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Resolutions of the Steinberg module for $GL(n)$ ☆

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ABSTRACT

We give several resolutions of the Steinberg representation St_n for the general linear group over a principal ideal domain, in particular over $\mathbb{Z}$. We compare them, and use these results to prove that the computations in Avner Ash et al. (2011) [AGM11] are definitive. In particular, in Avner Ash et al. (2011) [AGM11] we use two complexes to compute certain cohomology groups of congruence subgroups of $SL(4, \mathbb{Z})$. One complex is based on Voronoi's polyhedral decomposition of the symmetric space for $SL(n, \mathbb{R})$, whereas the other is a larger complex that has an action of the Hecke operators. We prove that both complexes allow us to compute the relevant cohomology groups, and that the use of the Voronoi complex does not introduce any spurious Hecke eigenclasses.

1. Introduction

In a series of papers [AGM02,AGM08,AGM10,AGM11] we computed the cohomology H^5 with constant coefficients of certain congruence subgroups $\Gamma \subset SL(4, \mathbb{Z})$, with 5 being chosen since this is the topmost degree that can contain classes corresponding to cuspidal automorphic forms [AGM02, §1]. We computed the action of the Hecke operators on the cohomology and studied connections with representations of the absolute Galois group of $\mathbb{Q}$.

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The main tool in our computations is the *Steinberg module* St_n, which is the dualizing module of any torsionfree congruence subgroup of $SL(n,\mathbb{Z})$ [BS73]. We used a variant of a resolution of the Steinberg module for $GL(4,\mathbb{Z})$ first published in [LS76]. This variant is called the *sharbly complex*. In our previous papers, we only asserted that the sharbly complex is a resolution of St_n; in this paper, we provide a proof.

When multiplication by 30 is not invertible in the coefficient module, our computations were not definitive for the following reason. To compute the cohomology we use the *Voronoi complex*, which is a chain complex built from an $SL(n,\mathbb{Z})$-equivariant polytopal tessellation of the symmetric space $SL(n,\mathbb{R})/SO(n)$. This tessellation is finite modulo $SL(n,\mathbb{Z})$, and thus provides a convenient tool for explicit computations. But to compute the action of the Hecke operators on cohomology, we must use the sharbly complex instead of the Voronoi, since the Hecke operators act on the former and not on the latter. There is a map from the Voronoi-based cohomology in degree 5 to the sharbly-based cohomology in that degree that we thought we had to assume was injective for our results to be meaningful. As far as we know, this map may fail to be injective. However, in this paper, we prove that if $n=4$, the Hecke eigenvalues we compute on H^5 are meaningful. We show that they are Hecke eigenvalues in the homology of Γ with coefficients in the sharbly complex. We prove a similar result for $n=3$.

Given any congruence subgroup $\Gamma\subset SL(n,\mathbb{Z})$ and any $\mathbb{Z}\Gamma$-module M, one can compute the homology of the sharbly complex with coefficients in M. If M is a vector space over $\mathbb{F}_p$ for a prime p that is not a torsion prime of Γ, or over a field of characteristic zero, then the sharbly homology is isomorphic to the group cohomology of Γ. For torsion primes, however, what one computes with the sharbly complex is not so clear. Our main result in this direction (Corollary 8) is that if p is any odd prime, then the sharbly homology is isomorphic to the *Steinberg homology* with coefficients in M; by definition this is $H_*(\Gamma, St_n\otimes_{\mathbb{Z}} M)$. Note that these latter groups are exactly the group cohomology of Γ with coefficients in M if Γ is torsionfree. Furthermore, we prove in Corollary 12 that if $n\leqslant 4$ the Voronoi homology (Definition 10) is isomorphic to the Steinberg homology. This was left open in [AGM11].

We remark that the key to making our results work in odd characteristics is to replace alternating chains with ordered chains in the sharbly complex. Our result on the meaningfulness of our Hecke eigenvalue computations depends on using Lee and Szczarba's original resolution of the Steinberg module.

2. The resolution of Lee and Szczarba of the Steinberg module

In this section we briefly review the relevant contents of [LS76, §3]. We phrase Lee and Szczarba's construction slightly differently, but equivalently, in a way that is more suitable for our purposes.

Let $n\geqslant 1$. For any ring A, let A^n denote the right $GL(n,A)$-module of row vectors. We say that $v\in A^n$ is *unimodular* if the entries of v generate the unit ideal in a. Let $M_{s,t}(A)$ denote the set of all $s\times t$ matrices with entries in A. For any set U, let $\mathbb{Z}(U)$ denote the free abelian group generated by U. All homology will be taken with $\mathbb{Z}$ coefficients unless otherwise indicated.

Now we assume A is a principal ideal domain. Let F be its field of fractions, and assume $n\geqslant 2$. The *Tits building* $T_n(A)$ is the simplicial complex whose vertices are the proper nonzero subspaces of F^n and whose simplices correspond to flags of subspaces. Note that $T_n(A)$ depends only on F. By the Solomon–Tits theorem $T_n(A)$ has the homotopy type of a wedge of $(n-2)$-dimensional spheres. It is a right $GL(n,F)$-module and therefore so is its homology. We define the *Steinberg module* to be the reduced homology of the Tits building:

$$St_n(A)=\widetilde{H}_{n-2}\big(T_n(A)\big). \tag{1}$$

For each hyperplane H of F^n let $S(H)$ be the subcomplex of $T_n(A)$ consisting of all simplices of $T_n(A)$ with H as a vertex. The $S(H)$ form an acyclic cover of $T_n(A)$, and for all $q\geqslant 0$ the nerve $\widetilde{N}$ of this cover satisfies

