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A family of regular graphs of girth 5[☆]

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Abstract

Murty [A generalization of the Hoffman–Singleton graph, Ars Combin. 7 (1979) 191–193.] constructed a family of $(p^m + 2)$-regular graphs of girth five and order $2p^{2m}$, where $p \geqslant 5$ is a prime, which includes the Hoffman–Singleton graph [A.J. Hoffman, R.R. Singleton, On Moore graphs with diameters 2 and 3, IBM J. (1960) 497–504]. This construction gives an upper bound for the least number $f(k)$ of vertices of a k-regular graph with girth 5. In this paper, we extend the Murty construction to k-regular graphs with girth 5, for each k. In particular, we obtain new upper bounds for $f(k)$, $k \geqslant 16$.

Keywords: Regular graphs; Cages

1. Introduction and preliminaries

A (k,g)-*cage* is a k-regular simple graph of girth g with the fewest possible number of vertices. Let $f(k, g)$ be the number of vertices in a (k, g)-cage and let $f(k)$ be the least number of vertices in a k-regular simple graph of girth 5, i.e. in a $(k, 5)$-cage. It is known that $f(k) \geqslant k^2 + 1$ and Hoffman and Singleton [2] have shown that $f(k) > k^2 + 1$ if $k \neq 2, 3, 7$ and possibly 57. They also constructed a (7,5)-cage on 50 vertices with $f(7) = 50$, known as the *Hoffman–Singleton graph*. Murty [4] constructed a family of $(p^m + 2)$-regular simple graphs of girth 5, $p \geqslant 5$ prime, including the Hoffman–Singleton graph, such that $f(k) \leqslant 2(k-2)^2$, with $k - 2 = p^m$. Note that, exact values for $f(k)$ are known in very few cases, e.g. the 5-cycle $(f(2) = 5)$, the Petersen graph $(f(3) = 10)$, the Hoffman–Singleton $(f(7) = 50)$. An up-to-date table with all so far known values is given by Royle [5].

We construct a matrix with elements over $GF(q)$, $q = p^m$, $p \geqslant 5$ prime, and then we "blow up" each of its entries in a square (0, 1)-block matrix $\overline{C}$. We obtain a regular graph G with girth 5 having $\overline{C}$ as an adjacency matrix. Furthermore, we rephrase the Murty construction in terms of such a matrix. We obtain a family of k-regular subgraphs of the graph G, still of girth 5, with $k = q + 2 - \lambda$, where each $0 \leqslant \lambda \leqslant q - 2$ determines a principal minor of the matrix $\overline{C}$. Given $k \geqslant 3$ and q the least prime power such that $q \geqslant k - 2$ and $\lambda = q - k + 2$ and $k \geqslant 3$, this family gives rise to a subfamily

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of graphs $H(k)$ that extends the Murty construction for $k\geqslant 3$. In particular, we obtain new upper bound for $f(k)$, with $k\geqslant 16$.

2. (0, 1)-Block matrices and $GF(q)$

Consider the additive group $(GF(q),+)$, with $q=p^m$ for some prime $p\geqslant 5$, whose elements can be expressed in terms of polynomials of degree at most $m-1$ as $GF(q)=\{\sum_{i=0}^{m-1} a_i x^i \,|\, a_i \in \mathbb{Z}_p\}$. Choose the lexicographic order of the elements of $GF(q)$ increasing with the degree and the coefficients of the elements. Fix the labelling $\{e_0,\ldots,e_i,\ldots,e_{q-1}\}$ for the elements of $GF(q)$, according to this order. In particular, $e_0=0$, $e_1=1,\ldots,e_{p-1}=p-1$.

