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Fuzzy n-ary polygroups related to fuzzy points

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ABSTRACT

Recently, fuzzy n-ary sub-polygroups were introduced and studied by Davvaz, Corsini and Leoreanu-Fotea [B. Davvaz, P. Corsini, V. Leoreanu-Fotea, Fuzzy n-ary sub-polygroups, Comput. Math. Appl. 57 (2008) 141–152]. Now, in this paper, the concept of $(\in, \in \vee q)$-fuzzy n-ary sub-polygroups, $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy n-ary sub-polygroups and fuzzy n-ary sub-polygroup with thresholds of an n-ary polygroup are introduced and some characterizations are described. Also, we give the definition of implication-based fuzzy n-ary sub-polygroups in an n-ary polygroup, in particular, the implication operators in Łukasiewicz system of continuous-valued logic are discussed.

1. Introduction

Hypergroup which is based on the notion of hyperoperation was introduced by Marty in [1] and studied extensively by many mathematicians. Hypergroup theory both extends some well-known group results and introduced new topics, leading thus to a wide variety of applications, as well as to a broadening of the investigation fields, see [2–5]. A recent book [3] contains a wealth of applications. There are applications in the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probability. Applications of hypergroups have mainly appeared in special subclasses. For example, polygroups which form an important subclass of hypergroups are studied by Comer [6–8]. Quasi-canonical hypergroups (called "polygroups" by Comer) were introduced for the first time in [9], as a generalization of canonical hypergroups, introduced in [10]. Zahedi, Bolurian and Hasankhani in 1995 [11] introduced the concept of a fuzzy subpolygroup of a polygroup. Davvaz and Poursalavati in 1999 [12] introduced matrix representations of polygroups over hyperrings. Also, they introduced the notion of a polygroup hyperring generalizing the notion of a group ring. In [13], Davvaz considered the factor polygroup and interpreted the lower and upper approximations as subsets of the factor polygroup, and then he introduced the concept of a factor rough subpolygroup. Using the concept of a fuzzy set, he introduced and discussed the concept of a fuzzy rough polygroup in [14]. Also, applications of the γ^*-relation to polygroups was given in [15].

The notion of an n-ary group is a natural generalization of the notion of a group and has many applications in different branches. The idea of investigations of such groups seems to be going back to E. Kasner's lecture at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904 [16]. But the first paper concerning the theory of n-ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (see [17]).

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n-ary generalizations of algebraic structures is the most natural way for further development and a deeper understanding of their fundamental properties. Ameri and Zahedi in [18] studied algebraic hypersystems. In [19], Davvaz and Vougiouklis introduced the concept of n-ary hypergroups as a generalization of hypergroups in the sense of Marty. Leoreanu-Fotea and Davvaz in [20] introduced and studied the notion of a partial n-hypergroupoid, associated with a binary relation. Some important results, concerning Rosenberg partial hypergroupoids, induced by relations, are generalized to the case of n-hypergroupoids. Ghadiri and Waphare [21] defined n-ary polygroups, as a subclass of n-ary hypergroups and as a generalization of polygroups.

After the introduction of fuzzy sets by Zadeh [22], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroups was defined by Rosenfeld [23] and its structure was investigated. A new type of fuzzy subgroup (viz, $(\in, \in \vee q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [24,25] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. This concept was studied further in [26–31]. Also, a generalization of Rosenfeld's fuzzy subgroup, and Bhakat and Das's fuzzy subgroup is given in [32].

Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now used in the real world, both from a theoretical point of view and for their many applications. The relations between fuzzy sets and hyperstructures have been already considered by Corsini, Davvaz, Leoreanu, Zahedi and others, for instance see [33–38]. In [35], Davvaz applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined fuzzy sub-hypergroup (resp. H_v-subgroup) of a hypergroup (resp. H_v-group) which is a generalization of the concept of Rosenfeld's fuzzy subgroup of a group. In [36], Davvaz and Corsini introduced the notion of a fuzzy and anti fuzzy n-ary subhypergroup of an n-ary hypergroup and to extend the fuzzy results of fundamental equivalence relations to n-ary hypergroups. In [37], the notion of a fuzzy n-ary subpolygroup of an n-ary polygroup is defined. In [28], using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of $(\in, \in \vee q)$-fuzzy sub-hyperquasigroups is introduced. Recently, the characterization of generalized fuzzy bi-ideals in semigroups were introduced by Kazancıand Yamak [39].

This paper is organized as follows. In Section 2, we first recall some basic definitions and results of n-ary hyperoperations. Since the concepts of $(\in, \in \vee q)$-fuzzy n-ary sub-polygroups are important and useful generalizations of ordinary fuzzy n-ary sub-polygroups of n-ary polygroup, some fundamental aspects of $(\in, \in \vee q)$-fuzzy n-ary sub-polygroups of n-ary polygroups will be discussed in Section 3. Finally, in Section 4, we consider the concepts of implication-based fuzzy n-ary sub-polygroups and some interesting properties are investigated.