$$H_q\big(T_n(A)\big)=H_q(\widetilde{N}).$$

For $k \geqslant 0$ let $\mathcal{S}_k \subset M_{n+k,n}(A)$ be the set of all matrices M with all rows unimodular. Let $\mathcal{P}_k \subset \mathcal{S}_k$ be the subset of matrices with rank $< n$. These sets have a natural right action by $GL(n, A)$. We define the boundary operator

$$\partial : \mathbb{Z}(\mathcal{S}_k) \to \mathbb{Z}(\mathcal{S}_{k-1})$$

by $\partial M = \sum_{i=1}^{n+k} (-1)^i M_i$, where M_i is the matrix formed from M by deleting its ith row. Since ∂ takes $\mathbb{Z}(\mathcal{P}_k)$ to $\mathbb{Z}(\mathcal{P}_{k-1})$, we can form the quotient complex

$$\mathcal{C}_k(A) := \mathbb{Z}(\mathcal{S}_k)/\mathbb{Z}(\mathcal{P}_k).$$

This is a complex of right $GL(n, A)$-modules.

Theorem 3.1 of [LS76] asserts that there is an epimorphism $\phi : \mathcal{C}_0(A) \to St_n(A)$ such that

$$\cdots \to \mathcal{C}_k(A) \to \mathcal{C}_{k-1}(A) \to \cdots \to \mathcal{C}_0(A) \to St_n(A) \to 0$$

is a free $GL(n, A)$-resolution of $St_n(A)$. For the convenience of the reader, and since we will need to use similar arguments in the sequel, we sketch the proof.

Let K be the simplicial complex whose vertices are the unimodular elements in A^n and whose simplices are all finite nonempty subsets of vertices. Let L be the subcomplex of K consisting of those simplices all of whose vertices lie in one and the same proper direct summand of A^n. The group $GL(n, A)$ acts on the right of K and L. Since K is acyclic, the exact sequence of the pair (K, L) implies $H_k(K, L) = \widetilde{H}_{k-1}(L)$ for all $k \geqslant 0$.

If M is a simplicial complex or a pair of such, let $C_*(M)$ denote the *ordered* simplicial chain complex on M [Spa81, §4.3]. The following is Lemma 3.2 of [LS76]:

Lemma 1. *$H_q(K, L) = 0$ if $q \neq n-1$ and $H_{n-1}(K, L) \approx St_n(A)$.*

Assume the lemma. The $(n-2)$-skeletons of L and K are the same, so $C_{n+k-1}(K, L) = 0$ if $k < 0$. We obtain an exact sequence

$$\cdots \to C_{n+k}(K, L) \to C_{n+k-1}(K, L) \to \cdots \to C_{n-1}(K, L) \to St_n(A) \to 0.$$

We can map a simplex σ in K to the matrix whose rows consist of the vertices of σ, in order. This gives us isomorphisms for $k \geqslant 0$:

$$C_{n+k-1}(K) \approx \mathbb{Z}(\mathcal{S}_k) \quad \text{and} \quad C_{n+k-1}(L) \approx \mathbb{Z}(\mathcal{P}_k).$$

Therefore $C_{n+k-1}(K, L) = \mathcal{C}_k(A)$ and we have an exact sequence

$$\cdots \to \mathcal{C}_{k+1}(A) \to \mathcal{C}_k(A) \to \cdots \to \mathcal{C}_0(A) \to St_n(A) \to 0.$$

It remains to prove Lemma 1.

Proof of Lemma 1. Let $\mathcal{H}$ be the set of direct summands of rank $n-1$ in A^n. Since A is a PID, H is isomorphic to A^{n-1}. For $H \in \mathcal{H}$, let K_H denote the subcomplex of L consisting of all simplices whose vertices lie in H. For the same reason that K is contractible, so is K_H. More generally, if $H_1, \ldots, H_q \in \mathcal{H}$, then $H_1 \cap \cdots \cap H_q$ is isomorphic to A^{n-q} and $K_{H_1} \cap \cdots \cap K_{H_q}$ is contractible.

Therefore, $\{K_H\}$ is an acyclic cover of L. Letting N denote its nerve, we have for all $q \geqslant 0$

$$H_q(L) \approx H_q(N).$$

The map $H \mapsto H \otimes_A F$ defines a simplicial isomorphism $N \to \widetilde{N}$. We obtain a sequence of $GL(n, A)$-equivariant isomorphisms:

$$H_q(K, L) \approx \widetilde{H}_{q-1}(L) \approx \widetilde{H}_{q-1}(N) \approx \widetilde{H}_{q-1}(\widetilde{N}) \approx \widetilde{H}_{q-1}\bigl(T_n(A)\bigr).$$

By the Solomon–Tits theorem and (1), the proof is complete. □

It is easy to see that $\mathcal{C}_*$ is free as a $GL(n, A)$-module.

In [AR79], to each matrix X in $GL(n, F)$ is associated the modular symbol $[\![X]\!] \in St_n(A)$, which is the fundamental class of the apartment corresponding to X in the Tits building. The map $\phi : \mathcal{C}_0(A) \to St_n(A)$ may be taken to be $M \mapsto [\![M]\!]$, and we will do so in the sequel.

3. A variant resolution of the Steinberg module

In this section we present a variant of the construction of Lee and Szczarba and prove that it gives a resolution of the Steinberg module as well. We keep the same notation as in the preceding section.

