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Modules over Crossed Products

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J. T. Stafford (1978, *J. London Math. Soc.* (2) **18**, 429–442) proved that any left ideal of the Weyl algebra $A_n(K)$ over a field K of characteristic 0 can be generated by two elements. In general, there is the problem of determining whether any left ideal of a Noetherian simple domain can be generated by two elements. In this work we show that this property holds for some crossed products of a simple ring with a supersolvable group and also for the tensor product of generalized Weyl algebras. We also prove that these rings are stably generated by 2 elements and that their finitely generated torsion left modules can be generated by two elements. Some results about stably 2-generated rings were found by V. A. Artamonov (1994, *Math. Sb.* **185**, No. 7, 3–12).

1. PRELIMINARIES

All rings considered throughout this paper will have an identity. Also, we denote by $U(S)$ the unit group of a ring S and by S^* the set of non-zero elements of S.

DEFINITION [5]. Let S be a ring and G a group. A *crossed product* of S and G is a G-graded ring

$$A = S * G = \bigoplus_{g \in G} A_g$$

such that $A_e = S$ (e is the identity element in G) and, for any $g \in G$, there is an element $\bar{g} \in A_g$ which is a unit in A.

Note that, under the conditions of the definition, $A_g = S\bar{g} = \bar{g}S$. So any element of $S * G$ can be written as $\sum_g s_g \bar{g}$, where the elements s_g belong

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to S and only finitely many of them are non-zero. Thus in order to describe the multiplication in $S * G$ it suffices to give the products of the form $(s_g \bar{g})(s_h \bar{h})$. For any element $g \in G$ we define an automorphism of S by means of $\tau_g(s) = \bar{g} s \bar{g}^{-1}$, so that the formula $\bar{g}s = \tau_g(s)\bar{g}$ holds for every $s \in S$ and $g \in G$. On the other hand, if $g, h \in G$ we define the element $\alpha_{g,h} = \bar{g}\bar{h}(\overline{gh})^{-1} \in U(S)$, so that $\bar{g}\bar{h} = \alpha_{g,h}\overline{gh}$. Then we have

$$(s_g \bar{g})(s_h \bar{h}) = s_g \tau_g(s_h)\alpha_{g,h}\overline{gh}.$$

We say that the maps $\tau\colon G \to \operatorname{Aut} S$ and $\alpha\colon G \times G \to U(S)$ given by $\tau(g) = \tau_g$ and $\alpha(g,h) = \alpha_{g,h}$ form a *crossed system* of the crossed product $S * G$.

If U is a subring of S we denote

$$\operatorname{Fix}(U) = \left\{ g \in G \,\middle|\, \text{there exists } s \in S \text{ such that } us = s\tau_g(u) \text{ for all } u \in U \right\}.$$

In the next sections, we shall use several times Propositions 1.1, 1.2, and 1.3, sometimes without mentioning them explicitly.

PROPOSITION 1.1 [7, Proposition 1.1.6]. *Let S be a left Noetherian ring and G a supersolvable group. Then $S * G$ is also left Noetherian.*

Throughout this paper $\mathcal{K}_l(S)$ is the left Krull dimension of S. We shall say that a left module has *finite length* if it is both left Noetherian and left Artinian. Recall that this happens when the module has a composition series. In this case, the length of a composition series will be called the *length* of the module. It is clear that if S is a ring and for any non-zero left ideal I of S the module S/I has finite length, then S is left Noetherian and $\mathcal{K}_l(S) \leq 1$. In the next result we see that the converse is true when S has no zero-divisors.

PROPOSITION 1.2 [6, Lemma 6.3.9]. *Suppose that S is a left Noetherian ring without zero-divisors and $\mathcal{K}_l(S) = 1$. Then for any non-zero left ideal I of S, S/I has finite length.*

PROPOSITION 1.3 [8]. *Let M be a left semisimple S-module, $B = \operatorname{End}_S(M)$ and $\phi \in \operatorname{End}_B(M)$. Then for every $u_1, u_2, \dots, u_n \in M$ there exists $x \in S$ such that $\phi(u_i) = xu_i$.*