2. Preliminaries

We start by giving some known and useful definitions and notations. Let H be a non-empty set and f be a mapping $f : H \times H \to P^*(H)$, where $P^*(H)$ is the set of all non-empty subsets of H. Then f is called a *binary hyperoperation* on H. We denote by H^n the Cartesian product $H \times \cdots \times H$, where H appears n times. An element of H^n will be denoted by $(x_1, \ldots, x_n)$, where $x_i \in H$ for any $1 \le i \le n$. In general, a mapping $f : H^n \to P^*(H)$ is called an *n-ary hyperoperation* and n is called the *arity* of the hyperoperation f. Let f be an n-ary hyperoperation on H and $A_1, \ldots, A_n$ be nonempty subsets of H. We define

$$f(A_1, \ldots, A_n) = \bigcup\{f(x_1, \ldots, x_n) \mid x_i \in A_i,\ i = 1, \ldots, n\}.$$

We shall use the following abbreviated notation: The sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty set. Thus

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$.

Also, $f(a_1^i, x*)$ mean $f(a_1^i, \underbrace{x, \ldots, x}_{n-i})$ for $a_1, \ldots, a_i, x \in P$ and $1 \le i \le n-1$.

A non-empty set H with an n-ary hyperoperation $f : H^n \to P^*(H)$ will be called an *n-ary hypergroupoid* and will be denoted by (H, f). An n-ary hypergroupoid (H, f) will be called an *n-ary semihypergroup* if and only if the following associative axiom holds: $f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{i-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$ for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$. If for all $(a_1, a_2, \ldots, a_n) \in H^n$, the set $f(a_1, a_2, \ldots, a_n)$ is singleton, then f is called an *n-ary operation* and (H, f) is called an *n-ary groupoid* (resp. *n-ary semigroup*).

Definition 2.1 (*[36]*). An n-ary semihypergroup (H, f) in which the equation

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$$

has a solution $x_i \in H$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in H$ and $1 \le i \le n$, is called an *n-ary hypergroup*. If f is n-ary operation then the equation becomes

$$b = f(a_1^{i-1}, x_i, a_{i+1}^n).$$

In this case (H, f) is an *n-ary group*.

Let (H, f) be an n-ary hypergroup and B be a non-empty subset of H. Then B is an *n-ary sub-hypergroup* of H if the following conditions hold:

(i) B is closed under the n-ary hyperoperation f, i.e., for every $(x_1^n) \in B^n$ we have $f(x_1^n) \subseteq B$.
(ii) Equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has a solution $x_i \in B$ for every $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n, b \in B$ and $1 \leq i \leq n$.

Definition 2.2 (*[21]*). An *n-ary polygroup* is a multivalued system $\langle P, f, e, {}^{-1}\rangle$, where $e \in P$, ${}^{-1}$ is a unitary operation on P, f is an n-ary hyperoperation on P, and the following axioms hold for all $i, j \in \{1, \ldots, n\}$ and $x_1, \ldots, x_{2n-1}, x \in P$:

(i) $f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{i-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$,
(ii) e is a unique element such that $f\left(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i}\right) = x$,
(iii) $x \in f(x_1^n)$ implies $x_i \in f\left(x_{i-1}^{-1}, \ldots, x_1^{-1}, x, x_n^{-1}, \ldots, x_{i+1}^{-1}\right)$.

It is clear that every 2-ary polygroup is a polygroup. Every n-ary polygroup is an n-ary hypergroup.

A non-empty subset S of an n-ary polygroup P is an n-ary sub-polygroup if $\langle S, f, e, {}^{-1}\rangle$ is an n-ary polygroup, i.e., if it is closed under the hyperoperation f, $e \in S$ and $x \in S$ implies that $x^{-1} \in S$. The following elementary facts about n-ary polygroups follow easily from the axioms,

(1) $e \in f\left(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{j-1-i}, x^{-1}, \underbrace{e, \ldots, e}_{n-j}\right)$ where $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$,
(2) $(x^{-1})^{-1} = x$,
(3) $f(x_1^i, e*) = f\left(x_1^{i-1}, \underbrace{e, \ldots, e}_{k}, x_i, e*\right)$ for $0 \neq k < n - i$,
(4) $f(x_1^n)^{-1} = f(x_n^{-1}, \ldots, x_1^{-1})$.

Now, we recall some structures commonly used in fuzzy set. In 1965, Zadeh [22] introduced the notion of a *fuzzy subset* A of a non-empty set X as a membership function $\mu_A : X \to [0, 1]$ which associates with each point $x \in X$ its "degree of membership" $\mu_A(x) \in [0, 1]$. The *complement* of A, denoted by A^c, is the fuzzy subset given by $\mu_{A^c}(x) = 1 - \mu_A(x)$ for all $x \in X$. In 1971, Rosenfeld [23] applied the concept of fuzzy sets to the theory of groups and studied fuzzy subgroups of a group. Davvaz [35] applied fuzzy sets to the theory of algebraic hyperstructures and defined the concept of fuzzy sub-hypergroups. Davvaz and Corsini [36] introduced the notion of a fuzzy n-ary sub-hypergroup of an n-ary hypergroup and then Davvaz and et al. [37] defined the notion of a fuzzy n-ary sub-polygroup of an n-ary polygroup. We shall use the following abbreviated notation: the sequence $\mu(a_i), \mu(a_{i+1}), \ldots, \mu(a_j)$ will be denoted by $\mu_{a_i}^{a_j}$.