Let $\mathbf{P}^{n-1}(F)$ be the set of lines in F^n. Set $\mathcal{S}'_k = (\mathbf{P}^{n-1}(F))^{n+k}$ and let $\mathcal{P}'_k$ be the subset of $\mathcal{S}'_k$ where the lines in the $(n+k)$-tuple do not span F^n. The right $GL(n, F)$-action on F^n induces an action on $\mathcal{S}'_k$ that preserves $\mathcal{P}'_k$.

There is an obvious $GL(n, A)$-equivariant quotient map $\theta_k : \mathcal{S}_k \to \mathcal{S}'_k$ that induces a map $\mathcal{P}_k \to \mathcal{P}'_k$. The boundary operator ∂ induces a boundary operator that we also denote by ∂:

$$\partial : \mathbb{Z}\bigl(\mathcal{S}'_k\bigr) \to \mathbb{Z}\bigl(\mathcal{S}'_{k-1}\bigr).$$

As before ∂ takes $\mathbb{Z}(\mathcal{P}'_k)$ to $\mathbb{Z}(\mathcal{P}'_{k-1})$, and we can form the quotient $GL(n, F)$-complex

$$\mathcal{C}'_k(A) := \mathbb{Z}\bigl(\mathcal{S}'_k\bigr)/\mathbb{Z}\bigl(\mathcal{P}'_k\bigr).$$

Note that $\phi : \mathcal{C}_0(A) \to St_n(A)$ is constant on the fibers of θ_0 and therefore induces an epimorphism $\phi' : \mathcal{C}'_0(A) \to St_n(A)$.

Theorem 2. *The exact sequence*

$$\cdots \to \mathcal{C}'_k(A) \to \mathcal{C}'_{k-1}(A) \to \cdots \to \mathcal{C}'_0(A) \xrightarrow{\phi'} St_n(A) \to 0 \tag{2}$$

is a $GL(n, F)$-resolution of $St_n(A)$.

We remark that (2) is not a free $GL(n, A)$-resolution if A has more than one unit.

Proof. Let K' be the simplicial complex whose vertices are the elements of $\mathbf{P}^{n-1}(F)$ and whose simplices are all finite nonempty subsets of vertices. Let L' be the subcomplex of K' consisting of those simplices all of whose vertices lie in one and the same proper direct summand of F^n. The group $GL(n, F)$ acts on the right of K' and L'. Since K' is acyclic, we have $H_k(K', L') = \widetilde{H}_{k-1}(L')$ for all $k \geqslant 0$ from the exact sequence of the pair (K', L').

Lemma 3. *$H_q(K', L') = 0$ if $q \neq n-1$ and $H_{n-1}(K', L') \approx St_n(A)$ via ϕ'.*

Proof. Let $\mathcal{H}'$ be the set of direct summands of rank $(n-1)$ in F^n. Since F is a field, any $H \in \mathcal{H}'$ is isomorphic to F^{n-1}. For $H \in \mathcal{H}'$, let K'_H denote the subcomplex of L' consisting of all simplices whose vertices lie in H'. For the same reason that K' is contractible, so is K'_H. More generally, if $H_1, \ldots, H_q \in \mathcal{H}$, then $H_1 \cap \cdots \cap H_q$ is isomorphic to F^{n-q}, and $K'_{H_1} \cap \cdots \cap K'_{H_q}$ is contractible.

Therefore $\{K'_H\}$ is an acyclic cover of L'. Letting N' denote its nerve, we have for all $q \geqslant 0$

$$H_q(L') \approx H_q(N').$$

The identity map $H \mapsto H$ defines a simplicial isomorphism $N' \to \widetilde{N}$. We obtain a sequence of $GL(n,F)$-equivariant isomorphisms

$$H_q(K',L') \approx \widetilde{H}_{q-1}(L') \approx \widetilde{H}_{q-1}(N') \approx \widetilde{H}_{q-1}(\widetilde{N}) \approx \widetilde{H}_{q-1}(T_n(A)).$$

Let $X \in GL(n,F)$ be associated to the modular symbol $[\![X]\!] \in St_n(A)$. The apartment corresponding to X in the Tits building depends only on the lines in F^n generated by the rows of X. Thus the map ϕ' behaves as claimed, and this proves the lemma. □

We return now to the proof of Theorem 2. The $(n-2)$-skeletons of L' and K' are the same, so $C_{n+k-1}(K',L') = 0$ if $k < 0$. We obtain an exact sequence

$$\cdots \to C_{n+k}(K',L') \to C_{n+k-1}(K',L') \to \cdots \to C_{n-1}(K',L') \to St_n(A) \to 0.$$

Clearly, for $k \geqslant 0$ we have isomorphisms

$$C_{n+k-1}(K') \approx \mathbb{Z}(\mathcal{S}'_k) \quad \text{and} \quad C_{n+k-1}(L') \approx \mathbb{Z}(\mathcal{P}'_k).$$

Therefore $C_{n+k-1}(K',L') = \mathcal{C}'_k(A)$, and we have an exact sequence

$$\cdots \to \mathcal{C}'_{k+1}(A) \to \mathcal{C}'_k(A) \to \cdots \to \mathcal{C}'_0(A) \xrightarrow{\phi'} St_n(A) \to 0.$$

This completes the proof of the theorem. □

4. The sharbly complex

For the purposes of computing Hecke operators, one needs to modify the construction of Lee and Szczarba to obtain a resolution of $St_n(A)$ by $GL(n,F)$-modules. One possibility is to take $A = F$ obtaining a free $GL(n,F)$-complex $\mathcal{C}(F)$, thereby allowing in principle an easy formula for the action of Hecke operators on the homology. However, this resolution is far too big to be used for practical computation.

