubes, we need the following preliminary results.

**LEMMA 1.** \( \text{LTQ}_3 \) is Hamiltonian-path-connected, 6-path-connected, and 4-pancyclic.

**PROOF.** For the first two assertions, in view of the symmetry of \( \text{LTQ}_3 \) shown in Figure 1b, it suffices to give a path of length 7 and a path of length 6, respectively, for each of the following node-pairs:

\[ (000, 100), \quad (000, 101), \quad (000, 011), \quad (000, 010). \]

The required paths are listed in Table 1.
Table 1. The required paths for typical node-pairs of LTQ₃.

<table>
<thead>
<tr>
<th>Node-Pair</th>
<th>A Path of Length 7</th>
<th>A Path of Length 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(000, 100)</td>
<td>(000, 001, 011, 010, 110, 111, 101, 100)</td>
<td>(000, 001, 011, 111, 110, 101, 100)</td>
</tr>
<tr>
<td>(000, 101)</td>
<td>(000, 010, 011, 001, 111, 110, 100, 101)</td>
<td>(000, 001, 011, 010, 111, 110, 101)</td>
</tr>
<tr>
<td>(000, 011)</td>
<td>(000, 001, 111, 101, 100, 110, 010, 011)</td>
<td>(000, 100, 101, 111, 110, 010, 011)</td>
</tr>
<tr>
<td>(000, 010)</td>
<td>(000, 100, 101, 011, 001, 111, 110, 100)</td>
<td>(000, 001, 111, 101, 100, 110, 010)</td>
</tr>
</tbody>
</table>

Table 2. The required cycles in LTQ₃.

<table>
<thead>
<tr>
<th>Length of Cycle</th>
<th>Cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(000, 100, 110, 010, 000)</td>
</tr>
<tr>
<td>5</td>
<td>(000, 100, 101, 011, 010, 000)</td>
</tr>
<tr>
<td>6</td>
<td>(000, 001, 011, 010, 110, 100, 000)</td>
</tr>
<tr>
<td>7</td>
<td>(000, 001, 011, 101, 111, 110, 100, 000)</td>
</tr>
<tr>
<td>8</td>
<td>(000, 100, 101, 011, 010, 110, 111, 001, 000)</td>
</tr>
</tbody>
</table>

For the third assertion, the required cycles are listed in Table 2.

**LEMMA 2.** LTQₙ is Hamiltonian-path-connected for n ≥ 3.

**PROOF.** We argue by induction on n. Lemma 1 ensures the correctness of the assertion for n = 3. Suppose the assertion is true for the case n = k (≥ 3). Assume n = k + 1. Let x and y be any two distinct nodes of LTQₖ₊₁. There are four possibilities for the locations of x and y.

**CASE 1.** x ∈ V(0LTQₖ) and y ∈ V(0LTQₖ). By the inductive hypothesis, 0LTQₖ contains a Hamiltonian path HP₀(x, y) from x to y. Let z be the node on HP₀(x, y) that is adjacent to x, and let HP₀(z, y) be the segment of HP₀(x, y) that starts from z and terminates at y. By the inductive hypothesis, 1LTQₖ contains a Hamiltonian path HP¹(N₁(x), N₁(z)) from N₁(x) to N₁(z). Hence, LTQₖ₊₁ contains a Hamiltonian path from x to y, which is of the following form:

(x, N₁(x), HP¹(N₁(x), N₁(z)), N₁(z), z, HP₀(z, y), y) (see Figure 2a).

**Figure 2.** Schematic explanations of the proof of Lemma 2.
CASE 2. \( x \in V(1LTQ_k) \) and \( y \in V(1LTQ_k) \). The argument is similar to that for Case 1.

CASE 3. \( x \in V(0LTQ_k) \) and \( y \in V(1LTQ_k) \). Let \( z \neq x \) be a node of \( 0LTQ_k \) such that \( N_1(z) \neq y \).

By the inductive hypothesis, \( 0LTQ_k \) contains a Hamiltonian path \( HP^0(x, z) \) from \( x \) to \( z \), and \( 1LTQ_k \) contains a Hamiltonian path from \( x \) to \( y \), which is of the following form:

\[ (x, HP^0(x, z), z, N_1(z), HP^1(N_1(z), y), y) \] (see Figure 3b).

CASE 4. \( x \in V(1LTQ_k) \) and \( y \in V(0LTQ_k) \). The argument is similar to that for Case 3.

This completes our inductive proof.

**LEMMA 3.** \( LTQ_n \) is \((2^n - 2)\)-path-connected for \( n \geq 3 \).

**PROOF.** By induction on \( n \). It follows from Lemma 1 that the assertion holds for \( n = 3 \). Suppose the assertion is true for the case \( n = k \) \((\geq 3)\). Assume \( n = k + 1 \). Let \( x \) and \( y \) be any two distinct nodes of \( LTQ_{k+1} \). Four possibilities are examined for the locations of \( x \) and \( y \).