Definition 2.3 (*[36]*). Let(H, f) be an n-ary hypergroup and μ be a fuzzy subset of H. Then μ is called a fuzzy *n-ary sub-hypergroup* of H if the following axioms hold:

(i) $\min\{\mu_{x_1}^{x_n}\} \leq \bigwedge_{z \in f(x_1^n)}\{\mu(z)\}$ for all $x_1^n \in H$,
(ii) for all $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, there exists $x_i \in H$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ and $\min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b)\} \leq \mu(x_i)$.

Definition 2.4 (*[37]*). Let P be an n-ary polygroup and μ be a fuzzy subset of P. Then μ is called a fuzzy *n-ary sub-polygroup* of P if the following axioms hold:

(i) $\min\{\mu_{x_1}^{x_n}\} \leq \bigwedge_{z \in f(x_1^n)}\{\mu(z)\}$ for all $x_1^n \in P$,
(ii) $\mu(x) \leq \mu(x^{-1})$ for all $x \in P$.

Let P be an n-ary polygroup and $B \subseteq P$. Then the characteristic function χ_B is a fuzzy n-ary sub-polygroup of P if and only if B is an n-ary sub-polygroup of P.

For any fuzzy subset A of a non-empty set X and any $t \in (0, 1]$, we define the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$, which is called a *$t$-level cut* of μ.

Theorem 2.5 (*[37]*). *Let P be an n-ary polygroup and μ a fuzzy subset of P. Then μ is a fuzzy n-ary sub-polygroup of P if and only if for every $t \in (0, 1]$, μ_t $(\neq \emptyset)$ is an n-ary sub-polygroup of P.*

3. Fuzzy *n*-ary sub-polygroups with thresholds

A fuzzy subset μ of P of the form

$$\mu(y) = \begin{cases} t \neq 0 & \text{if } y = x, \\ 0 & \text{if } y \neq x \end{cases}$$

is called a *fuzzy point* with support x and value t and is denoted x_t [40]. A fuzzy point x_t is said to be *belong* to (resp. be *quasi-coincident* with) a fuzz set μ, written as $x_t \in \mu$ (*resp.* $x_t q\mu$) if $\mu(x) \geq t$ (*resp.* $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q\mu$, then we write $x_t \in \vee q\mu$. The symbol $\overline{\in \vee q}$ means neither $\in$ nor q hold. Based on [25], we can extend the concept of $(\in, \in \vee q)$-fuzzy subgroups to the concept of $(\in, \in \vee q)$-fuzzy n-ary sub-polygroups in the following way:

Definition 3.1. A fuzzy subset μ of an n-ary polygroup P is called an $(\in, \in \vee q)$-*fuzzy n-ary sub-polygroup* of P if for all $t, t_1^n \in (0, 1]$ and $x, x_1^n \in P$,

(i) $(x_1)_{t_1}, (x_2)_{t_2}, \ldots, (x_n)_{t_n} \in \mu$ implies $z_{t_1 \wedge t_2 \wedge \cdots \wedge t_n} \in \vee q\mu$ for all $z \in f(x_1^n)$,
(ii) $x_t \in \mu$ implies $x_t^{-1} \in \mu$.

Proposition 3.2. *Conditions* (i) *and* (ii) *in Definition* 3.1 *are equivalent,respectively, to the following conditions.*

(1) $\min\{\min\{\mu_{x_1}^{x_n}\}, 0.5\} \leq \bigwedge_{z \in f(x_1^n)} \mu(z)$ *for all* $x_1^n \in P$,
(2) $\mu(x) \wedge 0.5 \leq \mu(x^{-1})$ *for all* $x \in P$.

Proof. (i $\Rightarrow$ 1): Suppose that $x_1^n \in P$. We consider the following cases:

(a) $\min\{\mu_{x_1}^{x_n}\} < 0.5$
(b) $\min\{\mu_{x_1}^{x_n}\} \geq 0.5$.

Case a: Assume that there exists $z \in f(x_1^n)$ such that $\mu(z) < \min\{\mu_{x_1}^{x_n}\} \wedge 0.5$, which implies that $\mu(z) < \min\{\mu_{x_1}^{x_n}\}$. Choose t such that $\mu(z) < t < \min\{\mu_{x_1}^{x_n}\}$. Then $(x_1)_t, (x_2)_t, \ldots, (x_n)_t \in \mu$, but $z_t \overline{\in \vee q\mu}$, which contradicts (i).