To make Lee and Szczarba's complex "smaller", we simultaneously antisymmetrize and factor out the action of scalars on row vectors, even though in this way we sacrifice freeness. We obtain the sharbly complex Sh_*, which gives a resolution of the Steinberg module by $GL(n,F)$-modules. In Section 5, we will introduce much smaller resolutions using Voronoi theory.

Let Γ be a subgroup of finite index in $GL(n,A)$. The sharbly resolution is Γ-free if Γ is torsionfree, but in general it is not even Γ-projective.

The sharbly complex was defined in [Ash94], in a slightly different form from the definition in [AGM11]. It is straightforward to see that the two different definitions give naturally isomorphic complexes of $GL(n,\mathbb{Z})$-modules. The advantage of the latter definition is that there is an obvious $GL(n,\mathbb{Q})$-action on the complex. We give here the latter form, generalized from $\mathbb{Z}$ to an arbitrary principal ideal domain.

We continue to use the notation of the preceding sections.

Definition 4. The *sharbly complex* $Sh_* = Sh_*(A)$ is the complex of $\mathbb{Z}GL(n,F)$-modules defined as follows. As an abelian group, $Sh_k(A)$ is generated by symbols $[v_1,\dots,v_{n+k}]$, where the v_i are nonzero vectors in F^n, modulo the submodule generated by the following relations:

(i) $[v_{\sigma(1)}, \ldots, v_{\sigma(n+k)}] - (-1)^{\sigma}[v_1, \ldots, v_{n+k}]$ for all permutations σ;
(ii) $[v_1, \ldots, v_{n+k}]$ if $v_1, \ldots, v_{n+k}$ do not span all of F^n; and
(iii) $[v_1, \ldots, v_{n+k}] - [av_1, v_2, \ldots, v_{n+k}]$ for all $a \in F^{\times}$.

The boundary map $\partial : Sh_k(A) \to Sh_{k-1}(A)$ is given by

$$\partial\big([v_1, \ldots, v_{n+k}]\big) = \sum_{i=1}^{n+k} [v_1, \ldots, \widehat{v_i}, \ldots, v_{n+k}],$$

where as usual $\widehat{v_i}$ means to delete v_i.

Writing $v_1, \ldots, v_n$ as row vectors, we let $X(v_1, \ldots, v_n)$ denote the matrix with the v_i as rows, and put $[\![v_1, \ldots, v_n]\!] = [\![X(v_1, \ldots, v_n)]\!]$ as in the proof of Lemma 3. The map $[v_1, \ldots, v_n] \mapsto [\![v_1, \ldots, v_n]\!]$ is constant on the cosets of the group generated by the relations (i)–(iii) and thus defines a surjective $GL(n, F)$-equivariant map $\phi_{Sh} : Sh_0(A) \to St_n(A)$.

For each line $\ell_i \in \mathbf{P}^{n-1}(F)$ choose a nonzero unimodular vector $v_i \in \ell_i \cap A^n$. The map $(\ell_1, \ldots, \ell_{n+k}) \mapsto [v_1, \ldots, v_{n+k}]$ extends to a $GL(n, F)$-equivariant chain map $f : \mathcal{C}'_*(A) \to Sh_*(A)$ and $\phi' = \phi_{Sh} \circ f$.

Theorem 5. *The following is an exact sequence of $GL(n, F)$-modules:*

$$\cdots \to Sh_k(A) \to Sh_{k-1}(A) \to \cdots \to Sh_0(A) \xrightarrow{\phi_{Sh}} St_n(A) \to 0.$$

Proof. We interpret f geometrically as follows. Since the $(n-2)$-skeletons of K' and L' are the same, $\mathcal{C}'_*(A)$ is naturally isomorphic to the complex of ordered chains of the pair of simplicial complexes (K', L') and the sharbly complex $Sh_*(A)$ is naturally isomorphic to the complex of oriented (antisymmetric) chains of (K', L'). A standard fact tells us that the complex of ordered chains on a simplicial complex Y is homotopy equivalent to the complex of oriented chains on Y (cf. [Spa81, Theorem 4.3.8]). We take the long exact sequence in ordered homology coming from the pair (K', L') with its natural map to the long exact sequence in oriented homology coming from (K', L'). Using Theorem 2, applying the "standard fact" to the homology of K' and L' respectively, plus repeated use of the Five Lemma gives the result. □

Definition 6. Let Γ be a subgroup of $GL(n, F)$ and M a right $\mathbb{Z}\Gamma$-module. Consider M to be a complex concentrated in dimension 0. We define the *Steinberg homology* of Γ with coefficients in M to be $H_*(\Gamma, St_n(A) \otimes_{\mathbb{Z}} M)$. We define the *sharbly homology* of Γ with coefficients in M to be $H_*(\Gamma, Sh_* \otimes_{\mathbb{Z}} M)$.

Note that if Γ is torsionfree, then Borel–Serre duality [BS73] implies that the Steinberg homology is isomorphic to the group cohomology of Γ. Also, if P is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, then by definition

$$H_*\big(\Gamma, Sh_*(A) \otimes_{\mathbb{Z}} M\big) = H_*\big(P \otimes_{\Gamma} \big(Sh_*(A) \otimes_{\mathbb{Z}} M\big)\big)$$

(well-defined up to canonical isomorphism). From Theorems 2 and 5 and from the standard spectral sequences of a double complex (e.g. [Bro94, p. 169]), we obtain spectral sequences

$$E^1_{p,q} = H_q(\Gamma, C_p \otimes_{\mathbb{Z}} M) \quad \Rightarrow \quad H_*\big(\Gamma, St_n(A) \otimes_{\mathbb{Z}} M\big),$$

where $C_* = \mathcal{C}'_*(A)$ or $Sh_*(A)$.

