PERGAMON

Applied Mathematics Letters 15 (2002) 845–854

Applied Mathematics Letters

www.elsevier.com/locate/aml

Nonlocal Cauchy Problem for Delay Integrodifferential Equations of Sobolev Type in Banach Spaces

K. BALACHANDRAN
Department of Mathematics, Bharathiar University
Coimbatore-641 046, India

J. Y. PARK
Department of Mathematics, Pusan National University
Pusan 609-735, Korea

M. CHANDRASEKARAN
Department of Mathematics, Bharathiar University
Coimbatore-641 046, India

(Received May 2000; revised and accepted October 2001)

Abstract—In this paper, we prove the existence of mild and strong solutions of nonlinear time varying delay integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces. The results are obtained by using the theory of compact semigroups and Schaefer's fixed-point theorem. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Existence theorem, Delay integrodifferential equations, Fixed-point theorem, Nonlocal condition.

1. INTRODUCTION

The problem of existence of solutions of semilinear differential equations and integrodifferential equations in Banach spaces has been studied by several authors [1–7]. Byszewski [8] has established the existence and uniqueness of mild, strong, and classical solutions of the following nonlocal Cauchy problem:

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)), \qquad t \in (0, a],$$
$$u(t_0) + g(t_1, t_2, \ldots, t_p, u(.)) = u_0,$$

where $0 \leq t_0 < t_1 < \cdots < t_p \leq a$, $a > 0$, $-A$ is the infinitesimal generator of a C_0-semigroup in a Banach space $X, u_0 \in X$, and $f : [0, a] \times X \to X$, $g : [0, a]^p \times X \to X$ are given functions. Subsequently, he has investigated the same problem for different types of evolution equations in Banach spaces [9–12]. Many papers have been written on nonlocal Cauchy problems for various

The authors are thankful to the referees for the improvement of the paper.
The work of the first author was supported by the KOSEF at the Pusan National University, South Korea during 1999.

PII: S0893-9659(02)00052-6

classes of differential and integrodifferential equations [13–19]. Physical motivation for this kind of problem is given in [8–11,20].

Brill [21] investigated the existence of solutions for a semilinear Sobolev evolution equation in a Banach space. Existence theorems for Sobolev type equations in Banach spaces have been proved in papers [22–25]. These types of equations arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics, and shear in second-order fluids (see [24]). Recently, Balachandran *et al.* [26] discussed the problem for nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces. In this paper, we shall establish the existence of solutions of time varying delay integrodifferential equations of Sobolev type with nonlocal conditions by using the compact semigroup and the Schaefer theorem.

2. BASIC ASSUMPTIONS

Consider the nonlinear time varying integrodifferential equation of Sobolev type with nonlocal condition of the form

$$(Eu(t))' + Au(t) = f\left(t, u(\sigma_1(t)), \dots, u(\sigma_n(t)), \int_0^t k(t,s)h(s, u(\sigma_{n+1}(s)))\, ds\right), \qquad t \in [0,a], \quad (1)$$

$$u(0) + g(u) = u_0, \quad (2)$$

where $f : I \times X^{n+1} \to Y$, $k : \Delta \to R$, $h : I \times X \to X$, and $g : X \to X$ are given functions. Moreover, $\sigma_i : I \to I$, $i = 1, \dots, n+1$, are continuous functions such that $\sigma_i(t) \le t$, $i = 1, \dots, n+1$, and $u_0 \in D(E)$. Let $I = [0,a]$ and $\Delta = \{(t,s) : 0 \le s \le t \le a\}$. We assume the following.

(i) For each $t \in I$, the function $f(t, ., \dots, .) : X^{n+1} \to Y$ is continuous and for each $u_1, \dots, u_{n+1} \in X$, the function $f(., u_1, \dots, u_{n+1}) : I \to Y$ is strongly measurable.

(ii) For each $t \in I$, the function $h(t, .) : X \to X$ is continuous and for each $u \in X$, the function $h(., u) : I \to X$ is strongly measurable.