Case b: Assume that $\mu(z) < 0.5$ for some $z \in f(x_1^n)$. Then

$$(x_1)_{0.5}, (x_2)_{0.5}, \ldots, (x_n)_{0.5} \in \mu,$$

but $z_{0.5} \overline{\in \vee q\mu}$, which is a contradiction. Therefore (1) holds.

(ii $\Rightarrow$ 2): Suppose that $x \in P$. We consider the following cases:

(a) $\mu(x) < 0.5$,
(b) $\mu(x) \geq 0.5$.

Case a: Assume that $\mu(x) = t < 0.5$ and $\mu(x^{-1}) = r < \mu(x)$. Choose s such that $r < s < t$ and $r + s < 1$. Then $x_s \in \mu$, but $(x^{-1})_s \overline{\in \vee q\mu}$ which contradicts (ii). So $\mu(x^{-1}) \geq \mu(x) = \mu(x) \wedge 0.5$.

Case b: Let $\mu(x) \geq 0.5$. If $\mu(x^{-1}) < \mu(x) \wedge 0.5$, then $x_{0.5} \in \mu$, but $(x^{-1})_{0.5} \overline{\in \vee q\mu}$, which contradicts (ii). Hence the conditions (2) holds.

(1 $\Rightarrow$ i): Let $(x_1)_{t_1}, (x_2)_{t_2}, \ldots, (x_n)_{t_n} \in \mu$. Then

$$\mu(x_1) \geq t_1, \ \mu(x_2) \geq t_2, \ \ldots, \mu(x_n) \geq t_n.$$

For every $z \in f(x_1^n)$, we have

$$\mu(z) \geq \min\{\mu_{x_1}^{x_n}\} \wedge 0.5 \geq t_1 \wedge t_2 \wedge \cdots \wedge t_n \wedge 0.5.$$

If $t_1 \wedge t_2 \wedge \cdots \wedge t_n > 0.5$, then $\mu(z) \geq 0.5$ which implies that $\mu(z) + (t_1 \wedge t_2 \wedge \cdots \wedge t_n) > 1$. If $t_1 \wedge t_2 \wedge \cdots \wedge t_n \leq 0.5$, then $\mu(z) \geq t_1 \wedge t_2 \wedge \cdots \wedge t_n$. Therefore $z_{t_1 \wedge t_2 \wedge \cdots \wedge t_n} \in \vee q\mu$ for all $z \in f(x_1^n)$.

(2 $\Rightarrow$ ii): Let $x_t \in \mu$, then $\mu(x) \geq t$. Now, we have $\mu(x^{-1}) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5$, which implies that $\mu(x^{-1}) \geq t$ or $\mu(x^{-1}) \geq 0.5$ according to $t \leq 0.5$ or $t > 0.5$. Therefore $x_t^{-1} \in \vee q\mu$. Hence (ii) holds. □

By Definition 3.1 and Proposition 3.2, we obtain immediately:

Corollary 3.3. *A fuzzy subset μ of an n-ary polygroup P is an $(\in, \in \vee q)$-fuzzy n-ary sub-polygroup of P if and only if the conditions* (1) *and* (2) *in Proposition* 3.2 *hold.*

We notice that if μ is a fuzzy n-ary sub-polygroup of P according to Definition 2.4, *then μ is an $(\in, \in \vee q)$-fuzzy n-ary sub-polygroup of P according to Definition* 3.1. *However, as the following example shows, the converse is not true.*

Example 3.4. Let $P = \{e, x, y\}$ be a set with a 3-ary hyperoperation f as follows:

$f(e,e,e) = e$	$f(x,x,e) = \{e,y\}$	$f(y,e,e) = y$
$f(e,e,x) = x$	$f(x,x,x) = \{x,y\}$	$f(y,e,x) = \{x,y\}$
$f(e,e,y) = y$	$f(x,x,y) = P$	$f(y,e,y) = \{e,x\}$
$f(e,x,e) = x$	$f(x,e,e) = x$	$f(y,x,e) = \{x,y\}$
$f(e,x,x) = \{e,y\}$	$f(x,e,x) = \{e,y\}$	$f(y,x,x) = P$
$f(e,x,y) = \{x,y\}$	$f(x,e,y) = \{x,y\}$	$f(y,x,y) = P$
$f(e,y,e) = y$	$f(x,y,e) = \{x,y\}$	$f(y,y,e) = \{e,x\}$
$f(e,y,x) = \{x,y\}$	$f(x,y,x) = P$	$f(y,y,x) = P$
$f(e,y,y) = \{e,x\}$	$f(x,y,y) = P$	$f(y,y,y) = \{x,y\}$.