Theorem 7. *Assume that $A^\times$ is finite of order o and let d be the product of all m such that Γ has a subgroup of order m. Assume that d is finite. Suppose that multiplication by the greatest common divisor $\gcd(o,d)$ is invertible on M. Then we have isomorphisms*

$$H_*(\Gamma, C_* \otimes_{\mathbb{Z}} M) \approx H_*\big(\Gamma, St_n(A) \otimes_{\mathbb{Z}} M\big)$$

where $C_* = \mathcal{C}'_*(A)$ or $Sh_*(A)$.

Proof. The chain map $\mathcal{C}'_*(A) \xrightarrow{f} Sh_*(A)$ is a weak equivalence. By [Bro94, Proposition VII.5.2], the map f induces an isomorphism on the homology of Γ with those coefficients. So it suffices to prove the theorem for $C_* = \mathcal{C}'_*(A)$.

Use the notation of the proof of Theorem 5. A basis for $\mathcal{C}'_k(A)$ is given by tuples $B = (\ell_1, \ldots, \ell_{n+k})$ such that the span of the lines $\ell_1, \ldots, \ell_{n+k}$ is all of F^n. Let $Stab_B$ be the stabilizer in Γ of B and let s_B be its order. Naturally, s_B divides d.

We claim that s_B also divides o^N for some sufficiently large integer N. Let $\gamma \in Stab_B$. Then $\ell_i\gamma = \ell_i$ for all i. Therefore, for each i there exists $a_i \in A^\times$ such that $v_i\gamma = a_i v_i$. The v_i span F^n. It follows that $\gamma^o = 1$. So any prime dividing the order of any element of $Stab_B$ divides o and the claim follows.

The Γ-module $\mathcal{C}'_k(A)$ is a direct sum of induced representations, each induced from $Stab_B$ for some B. By Shapiro's lemma, $H_q(\Gamma, \mathcal{C}'_p(A) \otimes M)$ is a direct sum of groups equal to $H_q(Stab_B, M)$ for various B. Since s_B is invertible on M, these groups vanish when $q > 0$. Thus the terms in the spectral sequence are 0 except when $q = 0$, which implies that the spectral sequence degenerates at E_2. This completes the proof. □

Since $\mathbb{Z}^\times = \{\pm 1\}$, Theorem 7 yields the following:

Corollary 8. *Let $A = \mathbb{Z}$. For any $\Gamma \subset GL(n,\mathbb{Z})$ and any coefficient module M in which 2 is invertible, the sharbly homology is isomorphic to the Steinberg homology.*

It follows from Corollary 8 that (a) and (d) of Conjecture 5 in [AGM11] are equivalent, if $p \neq 2$.

5. The Voronoi complex and its variants

From now on, we put $F = \mathbb{Q}$ and $A = \mathbb{Z}$. Let X_n^0 be the space of positive definite real $n \times n$ symmetric matrices. It is an open cone in the vector space Y_n^0 of all real $n \times n$ symmetric matrices. For each nonzero subspace W of $\mathbb{R}^n$ defined over $\mathbb{Q}$, set $b(W)$ to be the rational boundary component of X_n^0 consisting of the cone of all positive semidefinite real $n \times n$ symmetric matrices whose kernel is W. The (minimal) Satake bordification $(X_n^0)^*$ of X_n^0 is the union of X_n^0 with all the rational boundary components. It is convex and hence contractible.

Lemma 9. *Let $n \geqslant 2$. If $k \neq n-1$, $\widetilde{H}_k(\partial X_n^*) = 0$ and $\widetilde{H}_{n-1}(\partial X_n^*) \approx St_n(\mathbb{Z})$.*

Proof. Let W be a nonzero, proper subspace of $\mathbb{Q}^n$, thus a vertex of the Tits building $T_n(\mathbb{Z})$. Let $X(W)$ is the subset of X_n^* consisting of all semidefinite symmetric matrices whose kernel contains $W^\perp$. Note that $X(W)$ is homeomorphic to $X^*_{\dim W}$, and hence contractible. Then ∂X_n^* is covered by the set of $X(H)$ where H runs over hyperplanes of $\mathbb{Q}^n$. Also, if $H_1, \ldots, H_j$ are hyperplanes, then $X(H_1) \cap \cdots \cap X(H_j)$ is nonempty if and only if $H_1 \cap \cdots \cap H_j \neq \{0\}$, in which case $X(H_1) \cap \cdots \cap X(H_j) = X_{H_1 \cap \cdots \cap H_j}$. Thus, the $X(H)$ form an acyclic covering of ∂X_n^* whose nerve is $T_n(\mathbb{Z})$. The result now follows from the spectral sequence for a covering, see e.g. Theorem VII.4.4, p. 168 of [Bro94]. □

If $n \leqslant 4$, we will produce a resolution of $St_n(\mathbb{Z})$ based on the Voronoi decomposition of $(X_n^0)^*$. Expositions of the Voronoi decomposition are given in [AMRT10, II.6] and [Ste07, Appendix].

The positive real numbers $\mathbb{R}_+^\times$ act on $(X_n^0)^*$ by homotheties. Set $X_n^* = (X_n^0)^*/\mathbb{R}_+^\times$.