We say that a multiplicative subset $U \subseteq S^*$ has the *left Ore condition* in S if U has no zero-divisors and for every $u \in U$, $s \in S$ there are $v \in U$, $t \in S$ such that $vs = tu$. This condition permits the construction of the left ring of quotients $U^{-1}S$. The *right Ore condition* for U in S is defined symmetrically and enables us to construct SU^{-1}. As is well known, if U satisfies both Ore conditions then the corresponding left and right rings of quotients of S are isomorphic: $U^{-1}S \cong SU^{-1}$. We say that the ring S has the left (or right) Ore condition if that condition holds for S^*. It follows from Goldie's Theorem that a left Noetherian ring without zero-divisors necessarily has the left Ore condition. As a consequence of this result and Propositions 1.1 we obtain the following result, which shall be used freely in the paper: if S is a left Noetherian ring, G is a supersolvable group, and $S * G$ has no zero-divisors, then $S * G$ satisfies the left Ore condition.

If α is an automorphism of S, we denote by $S[x, x^{-1}, \alpha]$ the Ore extension of $S[x, \alpha]$ localized at the powers of x.

Next we introduce the concept of an almost simple ring, which appears naturally in the course of some proofs.

DEFINITION. A ring S is called *almost simple* if $I \cap Z(S) \neq \{0\}$ for any two-sided ideal $I \neq \{0\}$ of S, where $Z(S)$ denotes the center of S.

It is clear that if S is almost simple and has no zero-divisors, then the ring of quotients $(Z(S)^*)^{-1}S$ is simple.

2. CROSSED PRODUCTS

The main results of this section are Theorem 2.1 and Corollary 2.19.

THEOREM 2.1. *Let T be either a division ring or a left Noetherian simple ring with the right Ore condition and $\mathcal{K}_l(T) = 1$, G a supersolvable group, and $C = T * G$ a crossed product of T and G. Suppose that C is simple and has no zero-divisors. Then for any $a, b, c \in C$ and $s_1, s_2 \in C^*$ there are $f_1, f_2 \in C$ such that $a, b, c \in C(a + s_1 f_1 c) + C(b + s_2 f_2 c)$.*

This theorem is based on the following proposition:

PROPOSITION 2.2. *Let S_1 be an almost simple ring, G a supersolvable group, and $C = S_1 * G$ a crossed product of S_1 and G. Consider a multiplicative subset $U \subseteq S_1^*$ satisfying both Ore conditions in S_1 and let $S = U^{-1}S_1$ be the corresponding ring of quotients. Suppose that the following conditions hold*:

(i) *C is simple and has no zero-divisors.*

(ii) *For any $g \in G$ and $u \in U$, $\tau_g(u) \in U$.*

(iii) *S is left Noetherian and $\mathcal{K}_l(S) = 1$.*

Define now $A = U^{-1}C$, which is a crossed product of S and G. Then for any left ideal $I \neq \{0\}$ of S and any $u \in A$, $0 \neq v \in A$, there exists $f \in C$ such that $AI + A(u + vf) = A$.

We prove Proposition 2.2 arguing in three steps, which correspond to Subsections 2.1, 2.2, and 2.3.

Step 1. We prove the existence of f in A, satisfying $AI + A(u + vf) = A$. In fact, it follows immediately from the next proposition:

PROPOSITION 2.3. *Let $A = S * G$ be a crossed product of a left Noetherian ring S with $\mathscr{K}_l(S) = 1$ and a supersolvable group G. We suppose that A is simple and has no zero-divisors. Then if $I \neq \{0\}$ is a left ideal of S, for any $u \in A$ and $0 \neq v \in A$ there exists $f \in A$ such that $AI + A(u + vf) = A$.*

Step 2. We prove the existence of f in C, satisfying $AI + A(u + vf) = A$, in the case S_1 is simple.

Step 3. We finish the proof of Proposition 2.2 in the general case.

In order to derive Theorem 2.1 from Proposition 2.2 we need to prove that some particular rings are almost simple. We do this in Subsection 2.4, where we provide several families of almost simple rings. Finally, in Subsection 2.5 we prove Theorem 2.1 and its corollary.