For every $x_i \in P$ (i = 1, …, 5), we have

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5)),$$

i.e., f is associative. We suppose that $^{-1} : P \to P$ is the identity function on P. We have $x^{-1} = x, y^{-1} = y, e^{-1} = e$, and it is easy to see that $t \in f(x_1, x_2, x_3)$ implies $x_1 \in f(t, x_3^{-1}, x_2^{-1}), x_2 \in f(x_1^{-1}, t, x_3^{-1}), x_3 \in f(x_2^{-1}, x_1^{-1}, t)$ for every $x_i \in P, i = 1, 2, 3$. Therefore $\langle P, f, e, ^{-1}\rangle$ is a 3-ary polygroup.

Let $\mu : P \to [0, 1]$ be defined by

$$\mu(a) = \begin{cases} 0.5 & \text{if } a = e, \\ \alpha & \text{if } a = x \quad or \quad a = y, \end{cases}$$

where $\alpha \in [0, 1], 0.5 < \alpha$. Then μ is an $(\in, \in \vee q)$-fuzzy 3-ary sub-polygroup of P. But it is not a fuzzy 3-ary sub-polygroup of P. □

Theorem 3.5. *A non-empty subset I of P is an n-ary sub-polygroup of P if and only if χ_I is an $(\in, \in \vee q)$-fuzzy n-ary sub-polygroup of P.*

Proof. Assume that I is an n-ary sub-polygroup of P. Then χ_I is a fuzzy n-ary sub-polygroup in the sense of Definition 2.4 and so it is an $(\in, \in \vee q)$-fuzzy n-ary sub-polygroup.

Conversely, assume that χ_I is an $(\in, \in \vee q)$-fuzzy n-ary sub-polygroup of P. Then for every $x_1^n \in I$, we have

$$\bigwedge_{z \in f(x_1^n)} \chi_I(z) \geq \min\{\chi_{I_{x_1}}^{x_n}\} \wedge 0.5 = 0.5$$

and so $f(x_1^n) \subseteq I$. Now, let $x \in I$. $\chi_I(x^{-1}) \geq \chi_I(x) \wedge 0.5 = 0.5$ which implies that $x^{-1} \in I$. Therefore I is an n-ary sub-polygroup of P. □

Definition 3.6. A fuzzy subset μ of an n-ary polygroup P is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy n-ary sub-polygroup* of P if for all $t, t_1^n \in (0, 1]$ and $x, x_1^n \in P$,

(i) $z_{t_1 \wedge t_2 \wedge \cdots \wedge t_n} \overline{\in} \mu$ implies there exists $1 \leq i \leq n$ such that $(x_i)_{t_i} \overline{\in} \vee \overline{q} \mu$ for all $z \in f(x_1^n)$,
(ii) $x_t^{-1} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$.

Theorem 3.7. *A fuzzy subset μ of an n-ary polygroup P is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy n-ary sub-polygroup of P iff for all $x, x_1^n \in P$, it satisfies:*

(1) $\min\{\mu_{x_1}^{x_n}\} \leq \bigwedge_{z \in f(x_1^n)} (\mu(z) \vee 0.5)$,
(2) $\mu(x) \leq \mu(x^{-1}) \vee 0.5$.

Proof. We only prove (i)⟺(1). The proofs of (ii)⟺(2) is similar.

(i)⟹ (1): If there exist $x_1^n, z \in P$ with $z \in f(x_1^n)$ such that $\mu(z) \vee 0.5 < t = \min\{\mu_{x_1}^{x_n}\}$, then $t \in (0.5, 1]$, $z_t \overline{\in} \mu$ and $(x_i)_t \in \mu$. By (i), we have $(x_i)_t \overline{q} \mu$. Then $t \leq \mu(x_i)$ and $t + \mu(x_i) \leq 1$. Thus $t \leq 0.5$.This is a a contradiction with $t > 0.5$. So $\mu(z) \vee 0.5 \geq \min\{\mu_{x_1}^{x_n}\}$ for all $z \in f(x_1^n)$.

(1)⟹(i): Let $x_1^n \in P$ such that $z_{t_1 \wedge t_2 \wedge \cdots \wedge t_n} \overline{\in} \mu$ for some $z \in f(x_1^n)$. Then $\mu(z) < \min\{t_1, \ldots t_n\}$. Then we have the following.

(a) If $\min\{\mu_{x_1}^{x_n}\} \leq \bigwedge_{z \in f(x_1^n)} \mu(z)$, then $\min\{\mu_{x_1}^{x_n}\} < \min\{t_1, \ldots, t_n\}$, and consequently there exists $1 \leq i \leq n$ such that $\mu(x_i) < t_i$. It follows that $(x_i)_{t_i} \overline{\in} \mu$ which implies that $(x_i)_{t_i} \overline{\in} \vee \overline{q} \mu$.
(b) If $\min\{\mu_{x_1}^{x_n}\} > \bigwedge_{z \in f(x_1^n)} \mu(z)$, then by (1), we have $0.5 \geq \min\{\mu_{x_1}^{x_n}\}$. Hence there exist $1 \leq i \leq n$ such that $\mu(x_i) \leq 0.5$ Putting $(x_i)_{t_i} \in \mu$, then $t_i \leq \mu(x_i) \leq 0.5$, $1 \leq i \leq n$. it follows that $(x_i)_{t_i} \overline{\in} \mu$ and thus for $1 \leq i \leq n$, $(x_i)_{t_i} \overline{\in} \vee \overline{q} \mu$. So (i) holds. □