If $v \in \mathbb{Z}^n$ is a unimodular row vector, then ${}^t v v$ is a rank 1 matrix in $(X_n^0)^*$ and thus generates a rational boundary component. If $v_1, \dots, v_m$ are m such vectors, we let $\sigma(v_1, \dots, v_m)$ denote the image of the closed convex conical hull of ${}^t v_1 v_1, \dots, {}^t v_m v_m$ in X_n^*.

The Voronoi decomposition of X_n^* is the cellulation of X_n^* by the cells $\sigma_Q = \sigma(v_1, \dots, v_m)$, where Q runs over all positive definite real $n \times n$ quadratic forms, and where the nonzero integral vectors that minimize Q over all integral vectors are exactly $\pm v_1, \dots, \pm v_m$. This includes the cells in the rational boundary components of X_n^*. There is a right action of $GL(n, \mathbb{Z})$ on X_n^* induced by the action on X_n^0: $x \cdot \gamma = {}^t\gamma x \gamma$. The Voronoi decomposition is stable under this action.

A basic fact is that there are a finite number of Voronoi cells modulo $SL(n, \mathbb{Z})$. We will later need to refer to the representatives of the $SL(n, \mathbb{Z})$-orbits of some of the Voronoi cells for $n = 3, 4$ as tabulated in [McC91]. For $n \leqslant 4$, there is only one Voronoi cell (modulo $SL(n, \mathbb{Z})$) that is not a simplex, and that occurs in the top dimension when $n = 4$.

Let V_n denote X_n^* considered as a cell complex, with the Voronoi cellulation. Denote by $\mathbb{Z}V_*$ the oriented chain complex of V_n. That is, we fix an orientation on each cell of V_n. Then $(\mathbb{Z}V)_r$ is the free abelian group generated by the oriented cells of dimension r, and the boundary map from $(\mathbb{Z}V)_r$ to $(\mathbb{Z}V)_{r-1}$ sends an oriented r-cell to the linear combination of the $(r-1)$-cells on its boundary that keeps track of the chosen orientations.

A Voronoi cell $\sigma(v_1, \dots, v_k)$ lies in a boundary component of X_n^* if and only if $v_1, \dots, v_k$ do not span $\mathbb{Q}^n$. Let ∂X_n^* denote the union of all the boundary components, and let ∂V_n denote ∂X_n^* considered as a cell complex, with the Voronoi cellulation. Denote by $\mathbb{Z}\partial V_*$ the oriented chain complex of ∂V_n.

Let $\mathcal{V}_r = \mathbb{Z}V_r / \mathbb{Z}\partial V_r$. Write $((v_1, \dots, v_k))$ for the generator in $\mathcal{V}_*$ corresponding to the cell $\sigma(v_1, \dots, v_k)$. The boundary maps in all these chain complexes are induced by the boundary maps on the Voronoi cells.

Definition 10. For any coefficient module M and any subgroup $\Gamma \subset GL(n, \mathbb{Z})$, the *Voronoi homology* of Γ with coefficients in M is defined to be the homology $H_*(\Gamma, \mathcal{V}_* \otimes_{\mathbb{Z}} M)$.

Now let $n \leqslant 4$. Then there is only one $GL(n, \mathbb{Z})$-orbit of $(n-1)$-dimensional cells in V_n. (See [McC91].) They are all of the form $\sigma(w_1, \dots, w_n)$, where $w_1, \dots, w_n$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. We define a linear map $\phi : \mathcal{V}_{n-1} \to St_n(\mathbb{Z})$ by sending

$$\phi\big((w_1, \dots, w_n)\big) \mapsto [\![w_1, \dots, w_n]\!]. \tag{3}$$

This is clearly $GL(n, \mathbb{Z})$-equivariant.

Theorem 11. *Let $n \leqslant 4$. Then*

$$0 \to \mathcal{V}_{n(n+1)/2-1} \to \cdots \to \mathcal{V}_k \to \mathcal{V}_{k-1} \to \cdots \to \mathcal{V}_{n-1} \xrightarrow{\phi} St_n(\mathbb{Z}) \to 0 \tag{4}$$

is an exact sequence of $GL(n, \mathbb{Z})$-modules.

Proof. By Lemma 9 we know that $\widetilde{H}_k(\partial X_n^*) \approx St_n(\mathbb{Z})$ for $k = n-1$ and vanishes in all other degrees. Thus, the exact homology sequence of the pair $(X_n^*, \partial X_n^*)$ and the contractibility of X_n^* imply that $\widetilde{H}_k(\mathcal{V}_*) \approx \widetilde{H}_k(\partial X_n^*)$ for all k. The $(n-2)$-skeleton of X_n^* lies in ∂X_n^*. Therefore $\mathcal{V}_k = 0$ for $k \leqslant n-2$. Thus all of $\mathcal{V}_{n-1}$ consists of cycles, and its quotient by the image of the differential from $\mathcal{V}_n$ is isomorphic to $\widetilde{H}_{n-1}(\partial X_n^*) \approx St_n(\mathbb{Z})$.

Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{Q}^n$. Following out the isomorphisms in the paragraph above, we see that $((e_1, \dots, e_n)) \in \mathcal{V}_{n-1}$ is sent to the class of the submanifold of positive semidefinite diagonal matrices in ∂X_n^*, which goes to the modular symbol $[\![I_n]\!]$ in $St_n(\mathbb{Z})$. Therefore, using $GL(n, \mathbb{Z})$-equivariance, the composite is the map ϕ from (3) as desired.