(iii) For every positive integer r, there exists $h_r \in L^1(I)$ such that

$$\sup_{|u| \le r} \left\| f\left(t, u(\sigma_1(t)), \dots, u(\sigma_n(t)), \int_0^t k(t,s)h(s, u(\sigma_{n+1}(s)))\, ds\right) \right\| \le h_r(t).$$

DEFINITION 2.1. *(See [7].) A continuous solution $u(t)$ of the integral equation*

$$u(t) = E^{-1}T(t)Eu_0 - E^{-1}T(t)Eg(u) + \int_0^t E^{-1}T(t-s) f\left(s, u(\sigma_1(s)), \dots, u(\sigma_n(t)), \int_0^s k(s,\tau)h(\tau, u(\sigma_{n+1}(\tau)))\, d\tau\right) ds$$

is called a mild solution of (1),(2) on I.

DEFINITION 2.2. *(See [7].) A function u is said to be a strong solution of problem (1),(2) on I if u is differentiable almost everywhere on I, $u'(t) \in L^1(I, X)$, $u(0) + g(u) = u_0$ and*

$$(Eu(t))' + Au(t) = f\left(t, u(\sigma_1(t)), \dots, u(\sigma_n(t)), \int_0^t k(t,s)h(s, u(\sigma_{n+1}(s)))\, ds\right), \qquad \text{a.e. on } I.$$

In order to prove our main theorem, we assume certain conditions on the operators A and E. Let X and Y be Banach spaces with norm $|.|$ and $\|.\|$, respectively. The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the following hypotheses.

(H_1). *A and E are closed, linear operators.*

(H_2). *$D(E) \subset D(A)$ and E is bijective.*

(H_3). *$E^{-1} : Y \to D(E)$ is continuous.*

Hypotheses (H_1) and (H_2) and the closed graph theorem imply the boundedness of the linear operator $AE^{-1}: Y \to Y$ and $-AE^{-1}$ generates a uniformly continuous semigroup $T(t)$, $t \geq 0$, of bounded linear operators from Y into Y.

(H_4). *For some $\lambda \in \rho(-AE^{-1})$, the resolvent set of $-AE^{-1}$, the resolvent $R(\lambda, -AE^{-1})$ is a compact operator.*

Let $T(t)$ be a uniformly continuous semigroup and let A be its infinitesimal generator. If the resolvent set $R(\lambda : A)$ of A is compact for every $\lambda \in \rho(A)$, then $T(t)$ is a compact semigroup [7].

From the above fact that $-AE^{-1}$ generates a compact semigroup $T(t)$, $t \geq 0$, and so $\max_{t\in J} \|T(t)\|$ is finite and denote $\alpha = \|E^{-1}\|$. We need the following fixed-point theorem to prove our results.

SCHAEFER'S THEOREM. *(See [27].) Let Z be a normed linear space. Let $F : Z \to Z$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set, and let*

$$\zeta(F) = \{x \in Z : x = \lambda F x, \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

3. EXISTENCE THEOREMS

THEOREM 3.1. *Let $f : I \times X^{n+1} \to Y$ and $h : I \times X \to X$ be functions satisfying Conditions (i)–(iii). Assume that (H_1)–(H_4) hold. Further assume the following.*

(iv) *There exists a continuous function $m : I \to [0, \infty)$ such that*

$$|h(t,u)| \leq m(t)\Omega(|u|),$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(v) *There exists a continuous function $p : I \to [0,\infty)$ such that*

$$\|f(t,u_1,\dots,u_{n+1})\| \leq p(t)\Omega_0(|u_1| + \cdots + |u_{n+1}|),$$

where $\Omega_0 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(vi) *$k : \Delta \to R$ is a measurable function such that there exists a constant $L > 0$ such that*

$$|k(t,s)| \leq L, \qquad \text{for } t \geq s \geq 0.$$

(vii) *$T(t)$ is a compact semigroup and there exists a constant $M > 0$ such that*

$$\|T(t)\| \leq M.$$

(viii) *$g : C(I : X) \to D(E) \subset X$, is continuous, compact, and there exists a constant $G > 0$ such that*

$$\|Eg(u)\| \leq G, \qquad \text{for } u \in C(I : X).$$

Further, if

$$\int_0^a m^*(s)\,ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)},$$

where $c = \alpha n M(\|Eu_0\| + G)$ and $m^*(t) = \max\{n\alpha M p(t), L m(t)\}$, then problem (1),(2) has at least one mild solution on I.