Let $A=(a_{i,j})$ be the matrix of order q over $GF(q)$ defined by

$$a_{i,j} := (-e_i)+e_j \quad \text{for } i,j=0,\ldots,q-1.$$

Note that $(a_{i,j})$ represents an addition table for $GF(q)$, where the elements $e_0=0,-e_1,-e_2,\ldots,-e_{q-1}$ and $e_0,e_1,e_2,\ldots,e_{q-1}$ correspond to the $1,2,\ldots,q$th rows and the $1,2,\ldots,q$th columns, respectively. In particular, A is a skew symmetric matrix and hence the entries in the main diagonal are equal to 0.

For each $e\in GF(q)$, let P_e be the (0, 1)-matrix of order q whose entry in position (i,j) is defined by

$$(P_e)_{i,j} \quad := \quad \begin{cases} 1 & \text{if } a_{i,j}=e, \\ 0 & \text{otherwise.} \end{cases}$$

Since the element e appears in each row and column of the addition table $(a_{i,j})$ precisely once, P_e is a *permutation matrix* of order q. In particular, P_0 is the unit matrix of order q.

Definition 2.1. Let $B=(b_{i,j})$ be a square matrix of order $t\geqslant 2$ over $GF(q)$ and let $\overline{B}_{i,j}$ be (0, 1)-matrices of order q, for $0\leqslant i,j\leqslant t-1$. We call a *blow up* of B, the (0, 1)-matrix $\overline{B}$ of order qt obtained substituting each entry $b_{i,j}$ with the matrix $\overline{B}_{i,j}$.

We denote by O_q the *null matrix* of order q.

Proposition 2.2. *Let $B=(b_{i,j})$ be a square matrix of order $t\geqslant 2$ over $GF(q)$ (possibly with blank entries) and let $\overline{B}$ the blow up of B with blocks*

$$\overline{B}_{i,j} \quad := \quad \begin{cases} P_e & if\ b_{i,j}=e, \\ O_q & if\ b_{i,j}=\ blank. \end{cases}$$

(i) *$\overline{B}$ is symmetric if, and only if, B is a skew symmetric matrix over $GF(q)$.*
(ii) *The entries in the main diagonal of $\overline{B}$ are all equal to 0 if, and only if, no entry in the main diagonal of B is equal to* 0.

Proof. (i) Suppose B is a skew symmetric matrix. Let $\overline{b}_{i,j}$ be an entry of $\overline{B}$, using the Euclidean algorithm, we put $i=\alpha_1 q+\beta_1$ and $j=\alpha_2 q+\beta_2$ with $0\leqslant\beta_1\leqslant q-1$ and $0\leqslant\beta_2\leqslant q-1$. Then $\overline{b}_{i,j}$ belongs to the β_1th row and β_2th column of the block $\overline{B}_{\alpha_1,\alpha_2}$ of $\overline{B}$.

$$\overline{b}_{i,j} = \quad \begin{cases} 1 & \text{if } a_{\beta_1,\beta_2}=e \text{ and } b_{\alpha_1,\alpha_2}=e, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that a_{β_1,β_2} is an entry of the matrix A of order q over $GF(q)$ defined above. If b_{α_1,α_2} is a blank entry the result is trivial. If $b_{\alpha_1,\alpha_2}=e$ then $b_{\alpha_2,\alpha_1}=-e$ which implies that $\overline{B}_{\alpha_2,\alpha_1}=P_{-e}$ and

$$\overline{b}_{j,i} = \quad \begin{cases} 1 & \text{if } a_{\beta_2,\beta_1}=-e, \\ 0 & \text{otherwise.} \end{cases}$$

Since A, by construction, is skew symmetric, then $a_{\beta_1,\beta_2}=-a_{\beta_2,\beta_1}$. Therefore, $\overline{B}$ is symmetric.