2.1. *Step* 1

Throughout this subsection, we maintain the notation given in the statement of Proposition 2.3. We begin by considering the case in which I is a maximal left ideal of S, which is the key to the proof in the general case. We have that $AI = \oplus_{g \in G} \bar{g} I = \oplus_{g \in G} \tau_g(I)\bar{g}$ is a G-graded left ideal of A and A/AI is a G-graded A-module. If we consider A/AI as an S-module then

$$A/AI = \bigoplus_{g \in G} S\bar{g}/\tau_g(I)\bar{g} \cong \bigoplus_{g \in G} S/\tau_g(I).$$

Thus A/AI is a sum of simple S-modules and we have the following result.

LEMMA 2.4. *A/AI is a semisimple S-module.*

For any ring R, if M_1 and M_2 are two left R-modules and $M_1 = Rm$ is cyclic, $\mathrm{Hom}_R(M_1, M_2)$ can be embedded in M_2 via the $\mathbb{Z}$-module homomorphism f: $\mathrm{Hom}_R(M_1, M_2) \to M_2$ defined by $f(\phi) = (m)\phi$, and we can identify the homomorphisms in $\mathrm{Hom}_R(M_1, M_2)$ with their images. In particular, if we define $M = A/AI$ and $A_1 = \mathrm{End}_A(M)$, we identify any endomorphism $\phi \in A_1$ with its image $(1 + AI)\phi \in M$. Under this identification,

$$A_1 = \{a + AI \mid Ia \subseteq AI\}. \tag{1}$$

We also define $F_1 = \mathrm{End}_S(S/I)$ and make a similar identification of the elements of F_1 with the corresponding elements in S/I.

LEMMA 2.5. *A_1 is isomorphic to a crossed product $F_1 * H$ of the division ring F_1 and a subgroup H of G. In particular A_1 is left and right Noetherian.*

Proof. Let $\sum_i s_i\bar{g}_i + AI \in A_1$. From (1) and AI being G-graded, we get that $s_i\bar{g}_i + AI \in A_1$ for any i, so that

$$A_1 = \bigoplus_{g\in G} (A_1)_g, \qquad \text{where } (A_1)_g = A_1 \cap (A/AI)_g = \{s\bar{g} + AI \in A_1\}.$$

We rule out the trivial components in this decomposition by defining

$$\begin{aligned} H &= \{g \in G \mid (A_1)_g \neq 0\} \\ &= \{g \in G \mid \exists s \in S \setminus \tau_g(I) \text{ such that } s\bar{g} + AI \in A_1\}. \end{aligned}$$

Then $A_1 = \oplus_{h\in H}(A_1)_h$ as an abelian group. We have to prove that H is a subgroup of G and that this decomposition is an H-graduation of A_1.

First of all, we see that any non-zero element $s\bar{h} + AI \in (A_1)_h$ is a unit in A_1. Indeed, we have $s \in S \setminus \tau_h(I)$ and $Is\bar{h} \subseteq AI$. Then $Is \subseteq \tau_h(I)$ and so $s + \tau_h(I) \in \mathrm{Hom}_S(S/I, S/\tau_h(I))$. Since S/I and $S/\tau_h(I)$ are two simple S-modules, it follows that $s + \tau_h(I)$ is an isomorphism and we can consider its inverse $r + I \in \mathrm{Hom}_S(S/\tau_h(I), S/I)$. Then $\tau_h(I)r \subseteq I$, $sr \equiv 1 \pmod I$ and $rs \equiv 1 \pmod{\tau_h(I)}$. Now since $I(\bar{h})^{-1}r \subseteq A\tau_h(I)r \subseteq AI$, we have $(\bar{h})^{-1}r + AI \in A_1$ and it is the inverse of $s\bar{h} + AI$ in A_1. We also deduce that $(A_1)_{h^{-1}} \neq \{0\}$ and $h^{-1} \in H$.

On the other hand, let $h_1, h_2 \in H$, $s_1 \in S \setminus \tau_{h_1}(I)$, $s_2 \in S \setminus \tau_{h_2}(I)$ and suppose that $a_1 = s_1\bar{h}_1 + AI \in (A_1)_{h_1}$ and $a_2 = s_2\bar{h}_2 + AI \in (A_1)_{h_2}$. Then

$$(1 + AI)(a_1a_2) = s_1\tau_{h_1}(s_2)\left(\bar{h}_1\bar{h}_2\right) + AI \in (A_1)_{h_1h_2}.$$

Note that this element is not 0, since it is a product of two units in A_1. This proves that $h_1h_2 \in H$ and that the decomposition $A_1 = \oplus_{h\in H}(A_1)_h$ is an H-graduation.