In [32], Yuan, Zhang and Ren gave the definition of fuzzy subgroup with thresholds which is a generalization Rosenfeld's fuzzy subgroup, and Bhakat and Das's fuzzy subgroup. Based on [32], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy n-ary sub-polygroup with thresholds in the following way:

Definition 3.8. Let $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$. Let μ be a fuzzy subset of an n-ary polygroup P. Then μ is called a fuzzy n-ary sub-polygroup with thresholds of P, if for all $x, x_1^n \in P$

(i) $\min\{\mu_{x_1}^{x_n}\} \wedge \beta \leq \bigwedge_{z \in f(x_1^n)} (\mu(z) \vee \alpha)$,
(ii) $\mu(x) \wedge \beta \leq \mu(x^{-1}) \vee \alpha$.

Remark 3.9. If μ is a fuzzy n-ary sub-polygroup with thresholds $(\alpha, \beta]$, then by Definition 3.8, we can conclude that

(1) μ is ordinary fuzzy n-ary sub-polygroup when $\alpha = 0, \beta = 1$ (Definition 2.4);
(2) μ is an $(\in, \in \vee q)$-fuzzy n-ary sub-polygroup when $\alpha = 0, \beta = 0.5$ (Proposition 3.2);
(3) μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy n-ary sub-polygroup when $\alpha = 0.5, \beta = 1$ (Theorem 3.7).

Now, we give a characterization of fuzzy n-ary sub-polygroups with thresholds by using their level sub-polygroups.

Theorem 3.10. *A fuzzy subset μ of an n-ary polygroup P is a fuzzy n-ary sub-polygroup with thresholds (α, β) of P if and only if $\mu_t(\neq \emptyset)$ is an n-ary sub-polygroup of P for all $t \in (\alpha, \beta]$.*

Proof. Let μ be a fuzzy n-ary sub-polygroup with thresholds of P and $t \in (\alpha, \beta]$. Let $x_1^n \in \mu_t$. Then $\mu(x_1) \geq t$, $\mu(x_2) \geq t$, $\ldots$, $\mu(x_n) \geq t$. Now

$$\alpha < t = t \wedge \beta \leq \min\{\mu_{x_1}^{x_n}\} \wedge \beta \leq \bigwedge_{z \in f(x_1^n)} (\mu(z) \vee \alpha).$$

So for every $z \in f(x_1^n)$ we have $\mu(z) \vee \alpha \geq t > \alpha$ which implies that $\mu(z) \geq t$ and $z \in \mu_t$. Hence $f(x_1^n) \subseteq \mu_t$.

Now, let $x \in \mu_t$, then $\mu(x) \geq t$ and so

$$\alpha < t = t \wedge \beta \leq \mu(x) \wedge \beta \leq \mu(x^{-1}) \vee \alpha.$$

which implies that $\mu(x^{-1}) \geq t$ and so $x^{-1} \in \mu_t$. This prove that μ_t is an n-ary sub-polygroup of P for all $t \in (\alpha, \beta]$.

Conversely, let μ be a fuzzy subset of P such that $\mu_t(\neq \emptyset)$ is an n-ary sub-polygroup of P for all $\alpha < t \leq \beta$. If there exist $x_1^n, z \in P$ with $z \in f(x_1^n)$ such that $\mu(z) \vee \alpha < \min\{\mu_{x_1}^{x_n}\} \wedge \beta = t$, then $t \in (\alpha, \beta]$, $\mu(z) < t$, $x_1^n \in \mu_t$. Since μ_t is an n-ary sub-polygroup of P so $f(x_1^n) \subseteq \mu_t$. Hence $\mu(z) \geq t$ for all $z \in f(x_1^n)$. This is a contradiction with $\mu(z) < t$. Therefore $\min\{\mu_{x_1}^{x_n}\} \wedge \beta \leq \mu(z) \vee \alpha$ for all $x_1^n, z \in P$ which implies that

$$\min\{\mu_{x_1}^{x_n}\} \wedge \beta \leq \bigwedge_{z \in f(x_1^n)} (\mu(z) \vee \alpha)$$

for all $x_1^n \in P$. Hence the condition (1) of Definition 3.8 holds.