Since $\widetilde{H}_k(\partial X_n^*)$ vanishes for $k \geqslant n$, the rest of the sequence (4) is exact. □

Corollary 12. *Let $n \leqslant 4$ and $\Gamma \subset GL(n,\mathbb{Z})$. Let d be the greatest common divisor of the orders of the finite subgroups of Γ. For any coefficient module M on which multiplication by d is invertible, we have isomorphisms*

$$H_*(\Gamma, \mathcal{V}_* \otimes_{\mathbb{Z}} M) \approx H_*\big(\Gamma, St_n(\mathbb{Z}) \otimes_{\mathbb{Z}} M\big).$$

Proof. The proof runs along the same lines as the proof of Theorem 7. □

6. Voronoi sharbly homology classes and Hecke eigenvalues

For the remainder of this article, we take $n = 3$ or 4. We set $m = 3$ if $n = 3$ and $m = 5$ if $n = 4$. Then every Voronoi cell of dimension $\leqslant m$ is a simplex. To simplify notation, we regrade the Voronoi complex and put $\mathcal{W}_k = \mathcal{V}_{k+n-1}$. We also drop the $\mathbb{Z}$ from the notation for $\mathcal{C}'$, Sh and St_n and omit subscripts for complexes.

For $0 \leqslant k \leqslant m$, define the map of $\mathbb{Z}[GL(n,\mathbb{Z})]$-modules

$$\theta_k : \mathcal{W}_k \to Sh_k$$

as follows: if $\sigma(v_1,\dots,v_{k+n})$ is a Voronoi cell, it is in fact a simplex, and we set $\theta_k((v_1,\dots,v_{k+n})) = [v_1,\dots,v_{k+n}]$. Note that θ_0 commutes with the maps to St_n.

In [AGM11], where $n = 4$, we wished to compute the homology $H_1(Sh)$ (as a Hecke module) of the complex

$$Sh_2 \otimes_{\mathbb{Z}[\Gamma]} M \to Sh_1 \otimes_{\mathbb{Z}[\Gamma]} M \to Sh_0 \otimes_{\mathbb{Z}[\Gamma]} M$$

with Γ a congruence subgroup of $SL(4,\mathbb{Z})$ and $M = \mathbb{Z}$ with trivial Γ-action. Similar computations are anticipated for $n = 3$ in the near future.

What we actually computed in [AGM11] was the homology $H_1(\mathcal{W})$ of

$$\mathcal{W}_2 \otimes_{\mathbb{Z}[\Gamma]} M \to \mathcal{W}_1 \otimes_{\mathbb{Z}[\Gamma]} M \to \mathcal{W}_0 \otimes_{\mathbb{Z}[\Gamma]} M.$$

Let T be a Hecke operator and $\{x_i\}$ a basis of $H_1(\mathcal{W})$. Using the algorithm in [Gun00], we found elements y_i in $H_1(\mathcal{W})$ homologous in $H_1(Sh)$ to $\theta_{1,*}(x_i)$. Then we found the eigenvalues of the linear map that sends x_i to y_i. We did this because we do not have a good way to compute the action of T directly on the Voronoi homology. The reason is that a direct computation of T would require acting by integral matrices with determinant greater than 1, and such matrices do not stabilize the Voronoi decomposition. See [AGM11] for more details.

In [AGM11, §5], we stated that there would be a problem if $\theta_{1,*}$ is not injective, for then some of the Hecke "eigenvalues" we computed would be meaningless. However, this was not accurate. As far as we know now, $\theta_{1,*}$ could fail to be injective. (It would be interesting to decide this point.) Nevertheless, the Hecke eigenvalues we computed in [AGM11, §5] are in fact actual eigenvalues in the sharbly homology as defined in Definition 7. The rest of this section is devoted to proving this fact.

We refer to Lemma I.7.4 in [Bro94] as FLHA, or the fundamental lemma of homological algebra.

Let A be a ring and M a right $A[\Gamma]$-module. Let K stand for one of the complexes $\mathcal{C}$, $\mathcal{W}$ or Sh. Each of these is a resolution of St. We give $K \otimes_{\mathbb{Z}} M$ the diagonal Γ-action. Let F be a resolution of $\mathbb{Z}$ by free Γ-modules. Form the double complex

$$D(A) := F \otimes_{\mathbb{Z}[\Gamma]} (K \otimes_{\mathbb{Z}} M).$$

It is an A-module through the A-action on M. We have $D_{pq}(K) = F_q \otimes_{\mathbb{Z}[\Gamma]} (K_p \otimes_{\mathbb{Z}} M)$. The boundary maps are denoted as $\partial_1 : D_{pq}(K) \to D_{p-1,q}(K)$ and $\partial_2 : D_{pq}(K) \to D_{p,q-1}(K)$. The total differential ∂ on $D_{pq}(K)$ is given by $\partial = \partial_1 + (-1)^p \partial_2$. Then $H(K) := H(\Gamma, K \otimes_{\mathbb{Z}} M)$ is the total homology of $D(K)$.

Referring for example to Section 5 of Chapter VII of [Bro94], we have two spectral sequences that both abut to $H(K)$. We will refer to the "first spectral sequence" as the one whose E^2 page is

$$_{I}E^2_{pq}(K) = H_q\big(\Gamma, H_p(K)\big) \quad \Rightarrow \quad H_{p+q}(K),$$

and to the "second spectral sequence" as the one whose E^1 page is

$$_{II}E^1_{pq}(K) = H_q\big(\Gamma, C_p(K)\big) \quad \Rightarrow \quad H_{p+q}(K).$$

Let S be a subsemigroup of $GL(n, \mathbb{Q})$ such that (Γ, S) is a Hecke pair. From now on we assume that M is a right S-module and that F is a resolution of $\mathbb{Z}$ consisting of S-modules. Consider $s \in S$ and T the Hecke operator $\Gamma s \Gamma$. Then the action of T on $H(K)$ can be computed on the chain level as follows.