Finally, it can be easily checked that $(A_1)_e \cong F_1$ and so A_1 is isomorphic to a crossed product of the form $F_1 * H$. ∎

We define $B = \mathrm{End}_S(M)$.

LEMMA 2.6. *Let $\phi \in B$. Then for any $g \in G$ there exists $a \in A_1$ such that $(s\bar{g} + AI)\phi = (s\bar{g})a$ for all $s \in S$.*

Proof. We have $(s\bar{g} + AI)\phi = s(\bar{g} + AI)\phi = (s\bar{g})a$, where

$$a = (\bar{g})^{-1}(\bar{g} + AI)\phi.$$

Since $\tau_g(I)\bar{g} \subseteq AI$, it follows that $\tau_g(I)(\bar{g} + AI)\phi \subseteq AI$, whence $I(\bar{g})^{-1} \cdot (\bar{g} + AI)\phi \subseteq AI$ and we conclude that $a = (\bar{g})^{-1}(\bar{g} + AI)\phi \in A_1$. ∎

In the next lemma, we regard M as a right B- and A_1-module.

LEMMA 2.7. *M can be decomposed as a direct sum of submodules M_i, where each M_i is of the form $M_i = (s_i\bar{g}_i + AI)B = (s_i\bar{g}_i + AI)A_1$ for some $s_i \in S$ and $g_i \in G$.*

Proof. We consider a maximal direct sum of submodules M_i, each of them generated over B by some $s_i\bar{g}_i + AI$. We suppose that there exist $s \in S$, $g \in G$, and $b \in B$ such that $s\bar{g} + AI \notin \oplus_i M_i$ but $0 \neq (s\bar{g} + AI)b \in \oplus_i M_i$. By Lemma 2.6, this means that there exist non-zero elements $a, a_i \in A_1$ such that

$$(s\bar{g})a = \sum_i (s_i\bar{g}_i)a_i. \tag{2}$$

We can decompose the elements a, a_i according to the graduation of M and each homogeneous component will be an element in A_1. Let a_h be a non-zero homogeneous component of a. If we compare the gh-components on both sides of equality (2) we get that

$$(s\bar{g})a_h = \sum_i (s_i\bar{g}_i)c_i,$$

for some $c_i \in A_1$. Since a_h is invertible in A_1, it follows that $s\bar{g} + AI \in \oplus_i M_i$, which is a contradiction. This proves that $M = \oplus_i M_i$. ∎

LEMMA 2.8. *Let $m = \sum_i m_i$ be an element of M written according to the decomposition of Lemma 2.7. Then there exist elements $p_i \in S$ such that $p_i m = m_i$.*

Proof. Write $M_B = \oplus_i M_i$ as in Lemma 2.7. We denote by ϕ_i the projection of M over M_i. Since $\phi_i \in \operatorname{End}_B(M)$, we can apply Proposition 1.3 and find elements $p_i \in S$ such that $p_i m = \phi_i(m) = m_i$. ∎

LEMMA 2.9. *Proposition 2.3 holds if I is a maximal left ideal of S.*

Proof. As we have mentioned in the Introduction, A satisfies the left Ore condition: for any $0 \neq a$ and $0 \neq b \in A$, it holds $Aa \cap Ab \neq \{0\}$. Then if $u \neq 0$, there exists $t \neq 0$ such that $tu \in AI$. Note that if we prove the existence of an element f such that $AI + Atvf = A$, then we have $AI + A(u + vf) = A$. Thus, without loss of generality, we can suppose that $u = 0$.