Conversely, suppose $\overline{B}$ is symmetric and subdivide it into square blocks of order q. By definition of $\overline{B}$ the block

$$\overline{B}_{\alpha_1,\alpha_2} = \begin{cases} P_e & \text{if } b_{\alpha_1,\alpha_2} = e, \\ O_q & \text{if } b_{\alpha_1,\alpha_2} = \text{blank}. \end{cases}$$

If $b_{\alpha_1,\alpha_2} = \text{blank}$ the result is clear. We need to prove that if $\overline{B}_{\alpha_1,\alpha_2} = P_e$ then $\overline{B}_{\alpha_2,\alpha_1} = P_{-e}$. To this purpose, we stress that the block $\overline{B}_{\alpha_1,\alpha_2}$ is made exactly of the entries

$$\bar{b}_{i,j} = \begin{cases} 1 & \text{if } a_{\beta_1,\beta_2} = e, \\ 0 & \text{otherwise,} \end{cases}$$

where $i = \alpha_1 q + \beta_1$ and $j = \alpha_2 q + \beta_2$ with $0 \leqslant \beta_1 \leqslant q-1$ and $0 \leqslant \beta_2 \leqslant q-1$. Since $\overline{B}$ is symmetric then the block $\overline{B}_{\alpha_2,\alpha_1}$ is made exactly of the entries

$$\bar{b}_{j,i} = \begin{cases} 1 & \text{if } a_{\beta_1,\beta_2} = e, \\ 0 & \text{otherwise,} \end{cases}$$

Using again the fact that A is skew symmetric we get the desired result.

(ii) Let $i = \alpha_1 q + \beta_1$, with $0 \leqslant \beta_1 \leqslant q-1$. By construction of $\overline{B}$, it follows that $\bar{b}_{i,i} = 1$ if, and only if, $a_{\beta_1,\beta_1} = e$ and $\overline{B}_{\alpha_1,\alpha_1} = P_e$. Since A is skew symmetric, then this happens if, and only if, $e = 0$. □

Let $M := (m_{i,j})$ be the *full multiplication table* of $GF(q)$, defined by $m_{i,j} := e_i e_j$, for $i, j = 0, \ldots, q-1$, and let $D^{(k)} := \left(d^{(k)}_{i,j}\right)$, for $k \in \{1, 2\}$, be the matrix with entry $(k, p-k)$ on the main diagonal and "blank" entries elsewhere.

Definition 2.3. Let $\overline{D^{(k)}}$ be the blow up of $D^{(k)}$ with blocks

$$\left(\overline{D^{(k)}}\right)_{i,j} := \begin{cases} P_k + P_{p-k} & \text{if } d^{(k)}_{i,j} = (k, p-k), \\ O_q & \text{if } d^{(k)}_{i,j} = \text{blank entry.} \end{cases}$$

Let $C_1 := \left(\begin{array}{c|c} D^{(1)} & \\ \hline & D^{(2)} \end{array}\right)$, $C_2 := \left(\begin{array}{c|c} & M \\ \hline -M & \end{array}\right)$ and $C := C_1 + C_2 = \left(\begin{array}{c|c} D^{(1)} & M \\ \hline -M & D^{(2)} \end{array}\right)$. Then the corresponding blow ups are $\overline{C_1} = \left(\begin{array}{c|c} \overline{D^{(1)}} & O_{q^2} \\ \hline O_{q^2} & \overline{D^{(2)}} \end{array}\right)$, $\overline{C_2} = \left(\begin{array}{c|c} O_{q^2} & \overline{M} \\ \hline -\overline{M} & O_{q^2} \end{array}\right)$ and $\overline{C} := \overline{C_1} + \overline{C_2} = \left(\begin{array}{c|c} \overline{D^{(1)}} & \overline{M} \\ \hline -\overline{M} & \overline{D^{(2)}} \end{array}\right)$.