PROOF. We establish the existence of a mild solution of problem (1),(2) by applying the Schaefer fixed-point theorem. First, we obtain *a priori* bounds for the mild solutions of problem (3),(4), as in [18],

$$(Eu(t))' + Au(t) = \lambda f\left(t, u(\sigma_1(t)), \dots, u(\sigma_n(t)), \int_0^t k(t,s)h(s, u(\sigma_{n+1}(s)))\, ds\right), \tag{3}$$

$$u(0) = \lambda(u_0 - g(u)), \qquad \lambda \in (0,1). \tag{4}$$

Let u(t) be a mild solution of problem (3),(4). Then from the equation

$$\begin{aligned} u(t) &= \lambda E^{-1}T(t)Eu_0 - \lambda E^{-1}T(t)Eg(u) \\ &\quad + \lambda \int_0^t E^{-1}T(t-s) f\left(s, u(\sigma_1(s)), \dots, u(\sigma_n(s)), \int_0^s k(s,\tau)h(\tau, u(\sigma_{n+1}(\tau)))\, d\tau\right) ds, \end{aligned}$$

we have

$$\begin{aligned} |u(t)| &\le \alpha M \|Eu_0\| + \alpha MG \\ &\quad + M\int_0^t \alpha \left\| f\left(s, u(\sigma_1(s)), \dots, u(\sigma_n(s)), \int_0^s k(s,\tau)h(\tau, u(\sigma_{n+1}(\tau)))\, d\tau\right)\right\| ds \\ &\le \alpha M\|Eu_0\| + \alpha MG + M\int_0^t \alpha p(s)\Omega_0 \Big[|u(\sigma_1(s))| + \dots + |u(\sigma_n(s))| \\ &\quad + \int_0^s |k(s,\tau)|\ |h(\tau, u(\sigma_{n+1}(\tau)))\, d\tau|\Big] ds \\ &\le \alpha M\|Eu_0\| + \alpha MG + \alpha M\int_0^t p(s)\Omega_0 \Big[|u(\sigma_1(s))| + \dots + |u(\sigma_n(s))| \\ &\quad + L\int_0^s m(\tau)\Omega(|u(\sigma_{n+1}(\tau))|)\, d\tau\Big] ds. \end{aligned}$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have $v(0) = \alpha M(\|Eu_0\| + G)$, $|u(t)| \le v(t)$,

$$\begin{aligned} v'(t) &\le \alpha M p(t)\Omega_0 \left[|u(\sigma_1(t))| + \dots + |u(\sigma_n(t))| + L\int_0^t m(s)\Omega(|u(\sigma_{n+1}(s))|)\, ds\right] \\ &\le \alpha M p(t)\Omega_0 \left[v(\sigma_1(t)) + \dots + v(\sigma_n(t)) + L\int_0^t m(s)\Omega(v(\sigma_{n+1}(s)))\, ds\right] \\ &\le \alpha M p(t)\Omega_0 \left[v(t) + \dots + v(t) + L\int_0^t m(s)\Omega(v(\sigma_{n+1}(s)))\, ds\right] \\ &\le \alpha M p(t)\Omega_0 \left[nv(t) + L\int_0^t m(s)\Omega(v(s))\, ds\right], \end{aligned}$$

since v is obviously increasing and $\sigma_i(t) \le t$, $i = 1, \dots, n+1$.

Let $w(t) = nv(t) + L\int_0^t m(s)\Omega(v(s))\, ds$.

Then $w(0) = nv(0) = c$, $v(t) \le w(t)$,

$$\begin{aligned} w'(t) &= nv'(t) + Lm(t)\Omega(v(t)) \\ &\le n\alpha M p(t)\Omega_0(w(t)) + Lm(t)\Omega(w(t)) \\ &\le m^*(t)\left[\Omega_0(w(t)) + \Omega(w(t))\right]. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega(s)} \le \int_0^a m^*(s)\, ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}.$$

This inequality implies that there exists a constant K such that $w(t) \leq K$, $t \in I$, and hence, $u(t) \leq K$ where K depends only on a and on the functions m, p, Ω_0, and Ω.