Write $\Gamma s \Gamma = \coprod s_i \Gamma$ as a finite disjoint union of single cosets, with $s_i \in S$. If x is a cycle in $(D(K), \partial)$, denote the class in $H(K)$ that it represents by $[x]$. Then $[\sum x s_i] = [x]T$. This gives us a non-canonical lifting of T to the chain level. In other words, write $\mathbf{s} = \sum s_i$ and view it as an operator on the right on $D(K)$. Then $\mathbf{s}$ commutes with the boundary operators and $[x]T = [x\mathbf{s}]$ for any cycle x.

Let $\tau_m K$ be the truncation of K at degree m, i.e. the complex $K_m \to K_{m-1} \to \cdots \to K_0$. The maps θ_k above define a map of complexes $\theta : \tau_m \mathcal{W} \to \tau_m Sh$. Since $\mathcal{C}$ consists of free $\mathbb{Z}[\Gamma]$-modules, the FLHA gives us a map of $\mathbb{Z}[\Gamma]$-complexes $\phi_{\mathcal{W}} : \mathcal{C} \to \mathcal{W}$ that commutes with the augmentation maps to St. We obtain the map of $\mathbb{Z}[\Gamma]$-complexes $\theta \circ \phi_{\mathcal{W}} : \tau_m \mathcal{C} \to \tau_m Sh$. Using the FLHA, we can extend this to a map of $\mathbb{Z}[\Gamma]$-complexes $\phi_{Sh} : \mathcal{C} \to Sh$ so that $(\phi_{Sh})_k = \theta \circ (\phi_{\mathcal{W}})_k$ if $k \leqslant m$. Again, ϕ_{Sh} commutes with the augmentation maps to St.

It follows that $\phi_{\mathcal{W}}$ and ϕ_{Sh} are weak equivalences of Γ-chain complexes. Therefore, by Proposition VII.5.2 in [Bro94] (which uses the first spectral sequence), they induce isomorphisms $H(\mathcal{C}) \xrightarrow{\sim} H(\mathcal{W})$ and $H(\mathcal{C}) \xrightarrow{\sim} H(Sh)$ respectively.

Theorem 13. *Let $x \in D_{1,0}(\mathcal{W})$ be a chain such that $\partial_1(x) = 0$ and $[x] \in {}_{II}E^2_{1,0}(\mathcal{W}) - \{0\}$. Let T be a Hecke operator, $a \in K$ and assume that $[\theta(x)]T = a[\theta(x)]$. Then there exists $\xi \in D_{0,1}(Sh)$ such that* (i) *$\xi + \theta(x)$ is a cycle in $D_1(Sh)$ representing a nonzero class $z \in H_1(Sh)$, and* (ii) *$zT = az$.*

Proof. Since ${}_{II}E^2_{1,0}(\mathcal{W}) = {}_{II}E^\infty_{1,0}(\mathcal{W})$, we know that $[x]$ persists nonzero to ${}_{II}E^\infty(\mathcal{W})$. In other words there exists $\eta \in D_{0,1}(\mathcal{W})$ such that $\eta + x$ is a cycle in $D_1(\mathcal{W})$ and represents a nonzero class in $H_1(\mathcal{W})$. By the paragraph preceding the theorem, we obtain that $\theta_*(\eta + x) = \phi_{Sh,*} \circ \phi^{-1}_{\mathcal{W},*}(\eta + x)$ is a cycle in $D_1(Sh)$ and represents a nonzero class in $H_1(Sh)$.

We take $\xi = \theta_*(\eta)$. Then $\xi + \theta(x)$ is a cycle in $D_1(Sh)$ and it represents a nonzero class in $H_1(Sh)$. This proves assertion (i). Letting z denote this class, we compute zT on the chain level. It may help to refer to the following diagrams. In $D(\mathcal{W})$ we have our cycle

$$\begin{array}{ccc} \eta & \bullet & \\ \\ \bullet & x & \bullet \end{array}$$

so that $\partial_2 \eta = -\partial_1 x$. Mapping this to $D(Sh)$ we have the cycle

$$\begin{array}{ccc} \xi & \bullet & \\ \\ \bullet & \theta(x) & \bullet \end{array}$$

so that $\partial_2 \xi = -\partial_1 \theta(x)$.

By hypothesis, there exists $u \in D_{11}(Sh)$ and $y \in D_{20}(Sh)$ such that

$$\partial_1 y + \theta(x)\mathbf{s} - \partial_2 u = a\theta(x)$$

on the chain level. It follows that

$$\partial_2(a\xi) = -a\partial_1\theta(x) = -\partial_1\theta(x)\mathbf{s} + \partial_1\partial_2 u = \partial_2\xi\mathbf{s} + \partial_2\partial_1 u.$$

Since the columns of D are exact, there exists $w \in D_{02}(Sh)$ such that

$$a\xi = \xi\mathbf{s} + \partial_1 u + \partial_2 w.$$

In pictures,

w

$\xi\mathbf{s}$ u

$\bullet$ $\theta(x)\mathbf{s}$ y

shows that $(\xi + \theta(x))\mathbf{s}$ is homologous to $a(\xi + \theta(x))$ in $H_1(Sh)$. This proves (ii) and completes the proof of the theorem. □

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