Now, assume that there exists $x_0 \in P$ such that $\mu(x_0^{-1}) \vee \alpha < \mu(x_0) \wedge \beta = t$. Then $t \in (\alpha, \beta]$, $x_0 \in \mu_t$ and $\mu(x_0^{-1}) < t$. Since μ_t is an n-ary sub-polygruop, $x_0^{-1} \in \mu_t$, we obtain $\mu(x_0^{-1}) \geq t$. This is a contradiction with $\mu(x_0^{-1}) < t$. Hence $\mu(x) \wedge \beta \leq \mu(x^{-1}) \vee \alpha$. Hence condition (2) of Definition 3.8 holds. □

By Theorem 3.10 and Remark 3.9, we have the following Corollary:

Corollary 3.11. *Let μ be a fuzzy subset of an n-ary polygroup P. Then*

(i) *μ is an $(\in, \in \vee q)$-fuzzy n-ary sub-polygroup of P if and only if the set $\mu_t(\neq \emptyset)$ is an n-ary sub-polygroup of P for all $t \in (0, 0.5]$.*
(ii) *μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy n-ary sub-polygroup of P if and only if the set $\mu_t(\neq \emptyset)$ is an n-ary sub-polygroup of P for all $t \in (0.5, 1]$.*

Remark 3.12. (1) By Definition 3.8, we can define other fuzzy n-ary sub-polygroup of P, such as n-ary sub-polygroup with thresholds (0.3, 0.9], with thresholds (0.4,0,7] of P, etc.
(2) However, the fuzzy n-ary sub-polygroup with thresholds of P may not be an ordinary fuzzy n-ary sub-polygroup, may not be an $(\in, \in \vee q)$-fuzzy n-ary sub-polygroup, and may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy n-ary sub-polygroup, respectively, as shown by the following example.

Example 3.13. Consider the 3-ary polygroup P as defined in Example 3.4. Let us define a fuzzy subset $\mu : P \to [0, 1]$ as follows:

$$\mu(x) = 0.3, \qquad \mu(e) = 0.6, \qquad \mu(y) = 0.7.$$

We have

$$\mu_t = \begin{cases} P & \text{if } 0 \leq t \leq 0.3, \\ \{e, y\} & \text{if } 0.3 < t \leq 0.6, \\ \{y\} & \text{if } 0.6 < t \leq 0.7, \\ \emptyset & \text{if } 0.7 < t \leq 1. \end{cases}$$

Then μ is a fuzzy 3-ary sub-polygroup with thresholds for $(\alpha = 0, \beta = 0.3)$. But μ

- is not a fuzzy 3-ary sub-polygroup,
- is not an $(\in, \in \vee q)$-fuzzy 3-ary sub-polygroup of P,
- is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy n-ary sub-polygroup of P.

4. Applications of fuzzy implications

Fuzzy logic is an extension of set theoretic variables in terms of the linguistic variable truth. Some operators, like $\wedge, \vee, \neg, \to$ in fuzzy logic are also defined by using truth tables, the extension principle can be applied to derive definitions of the operators.

In the fuzzy logic, truth value of fuzzy proposition P is denoted by $[P]$. In the following, we display the fuzzy logical and corresponding set-theoretical notions used in this paper:

$[x \in A] = A(x)$,

$[x \notin A] = 1 - A(x)$,

$[P \wedge Q] = \min\{[P], [Q]\}$,

$[P \rightarrow Q] = \min\{1, 1 - [P] + [Q]\}$,

$[\forall x P(x)] = \inf[P(x)]$,

$\models P$ if and only if $[P] = 1$.

A function $I : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called fuzzy implication if it is monotonic with respect to both variables (separately) and fulfils the binary implication truth table:

$$I(0, 0) = I(0, 1) = I(1, 1) = 1, \qquad I(1, 0) = 0.$$

By monotonicity

$$I(0, x) = I(x, 1) = 1 \quad \text{for all } x \in [0, 1],$$

where I is decreasing with respect to the first variable ($I(1, 0) < I(0, 0)$) and I is increasing with respect to the second variable ($I(1, 0) < I(1, 1)$).

Of course, various implication operators have been defined. We only show a selection of the most important multi-valued implications in the next table, α denotes the degree of truth (or degree of membership) of the premise, β the respective values for the consequence, and I the resulting degree of truth for the implication:

Name	Definition of implication operators
Early Zadeh	$I_m(\alpha, \beta) = \max\{1 - \alpha, \min\{\alpha, \beta\}\}$
Łukasiewicz	$I_a(\alpha, \beta) = \min\{1, 1 - \alpha + \beta\}$
Standard star(Gödel)	$I_g(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{if } \alpha > \beta \end{cases}$
Contraposition of Gödel	$I_{cg}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 1 - \alpha & \text{if } \alpha > \beta \end{cases}$
Gaines–Rescher	$I_{gr}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 0 & \text{if } \alpha > \beta \end{cases}$
Kleene–Dienes	$I_b(\alpha, \beta) = \max\{1 - \alpha, \beta\}$
Goguen	$I_{gg}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \dfrac{\beta}{\alpha} & \text{if } \alpha > \beta \end{cases}$

In the following definition, we consider the definition of implication operator in the Łukasiewicz system of continuous-valued logic.