Proposition 2.4. *The* $(0, 1)$*-matrix* $\overline{C}$ *is symmetric and contains only zeros on the main diagonal.*

Proof. Subdivide $\overline{C_1}$ into square blocks of order q and note that all blocks are zero except for the diagonal ones which are $P_k + P_{p-k}$, for $k \in \{1, 2\}$. Since $P_k + P_{p-k}$ are symmetric and contain all zeros on the main diagonal, $\overline{C_1}$ has the desired property. Furthermore, note that C_2 is skew symmetric and does not contain zeros on the main diagonal. Therefore, we apply Proposition 2.2 to get that $\overline{C_2}$ has the desired property. Hence, the result follows for $\overline{C} = \overline{C_1} + \overline{C_2}$ □

A $(0, 1)$-matrix corresponds to the adjacency matrix of simple graph without loops if, and only if, it is symmetric and contains only zeros on the main diagonal. Let G be the graph having adjacency matrix $\overline{C}$.

Lemma 2.5. *The graph G does not have girth* 3.

Proof. Consider $\overline{C}$ as a $2q \times 2q$ block matrix. We suppose by contradiction that (i, j, k) is a triangle in G with $i, j, k \in \{0, \ldots, 2q^2 - 1\}$.

Let $i < j < k$, using the Euclidean algorithm, we may write $i = \alpha_1 q + \beta_1$, $j = \alpha_2 q + \beta_2$ and $k = \alpha_3 q + \beta_3$ with $0 \leqslant \beta_1, \beta_2, \beta_3 \leqslant q-1$.

If $\alpha_1 = \alpha_2 = \alpha_3$, then the triangle should lie all in a single block, but this cannot happen since each block in $\overline{M}$ and in $-\overline{M}$ has exactly one entry 1, and each block in $\overline{D^{(1)}}$ and in $\overline{D^{(2)}}$ is either blank or has only p-cycles (with $p \geqslant 5$).

If $\alpha_1 = \alpha_2$, then i and j belong to a block on the main diagonal of $\overline{C}$, but, then k corresponds to a blank block or to a block in M. In both cases, we get a contradiction, since in the first case there cannot be entries 1 and in the second case there cannot be two entries 1 in a single row (or column). Therefore, the six entries forming the triangle must belong to different blocks. Then, two of these lie in either $\overline{D^{(1)}}$ or in $\overline{D^{(2)}}$, implying that there will be some entry 1 in the null blocks, a contradiction. □

A square (0, 1)-matrix is *linear* if it does not contain any 2×2 submatrix S all of whose entries are 1 [1]. Note that, a graph whose adjacency matrix is linear does not contain quadrangles.

Proposition 2.6. *The* (0, 1)*-matrix* $\overline{C}$ *is linear.*

Proof. Suppose, by contradiction, that in $\overline{C}$ there are four elements $\overline{c}_{i,j} = \overline{c}_{i,k} = \overline{c}_{l,j} = \overline{c}_{l,k} = 1$, pairwise in the same row and the same column. Let $i = \alpha_1 q + \beta_1$, $j = \alpha_2 q + \beta_2$, $k = \alpha_3 q + \beta_3$ and $l = \alpha_4 q + \beta_4$, for $0 \leqslant \beta_1, \beta_2, \beta_3, \beta_4 \leqslant q - 1$. We distinguish two cases according whether $\alpha_1 = \alpha_2$ or $\alpha_1 \neq \alpha_2$.

Case 1: $\alpha_1 = \alpha_2$. We have different subcases, the only non-trivial is the following: $\alpha_1 \neq \alpha_4$ and $\alpha_2 \neq \alpha_3$. In this case, all entries are in different blocks. We can deduce by construction that $\alpha_3 = \alpha_4$, $\beta_1 = \beta_3$, and $\beta_2 = \beta_4$. Thus, we have that $-e_{\beta_1} + e_{\beta_2} = \pm 1$ and $-e_{\beta_4} + e_{\beta_3} = \pm 2$, this implies that $-e_{\beta_1} + e_{\beta_3} \neq 0$, a contradiction.