Next we prove that the operator $F : B = C(I, X) \to B$ defined by

$$(Fy)(t) = E^{-1}T(t)Eu_0 - E^{-1}T(t)Eg(y) + \int_0^t E^{-1}T(t-s)f\left(s, y(\sigma_1(s)), \dots, y(\sigma_n(s)), \int_0^s k(s,\tau)\, h(\tau, y(\sigma_{n+1}(\tau)))\, d\tau\right) ds$$

is a completely continuous operator.

Let $B_r = \{y \in B : \|y\| \leq r\}$ for some $r \geq 1$. We first show that F maps B_r into an equicontinuous family. Let $y \in B_r$ and $t_1, t_2 \in I$ and $\epsilon > 0$. Then if $0 < \epsilon < t_1 < t_2 \leq a$,

$$\begin{aligned}
&\|(Fy)(t_1) - (Fy)(t_2)\| \\
&\leq \|T(t_1) - T(t_2)\|\, \alpha\, (\|Eu_0\| + \|Eg(y)\|) + \int_0^{t_1} \alpha\, \|(T(t_1 - s) - T(t_2 - s))\| \\
&\quad \times \left\| f\left(s, y(\sigma_1(s)), \dots, y(\sigma_n(s)), \int_0^s k(s,\tau) h(\tau, y(\sigma_{n+1}(\tau)))\, d\tau\right)\right\| ds \\
&\quad + \int_{t_1}^{t_2} \alpha\, \|T(t_2 - s)\| \left\| f\left(s, y(\sigma_1(s)), \dots, y(\sigma_n(s)), \int_0^s k(s,\tau) h(\tau, y(\sigma_{n+1}(\tau)))\, d\tau\right)\right\| ds \\
&\leq \|T(t_1) - T(t_2)\|\, \alpha\, (\|Eu_0\| + G) + \int_0^{t_1} \alpha\, \|(T(t_1 - s) - T(t_2 - s))\|\, h_r(s)\, ds \\
&\quad + \int_{t_1}^{t_2} \alpha\, \|T(t_2 - s)\|\, h_r(s)\, ds.
\end{aligned}$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero since the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus, F maps B_r into an equicontinuous family of functions. It is easy to see that the family FB_r is uniformly bounded.

Next we show that $\overline{FB_r}$ is compact. Since we have proved that FB_r is an equicontinuous family, it is sufficient, by the Arzela-Ascoli theorem, to show that F maps B_r into a precompact set in X. This is clear when $t = 0$, the set $Fy(0) = \{u_0 - g(y)\}$ is precompact in X, since g is compact.

Let $0 < t \leq a$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_r$, we define

$$(F_\epsilon y)(t) = E^{-1}T(t)Eu_0 - E^{-1}T(t)Eg(y) + \int_0^{t-\epsilon} E^{-1}T(t-s)f\left(s, y(\sigma_1(s)), \dots, y(\sigma_n(s)), \int_0^s k(s,\tau)\, h(\tau, y(\sigma_{n+1}(\tau)))\, d\tau\right) ds.$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_r\}$ is precompact in X, for every ϵ, $0 < \epsilon < t$. Moreover, for every $y \in B_r$, we have

$$\begin{aligned}
&\|(Fy)(t) - (F_\epsilon y)(t)\| \\
&\leq \int_{t-\epsilon}^t \alpha \left\| T(t-s) f\left(s, y(\sigma_1(s)), \dots, y(\sigma_n(s)), \int_0^s k(s,\tau) h(\tau, y(\sigma_{n+1}(\tau)))\, d\tau\right)\right\| ds \\
&\leq \int_{t-\epsilon}^t \alpha \|T(t-s)\| h_r(s)\, ds.
\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_r\}$.

Hence, the set $\{(Fy)(t) : y \in B_r\}$ is precompact in X.