Definition 4.1. A fuzzy subset μ of an n-ary polygroup P is called a *fuzzifying n-ary sub-polygroup* of P if and only if it satisfies:

(i) For any $x_1^n \in P$,

$$\models [[x_1 \in \mu] \wedge [x_2 \in \mu] \wedge \ldots \wedge [x_n \in \mu] \longrightarrow [\forall z \in f(x_1^n), z \in \mu]],$$

(ii) For any $x \in P$,

$$\models [[x \in \mu] \longrightarrow [x^{-1} \in \mu]].$$

Clearly Definition 4.1 is equivalent to Definition 3.1. Therefore, a fuzzyfying n-ary sub-polygroup is an ordinary fuzzy n-ary sub-polygroup.

In [40], the concept of t- tautology is used, i.e.,

$\models_t$ T if and only if $[T] \geq t$ for all valuations.

Now, we can extend the concept of implication-based fuzzy subgroup to the concept of implication-based fuzzy n-ary sub-polygroups in the following way.

Definition 4.2. A fuzzy subset μ of an n-ary polygroup P is called a *t-implication-based fuzzy n-ary sub-polygroup* of P with respect to implication $\longrightarrow$ if and only if satisfies:

(i) For any $x_1^n \in P$

$$\models_t [[x_1 \in \mu] \wedge [x_2 \in \mu] \wedge \cdots \wedge [x_n \in \mu] \longrightarrow [\forall z \in f(x_1^n), z \in \mu]]$$

(ii) For any $x \in P$,

$$\models_t [[x \in \mu] \longrightarrow [x^{-1} \in \mu]].$$

Corollary 4.3. *A fuzzy subset μ of an n-ary polygroup P is a t-implication-based fuzzy n-ary sub-polygroup of P with respect to implication I if and only if*

(i) $I(\min\{\mu_{x_1}^{x_n}\}, \quad \bigwedge_{z\in f(x_1^n)} \mu(z)) \geq t$ *for all* $x_1^n \in P$,

(ii) *for* $x \in P$, $I(\mu(x), \mu(x^{-1})) \geq t$.

Example 4.4. Consider the 3-ary polygroup P as defined in Example 3.4. Let us define a fuzzy subset $\mu : P \to [0, 1]$ as follows:

$$\mu(e) = 0.4, \qquad \mu(x) = 1, \qquad \mu(y) = 0.2.$$

Then μ is a t-implication-based fuzzy 3-ary sub-polygroup of P with respect to Łukasiewicz implication, for all $0 < t \leq 0.2$. But μ

- is not a t-implication-based fuzzy 3-ary sub-polygroup of P with respect to Contraposition of Gödel implication, for all $t \in (0, 1]$,
- is not an $(\in, \in \vee q)$-fuzzy 3-ary sub-polygroup of P.

Theorem 4.5. *Let μ be fuzzy subset of an n-ary polygroup P.*

(i) *Let $I = I_{gr}$. Then μ is a 0.5-implication-based fuzzy n-ary sub-polygroup of P if and only if μ is a fuzzy n-ary sub-polygroup with thresholds $\alpha = 0$ and $\beta = 1$ of P.*

(ii) *Let $I = I_g$. Then μ is a 0.5-implication-based fuzzy n-ary sub-polygroup of P if and only if μ is a fuzzy n-ary sub-polygroup with thresholds $\alpha = 0$ and $\beta = 0.5$ of P.*

(iii) *Let $I = I_{cg}$. Then μ is a 0.5-implication-based fuzzy n-ary sub-polygroup with thresholds if and only if μ is a fuzzy n-ary sub-polygroup with thresholds $\alpha = 0.5$ and $\beta = 1$ of P.*

Proof. We prove only (ii) and the proof of (i) and (iii) are similar. Suppose that μ is a 0.5-implication-based fuzzy n-ary sub-polygroup of P. Then by Corollary 4.3(i), we have

(1) $I(\min\{\mu_{x_1}^{x_n}\}, \bigwedge_{z\in f(x_1^n)} \mu(z)) \geq 0.5$ for all $x_1^n \in P$,

(2) for $x \in P$, $I(\mu(x), \mu(x^{-1})) \geq 0.5$.

From (1), we have

$$\min\{\mu_{x_1}^{x_n}\} \leq \bigwedge_{z\in f(x_1^n)} \mu(z) \quad or \quad 0.5 \leq \bigwedge_{z\in f(x_1^n)} \mu(z) < \min\{\mu_{x_1}^{x_n}\}.$$

Then $\min\{\min\{\mu_{x_1}^{x_n}\}, 0.5\} \leq \bigwedge_{z\in f(x_1^n)} \mu(z)$ which implies that

$$\min\{\min\{\mu_{x_1}^{x_n}\}, 0.5\} \leq \left\{ \bigwedge_{z\in f(x_1^n)} \mu(z) \vee 0 \right\}.$$

From (2), we have $\mu(x) \leq \mu(x^{-1})$ or $0.5 \leq \mu(x^{-1}) < \mu(x)$. Thus $\min\{\mu(x), 0.5\} \leq \mu(x^{-1})$ which implies that $\mu(x) \wedge 0.5 \leq \mu(x^{-1} \vee 0)$.

The converse is clear. □

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