Case 2: $\alpha_1 \neq \alpha_2$. We have different subcases, the only non-trivial one is the same as in the previous case: $\alpha_1 \neq \alpha_4$ and $\alpha_2 \neq \alpha_3$. The matrix $\overline{C}_{\alpha_1,\alpha_2}$ is either a block in M or in $-M$. This implies that $e_{\alpha_1} \neq e_{\alpha_4}$ and $e_{\alpha_2} \neq e_{\alpha_3}$. Let $a := e_{\alpha_1} e_{\alpha_2}$, $b := e_{\alpha_1} e_{\alpha_3}$, $c := e_{\alpha_4} e_{\alpha_3}$ and $d := e_{\alpha_4} e_{\alpha_2}$. Hence, $a - b + c - d = (e_{\alpha_1} - e_{\alpha_4})(e_{\alpha_2} - e_{\alpha_3}) \neq 0$. On the other hand, from the construction of $\overline{C}$, we have $-e_{\beta_1} + e_{\beta_2} = a$, $-e_{\beta_1} + e_{\beta_3} = b$, $-e_{\beta_4} + e_{\beta_3} = c$ and $-e_{\beta_4} + e_{\beta_2} = d$. Subtracting the second and fourth equations from the sum of the first and third, we obtain $0 = a - b + c - d$, a contradiction. □

Corollary 2.7. *The graph G does not have girth* 4.

Theorem 2.8. *The graph G has girth* 5.

Proof. By Lemma 2.5 and Corollary 2.7, the girth of G is 5 or greater. The thesis follows since there is at least the pentagon given by $(0, q^2 + q, 2q - 1, q^2 + q - 1, q - 1, 0)$ in G. □

3. The Murty construction

Consider the following two sets V_1 and V_2, where, for $i = 1, 2$:

$$V_i = \{(i, x_1, x_2) : x_1, x_2 \in GF(p^m)\}.$$

Denote with $G = G(p, m)$ the graph on $V_1 \cup V_2$ vertices with adjacencies determined via the following rules:

(1) For $i = 1, 2$, and $x_1, x_2, y_1, y_2 \in GF(p^m)$, (i, x_1, x_2) and (i, y_1, y_2) are adjacent if
  (i) $x_2 = y_2$,
  (ii) $x_1 - y_1 \in GF(p)$,
  (iii) $x_1 - y_1 \equiv \pm i \pmod p$.
(2) For $x_1, x_2, y_1, y_2 \in GF(p^m)$ $(1, x_1, x_2)$ and $(2, y_1, y_2)$ are adjacent if $x_1 + x_2 y_2 = y_1$.

Murty [4] proves that $G = G(p, m)$ is a $(p^m + 2)$-regular graph of girth 5. Note that $G(5, 1)$ is the Hoffman–Singleton graph i.e. the (7, 5)-cage or the smallest 7-regular graph of girth 5 [2]. Let $\overline{C}$ be the (0, 1)-matrix defined in Section 2.

Theorem 3.1. *The matrix* $\overline{C}$ *is an adjacency matrix for the graph* $G(p, m)$.

Proof. We redefine V_1 and V_2 as follows: $V_i' = \{(i, x_2, x_1) : x_1, x_2 \in GF(p^m)\}$ keeping the same adjacency rules. Fix a labelling for the vertices of $G(p, m)$ taking lexicographic order on the coordinates of the elements of $V_1' \cup V_2'$. Note that, the entries 1 in $\overline{C}$ appearing in $\overline{D^{(1)}}$ and $\overline{D^{(2)}}$ correspond exactly to the adjacencies determined by rule

(1). Moreover, the entries 1 in $\overline{C}$ appearing in $\overline{M}$ and $-\overline{M}$ correspond exactly to the adjacencies determined by rule (2). $\square$