It remains to be shown that $F : B \to B$ is continuous. Let $\{u_j\}$ be a sequence such that $u_j \to u$ in B. Then there is an integer q such that $\|u_j\| \leq q$ for all j and $\|u\| \leq q$, $t \in I$, and so

$u_j \in B_q$ and $u \in B_q$. By (i) and (ii),

$$f\left(t, u_j(\sigma_1(t)), \dots, u_j(\sigma_n(t)), \int_0^t k(t,s)h(s, u_j(\sigma_{n+1}(s)))\,ds\right)$$
$$\to f\left(t, u(\sigma_1(t)), \dots, u(\sigma_n(t)), \int_0^t k(t,s)h(s, u(\sigma_{n+1}(s)))\,ds\right),$$

for each $t \in I$ and since

$$\left\| f\left(t, u_j(\sigma_1(t)), \dots, u_j(\sigma_n(t)), \int_0^t k(t,s)h(s, u_j(\sigma_{n+1}(s)))\,ds\right)\right.$$
$$\left. - f\left(t, u(\sigma_1(t)), \dots, u(\sigma_n(t)), \int_0^t k(t,s)h(s, u(\sigma_{n+1}(s)))\,ds\right)\right\| \le 2h_q(t),$$

we have, by dominated convergence theorem,

$$\begin{aligned}
&\|Fu_j - Fu\| \\
&= \sup_{t\in I}\left\| \int_0^t E^{-1}T(t-s)\left\{ f\left(s, u_j(\sigma_1(s)), \dots, u_j(\sigma_n(t)), \int_0^s k(s,\tau)h(\tau, u_j(\sigma_{n+1}(\tau)))\,d\tau\right)\right.\right. \\
&\quad \left.\left. - f\left(s, u(\sigma_1(s)), \dots, u(\sigma_n(t)), \int_0^s k(s,\tau)h(\tau, u(\sigma_{n+1}(\tau)))\,d\tau\right)\right\} ds\right\| \\
&\le \sup_{t\in I}\int_0^t \|E^{-1}T(t-s)\| \left\| \left\{ f\left(s, u_j(\sigma_1(s)), \dots, u_j(\sigma_n(t)), \int_0^s k(s,\tau)h(\tau, u_j(\sigma_{n+1}(\tau)))\,d\tau\right)\right.\right. \\
&\quad \left.\left. - f\left(s, u(\sigma_1(s)), \dots, u(\sigma_n(t)), \int_0^s k(s,\tau)h(\tau, u(\sigma_{n+1}(\tau)))\,d\tau\right)\right\}\right\| ds \to 0, \quad \text{as } j \to \infty.
\end{aligned}$$

Thus, F is continuous. This completes the proof that F is completely continuous.

We have already proved that the set $\zeta(F) = \{y \in B : y = \lambda F y, \lambda \in (0,1)\}$ is bounded and, by Schaefer's theorem, the operator F has a fixed point in B. This means that problem (1),(2) has a mild solution.

THEOREM 3.2. *Let Assumptions (i)–(viii) in Theorem 3.1 be satisfied and the following additional assumptions hold.*

(ix) *Y is a reflexive Banach space and $B_r = \{y \in B : \|y\| \le r\}$.*

(x) *$f : I \times X^{n+1} \to Y$ is continuous in t on I and there exists constants $N_0 > 0$ and $N > 0$ such that*

$$\|f(t, u_1, \dots, u_{n+1})\| \le N_0,$$
$$\|f(s, u_1, \dots, u_{n+1}) - f(t, v_1, \dots, v_{n+1})\| \le N\left[|s-t| + |u_1 - v_1| + \dots + |u_{n+1} - v_{n+1}|\right],$$
$$s, t \in I, \quad u_i, v_i \in B_r, \qquad i = 1, \dots, n+1.$$

(xi) *$k : \Delta \to R$ is such that there exists a constant $L^* > 0$ such that*

$$|k(t,\tau) - k(s,\tau)| \le L^*|t-s|.$$

(xii) *u is the unique mild solution of problem (1),(2) and there is a constant γ such that*

$$|u(\sigma_i(s)) - u(\sigma_i(t))| \le \gamma|u(s) - u(t)|, \qquad \text{for } t, s \in I \text{ and } i = 1, \dots, n.$$

Then u is the unique strong solution of problem (1),(2) on I.

PROOF. Since all the assumptions of Theorem 3.1 are satisfied, then problem (1),(2) possesses a mild solution u which, according to Assumption (xii), is the unique mild solution of problem (1),(2).