CASE 1. \( x \in V(0LTQ_{k}) \) and \( y \in V(0LTQ_{k}) \). By the inductive hypothesis, \( 0LTQ_k \) contains a path \( P^0(x, y) \) of length \( 2^k - 2 \) from \( x \) to \( y \). Let \( z \) be the node on \( P^0(x, y) \) that is adjacent to \( x \), and let \( P^0(z, y) \) be the segment of \( P^0(x, y) \) that starts from \( z \) and terminates at \( y \). By Lemma 2, \( 1LTQ_k \) contains a Hamiltonian path \( HP^1(N_1(z), y) \) from \( N_1(z) \) to \( y \). Hence, \( LTQ_{k+1} \) contains a path from \( x \) to \( y \), which is of the following form:

\[ (x, N_1(x), HP^1(N_1(x), N_1(z)), N_1(z), z, P^0(z, y), y) \] (see Figure 3a).

The length of this path is \((2^k - 3) + (2^k - 1) + 2 = 2^{k+1} - 2\).

CASE 2. \( x \in V(1LTQ_{k}) \) and \( y \in V(1LTQ_{k}) \). The argument is similar to that for Case 1.

CASE 3. \( x \in V(0LTQ_{k}) \) and \( y \in V(1LTQ_{k}) \). Let \( z \neq x \) be a node of \( 0LTQ_k \) such that \( N_1(z) \neq y \).

By the inductive hypothesis, \( 0LTQ_k \) contains a path \( P^0(x, z) \) of length \( 2^k - 2 \) from \( x \) to \( z \). By Lemma 2, \( 1LTQ_k \) contains a Hamiltonian path \( HP^1(N_1(z), y) \) from \( N_1(z) \) to \( y \). Hence, \( LTQ_{k+1} \) contains a path from \( x \) to \( y \), which is of the following form:

\[ (x, P^0(x, z), z, N_1(z), HP^1(N_1(z), y), y) \] (see Figure 3b).

The length of the path is \((2^k - 2) + (2^k - 1) + 1 = 2^{k+1} - 2\).

CASE 4. \( x \in V(1LTQ_{k}) \) and \( y \in V(0LTQ_{k}) \). The argument is similar to that for Case 3.

Our inductive proof is accomplished.
Now we are ready to present the main result of this paper.

**THEOREM 4.** LTQ\(_n\) is 4-pancyclic for \(n \geq 3\).

**PROOF.** By induction on \(n\). As the inductive basis, the 4-pancyclicity of LTQ\(_3\) is ensured by Lemma 1. Suppose the assertion is true for the case \(n = k \geq 3\). That is, LTQ\(_k\) is 4-pancyclic. Assume \(n = k + 1\). Let \(l\) be an integer satisfying \(4 \leq l \leq 2^{k+1}\). We examine three possibilities.

**CASE 1.** \(4 \leq l \leq 2^k\). By the inductive hypothesis, 0LTQ\(_{k}\) (and hence, LTQ\(_{k+1}\)) contains a cycle of length \(l\).

**CASE 2.** \(2^k + 1 \leq l \leq 2^{k+1}\). Let \(l_0 = l - 2^k - 1\), then \(1 \leq l_0 \leq 2^k - 1\). By the inductive hypothesis, 0LTQ\(_{k}\) contains a Hamiltonian cycle. So 0LTQ\(_{k}\) contains a path \(P^0(x,y)\) of length \(l_0\) that starts from some node \(x\) and terminates at some node \(y\). By Lemma 2, 1LTQ\(_k\) contains a Hamiltonian path \(HP^1(N_1(y), N_1(x))\) from \(N_1(y)\) to \(N_1(x)\) (which is of length \(2^k - 1\)). Hence, LTQ\(_{k+1}\) contains a cycle of the form

\[
(x, P^0(x,y), y, N_1(y), HP^1(N_1(y), N_1(x)), N_1(x), x) \quad \text{(see Figure 4a).}
\]

The length of the cycle is \(l_0 + (2^k - 1) + 2 = (l - 2^k - 1) + (2^k - 1) + 2 = l\).

**CASE 3.** \(l = 2^k + 1\). Let \(x\) and \(y\) be two neighboring nodes of 0LTQ\(_k\). By Lemma 3, 1LTQ\(_k\) contains a path \(P^1(N_1(y), N_1(x))\) of length \(2^k - 2\) from \(N_1(y)\) to \(N_1(x)\). Hence, LTQ\(_{k+1}\) contains a cycle of the form

\[
(x, y, N_1(y), P^1(N_1(y), N_1(x)), N_1(x), x) \quad \text{(see Figure 4b).}
\]

The length of the cycle is \(1 + 2 + (2^k - 2) = 2^k + 1 = l\).

By the inductive principle, the assertion is true for any \(n \geq 3\). \(\blacksquare\)

**COROLLARY 5.** For \(n \geq 3\), LTQ\(_n\) is Hamiltonian.

**REFERENCES**