4. A construction of new regular graphs of girth 5

Let $C := (c_{i,j})$ be a matrix defined as in Section 3, and let $R := (r_{i,j})$ be the block matrix of order $2q$ with blocks $R_{i,j} := \begin{pmatrix} c_{i,j} & c_{i,j+q} \\ c_{i+q,j} & c_{i+q,j+q} \end{pmatrix}$ of order 2. We blow up R to $\overline{R}$ having blocks of order q of type

$$\overline{R}_{i,j} := \begin{cases} P_k + P_{p-k} & \text{if } r_{i,j} = (k, p-k), \\ P_e & \text{if } r_{i,j} = e, \\ O_q & \text{if } r_{i,j} = \text{blank entry}. \end{cases}$$

We define *the principal minor* R_λ of R the matrix of order $2q - 2\lambda$ obtained from R by deleting the last 2λ rows and columns, for each $0 \leqslant \lambda \leqslant q-2$. Let $\overline{R}_\lambda$ be the corresponding blow up. Note that, $\overline{R}_\lambda$ is the principal minor obtained from $\overline{R}$ by deleting its last $2q\lambda$ rows and columns.

Theorem 4.1. *The matrices $\overline{R}_\lambda$ give rise to an infinite family of $(q+2-\lambda)$-regular graphs of girth 5 on $2q(q-\lambda)$ vertices, with $q = p^m$, $p \geqslant 5$ and $0 \leqslant \lambda \leqslant q-2$.*

Proof. Note that $\overline{R}$ is another adjacency matrix for the $(q+2)$-regular graph G for which $\overline{C}$ is an adjacency matrix. Deleting rows and columns of $\overline{R}$ does not add any vertex or edge in the corresponding graph G'. Hence, the deletion does not give rise to triangles or quadrangles in G'. On the other hand, the pentagon $(0, q^2+q, 2q-1, q^2+q-1, q-1, 0)$ belongs to the first $4q$ rows and columns of $\overline{R}$. Thus, $\overline{R}_\lambda$ is the adjacency matrix of a graph of girth 5.

The order of the graph G' corresponding to $\overline{R}_\lambda$ is $2q(q-\lambda)$, since deleting the last $2q\lambda$ rows and columns of $\overline{R}$ corresponds to delete $2q\lambda$ vertices. In the last $2q\lambda$ columns of $\overline{R}$, there are exactly λ entries 1 in each of the first $2q^2 - 2q\lambda$ rows, therefore, the remaining vertices have degree $q+2-\lambda$ in $\overline{R}_\lambda$. $\square$

Note: The block $\overline{R}_{ii}$ of $\overline{R}$ is also the adjacency matrix of p^{m-1} disjoint copies of the *generalized Petersen graph* $P(p, 2)$ (cf. e.g. [3]).

Given an integer $k \geqslant 3$, let $H(k) := H(q, \lambda)$ be a graph in the family of graphs described in Theorem 4.1, such that q is the least prime power with $q \geqslant k-2$ and $\lambda = q-k+2$. Recall that, $q = p^m$, for some prime $p \geqslant 5$.

Comparing the possible orders of $H(k)$ with those from the table which appears in Royle's web page [5], we get the table below. In this table k, $|V(H(k))|$ and $\rho(k)$ stand for the regularity of the graphs, the order of the graph $H(k)$ and the order of the smallest known k-regular simple graphs of girth 5, respectively. Note that, $f(k) \leqslant \rho(k)$ and equality holds in the case of $(k, 5)$-cages, with $k \geqslant 2$.

k	$\lvert V(H(k))\rvert$	$\rho(k)$
3	10	10
4	20	19
5	30	30
6	40	40
7	50	50
8	84	80
9	98	98
10	176	126
11	198	160
12	220	203
13	242	240
14	312	312
15	338	406
16	476	480
17	510	576
18	544	

Remark 4.2. The case $k=15$ gives a graph from Murty's family. Furthermore, for $k=16$, 17, we obtain better bounds for possible cages with respect to those known, and for each $k \geqslant 18$ we obtain new values. Note that, the graphs with $k-2=p^m$, $p \geqslant 5$ prime, are those from the Murty construction.

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