Now we show that this mild solution is the unique strong solution of problem (1),(2) on I. For any $t \in I$, we have

$$\begin{aligned} u(t+h)-u(t) &= E^{-1}[T(t+h)-T(t)]Eu_0 - E^{-1}[T(t+h)-T(t)]Eg(u) \\ &+ \int_0^h E^{-1}T(t+h-s)f(s,u(\sigma_1(s)),\dots,u(\sigma_n(s)), \int_0^s k(s,\tau)h(\tau,u(\sigma_{n+1}(\tau)))\,d\tau)\,ds \\ &+ \int_h^{t+h} E^{-1}T(t+h-s)f(s,u(\sigma_1(s)),\dots,u(\sigma_n(s)), \int_0^s k(s,\tau)h(\tau,u(\sigma_{n+1}(\tau)))\,d\tau)\,ds \\ &- \int_0^t E^{-1}T(t-s)f(s,u(\sigma_1(s)),\dots,u(\sigma_n(s)), \int_0^s k(s,\tau)h(\tau,u(\sigma_{n+1}(\tau)))\,d\tau)\,ds. \end{aligned}$$

From our assumptions, we have

$$\begin{aligned} \|u(t+h)-u(t)\| &\le \alpha\,\|T(t+h)-T(t)\|\,(\|Eu_0\|+\|Eg(u)\|) + \alpha M N_0 h \\ &+ \int_0^t \Bigg\| E^{-1}T(t-s)\Bigg[f\Bigg(s+h, u(\sigma_1(s+h)),\dots,u(\sigma_n(s+h)), \\ &\times \int_0^{s+h} k(s+h,\tau)\,h(\tau,u(\sigma_{n+1}(\tau)))\,d\tau\Bigg) \\ &- f\Bigg(s, u(\sigma_1(s)),\dots,u(\sigma_n(s)), \int_0^s k(s,\tau)h(\tau,u(\sigma_{n+1}(\tau)))\,d\tau\Bigg)\Bigg]\Bigg\|\,ds \\ &\le \alpha\,\|T(t+h)-T(t)\|\,(\|Eu_0\|+\|Eg(u)\|) + \alpha M h\,(N_0+Na) \\ &+ \alpha MN\Bigg[\int_0^t (|u(\sigma_1(s+h))-u(\sigma_1(s))| + \cdots + |u(\sigma_n(s+h))-u(\sigma_n(s))|)\,ds \\ &+ a\left|\int_0^{s+h} k(s+h,\tau)h(\tau,u(\sigma_{n+1}(\tau)))\,d\tau - \int_0^s k(s,\tau)h(\tau,u(\sigma_{n+1}(\tau)))\,d\tau\right|\Bigg] \\ &\le \alpha h M\,\|AE^{-1}\|\,(\|Eu_0\|+G) + \alpha M h\,(N_0+Na) \\ &+ \alpha MNah\,(L^*a+L) + \alpha MN\int_0^t n\gamma|u(s+h)-u(s)|\,ds \\ &\le Qh + P\int_0^t |u(s+h)-u(s)|\,ds, \end{aligned}$$

where

$$\begin{aligned} Q &= \alpha M\,\|AE^{-1}\|\,(\|Eu_0\|+G) + \alpha M\,(N_0+Na) + \alpha MNa\,(L^*a+L), \\ P &= \alpha\gamma MNn. \end{aligned}$$

Using Gronwall's inequality, we get

$$\|u(t+h)-u(t)\| \le hQe^{Pa}, \qquad t \in I.$$

Therefore, u is Lipschitz continuous on I.

The Lipschitz continuity of u on I, combined with (x), gives that

$$t \to f\left(t, u(\sigma_1(t)),\dots,u(\sigma_n(t)), \int_0^t k(t,s)h(s,u(\sigma_{n+1}(s)))\,ds\right)$$

is Lipschitz continuous on I. Using Corollary 2.11 in Section 4.2 in [7] and the definition of strong solution, we observe that the linear Cauchy problem

$$(Ev(t))' + Av(t) = f\left(t, u(\sigma_1(t)), \dots, u(\sigma_n(t)), \int_0^t k(t,s)h(s, u(\sigma_{n+1}(s)))\, ds\right),$$
$$v(0) = u_0 - g(u),$$

has a unique strong solution v satisfying the equation

$$\begin{aligned} v(t) &= E^{-1}T(t)Eu_0 - E^{-1}T(t)Eg(u) \\ &\quad + \int_0^t E^{-1}T(t-s)f\left(s, u(\sigma_1(s)), \dots, u(\sigma_n(t)), \int_0^s k(s,\tau)h(\tau, u(\sigma_{n+1}(\tau)))\, d\tau\right) ds \\ &= u(t). \end{aligned}$$

Consequently, u is the unique strong solution of problem (1),(2) on I.

4. EXAMPLE

Consider the partial integrodifferential equation of the form

$$\frac{\partial}{\partial t}[z(t,x) - z_{xx}(t,x)] = \frac{\partial^2}{\partial x^2}z(t,x) + \frac{1}{(1+t)(1+t^2)}\left[z(\sin t, x) + \sin z(t,x)\int_0^t e^{-z(\sin s, x)}\, ds\right],$$
$$0 \le x \le \pi, \qquad t \in J = [0,1], \tag{5}$$
$$z(0,t) = z(\pi,t) = 0, \quad t \in J, \qquad z(x,0) + g(z) = z_0(x) \in C^2[0,\pi],$$

where $g(z) = \int_0^a z(s,x)\, ds$, $a < 1$, satisfies the Lipchitz condition.

Take $X = Y = L^2[0,\pi]$ and let

$$\int_0^t k(t,s)h(s, z(\sigma(s))(x)\, ds = \frac{\sin z(t,x)}{(1+t)(1+t^2)}\int_0^t e^{-z(\sin s,x)}\, ds,$$
$$f(t, z(\sigma(t)) \int_0^t k(t,s)h(s, z(\sigma(s))\, ds(x) = \frac{1}{(1+t)(1+t^2)},$$
$$\left[z(\sin t, x) + \sin z(t,x)\int_0^t e^{-z(\sin s,x)}\, ds\right].$$

Define the operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ by

$$Aw = w'' \qquad \text{and} \qquad Ew = w - w'',$$

where each domain $D(A)$ and $D(E)$ is given by

$$\{w \in X : w, w' \text{ are absolutely continuous, } w'' \in X,\ w(0) = w(\pi) = 0\}.$$

Then A and E can be written, respectively, as

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n)\, w_n, \qquad w \in D(A),$$
$$Ew = \sum_{n=1}^{\infty} (1+n^2)(w, w_n)\, w_n, \qquad w \in D(E),$$

where $w_n(x) = \sqrt{2/\pi}\sin nx$, $n = 1, 2, \dots$, is the orthogonal set of vectors of A. Furthermore, for $w \in X$, we have

$$E^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1+n^2} (w, w_n) w_n,$$
$$AE^{-1}w = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2} (w, w_n) w_n,$$
$$T(t)w = \sum_{n=1}^{\infty} \exp\left(\frac{-n^2 t}{1+n^2}\right) (w, w_n) w_n.$$

It is easy to see that AE^{-1} generates a strongly continuous semigroup $T(t)$ on Y and $T(t)$ is compact such that $|T(t)| \le e^{-t}$ for each $t > 0$. Further, we have

$$\left|\frac{1}{(1+t)(1+t^2)}\left[z(\sin t, x) + \sin z(t,x)\int_0^t e^{-z(\sin s, x)}\, ds\right]\right| \le \frac{1}{(1+t^2)}|z|.$$

Moreover, all the other conditions stated in Theorem 3.1 are satisfied. Hence, equation (5) has a mild solution on $[0, 1]$.

REFERENCES

1. D. Bahuguna, Quasilinear integrodifferential equations in Banach spaces, *Nonlinear Analysis, Theory, Methods and Applications* **24**, 175–183 (1995).
2. M.B. Dhakne and B.G. Pachpatte, On a general class of abstract functional integrodifferential equations, *Indian Journal of Pure and Applied Mathematics* **19**, 728–746 (1988).
3. G. Gripenberg, Global existence of solutions of Volterra integrodifferential equations of parabolic type, *Journal of Differential Equations* **102**, 382–390 (1993).
4. M.L. Heard, A class of hyperbolic Volterra integrodifferential equations, *Nonlinear Analysis, Theory, Methods and Applications* **8**, 79–93 (1984).
5. M.L. Heard and S.M. Rankin III, A semilinear parabolic Volterra integrodifferential equation, *Journal of Differential Equations* **71**, 201–233, (1988).
6. M.L. Heard and S.M. Rankin III, Nonlinear Volterra integrodifferential equations, *Journal of Mathematical Analysis and Applications* **188**, 569–589, (1994).
7. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, (1983).
8. L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *Journal of Mathematical Analysis and Applications* **162**, 494–505 (1991).
9. L. Byszewski, Existence, uniqueness and asymptotic stability of weak and strong solutions of functional-differential abstract nonlocal Cauchy problems, *Dynamic Systems and Applications* **5**, 595–606 (1996).
10. L. Byszewski, On weak solutions of functional-differential abstract nonlocal Cauchy problems, *Annales Polonici Mathematici* **2**, 163–170, (1997).
11. L. Byszewski, On a mild solution of a semilinear functional-differential evolution nonlocal problems, *Journal of Applied Mathematics and Stochastic Analysis* **10**, 265–271 (1997).
12. L. Byszewski and H. Akca, Existence of solutions of a semilinear functional differential evolution nonlocal problem, *Nonlinear Analysis, Theory, Methods and Applications* **34**, 65–712 (1998).
13. K. Balachandran, Existence and uniqueness of mild and strong solutions of nonlinear integrodifferential equation with nonlocal condition, *Differential Equations and Dynamical Systems* **6**, 159–165 (1998).
14. K. Balachandran and S. Ilamaran, Existence and uniqueness of mild and strong solutions of a Volterra integrodifferential equation with nonlocal conditions, *Tamkang Journal of Mathematics* **28**, 93–100 (1997).
15. J.P. Dauer and K. Balachandran, Existence of solutions for an integrodifferential equation with nonlocal condition in Banach spaces, *Libertas Mathematica* **16**, 133–143 (1996).
16. S. Ilamaran and K. Balachandran, Existence and uniqueness of mild and strong solutions of an integrodifferential equation with nonlocal condition, *Libertas Mathematica* **15**, 119–127 (1995).
17. Y. Lin and J.H. Liu, Semilinear integrodifferential equations with nonlocal Cauchy problem, *Nonlinear Analysis, Theory, Method and Applications* **26**, 1023–1033 (1996).
18. S.K. Ntouyas and P.Ch. Tsamatos, Global existence for semilinear evolution equations with nonlocal conditions, *Journal of Mathematical Analysis and Applications* **210**, 679–987 (1997).
19. S.K. Ntouyas and P.Ch. Tsamatos, Global existence for semilinear evolution integrodifferential equations with delay and nonlocal conditions, *Applicable Analysis* **64**, 99–105 (1997).

20. K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *Journal of Mathematical Analysis and Applications* **179**, 630–637 (1993).
21. H. Brill, A semilinear Sobolev evolution equation in a Banach space, *Journal of Differential Equations* **24**, 412–425 (1977).
22. J.E. Lagnese, Exponential stability of solutions of differential equation of Sobolev type, *SIAM Journal of Mathematical Analysis* **3**, 625–636 (1972).
23. J.H. Lightboure III and S.M. Rankin III, A partial functional differential equation of Sobolev Type, *Journal of Mathematical Analysis and Applications* **93**, 328–337 (1983).
24. R.E. Showalter, Existence and representation theorems for a semilinear Sobolev equation in Banach space, *SIAM Journal of Mathematical Analysis* **3**, 527–543 (1972).
25. R.E. Showalter, A nonlinear parabolic-Sobolev equation, *Journal of Mathematical Analysis and Applications* **50**, 183–190 (1975).
26. K. Balachandran, D.G. Park and Y.C. Kwun, Nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces, *Communications of the Korean Mathematical Society* **14**, 223–231 (1999).
27. H. Schaefer, Uber die methode der *a priori* schranken, *Mathematische Annalem* **129**, 415–416 (1955).