We decompose $M_{A_1} = \oplus_i M_i$ as in Lemma 2.7. Then we can write each $m \in M$ in the form $m = \sum_i m_i$ with $m_i \in M_i$. If we set $\operatorname{supp} m = \{i \mid m_i \neq 0\}$, it follows from Lemma 2.8 that

$$Am = \sum_{i \in \operatorname{supp} m} Am_i. \tag{3}$$

Choose now $t \in vM$ such that At is maximal in the set $\{Am \mid m \in vM\}$. Note that we can do this because M is a left Noetherian A-module. Let $N = vM \cap (\bigoplus_{i \notin \operatorname{supp} t} M_i)$ and suppose $z \in N$. If $z \notin At$ then according to (3) we have $At \subset At + Az = A(t+z)$ and we obtain a contradiction with the maximality of At. So $N \subseteq At$. We have that

$$vM \Bigg/ \left(vM \cap \left(\bigoplus_{i \notin \operatorname{supp} t} M_i \right) \right) \cong \left(vM + \left(\bigoplus_{i \notin \operatorname{supp} t} M_i \right) \right) \Bigg/ \left(\bigoplus_{i \notin \operatorname{supp} t} M_i \right)$$

can be embedded in $\bigoplus_{i \in \operatorname{supp} t} M_i$. Hence vM/N is finitely generated over A_1 because A_1 is right Noetherian. Then, there is a finite set $\{a_j \in vM\}$ such that

$$vM = \sum_j a_j A_1 + N \subseteq \sum_j a_j A_1 + At.$$

Since $\mathscr{K}_l(S) = 1$ and M is a semisimple S-module, we can choose $0 \neq s \in S$ such that $sa_j = 0$ in M for all j and consequently $svM \subseteq At$. Since A has no zero-divisors we have $sv \neq 0$. It follows from the simplicity of A that $AsvA = A$, so

$$M = AM = AsvAM \subseteq At.$$

Thus if we write $t = vf + AI$ we obtain $AI + Avf = A$. ∎

Proof of Proposition 2.3. We proceed now by induction on the length of the S-module S/I. The case when I is maximal has been demonstrated in Lemma 2.9. We suppose now that I is not a maximal left ideal. By Proposition 1.2 we know that S/I is a left S-module of finite length and so there exists a left ideal I_2 of S such that $I \subset I_2 \subset S$ and I_2/I is a simple module over S. We have $I_2 = Sa_2 + I$ for a suitable $a_2 \in I_2$. Set $I_1 = \{t \in S \mid ta_2 \in I\}$. This is a maximal left ideal of S. The length of the module S/I_2 is smaller than that of S/I and so by the inductive hypothesis there exists $f_2 \in A$ such that

$$AI_2 + A(u + vf_2) = A.$$

Let $u_1 = u + vf_2$. We can find $t, u_2 \in A$ such that $tu_1 = u_2 a_2$ and there exists $f_1 \in A$ such that

$$AI_1 + A(u_2 + tvf_1) = A.$$

Then

$$Aa_2 \subseteq AI_1 a_2 + A(u_2 + tvf_1)a_2 \subseteq AI + At(u_1 + vf_1 a_2).$$

Hence $AI + A(u_1 + vf_1a_2)$ contains AI, Aa_2, and Au_1. But we have $A = AI_2 + A(u + vf_2) \subseteq AI + Aa_2 + Au_1$ and so

$$AI + A\big(u + v(f_2 + f_1a_2)\big) = A.$$

∎

2.2. *Step* 2

In the following we suppose that S_1 is a simple ring without zero-divisors and $C = S_1 * G$ is a crossed product of S_1 and G. Consider a multiplicative subset $U \subseteq S_1^*$ satisfying both Ore conditions in S_1 and let $S = U^{-1}S_1$ be the corresponding ring of quotients. Suppose that for any $g \in G$ and $u \in U$, $\tau_g(u) \in U$. Then we have that U satisfies both Ore conditions in C, and $U^{-1}C \cong CU^{-1}$ is a crossed product of S and G [7, Lemma 37.7].

We write $g_1 \approx g_2$ if there exists $t \in S_1$ such that $ft = t\tau_{g_1} \circ \tau_{g_2}^{-1}(f)$ for all $f \in S_1$. Set $H = \mathrm{Fix}(S_1) = \{h \in G \mid h \approx e\}$, which is a normal subgroup of G. From the definition of $\approx$ it follows that for any $h \in H$ there exists $t_h \in S_1$ such that $ft_h = t_h\tau_h(f)$ for all $f \in S_1$. Since S_1 is a simple ring, t_h is invertible. By substituting $\bar{h}$ by $t_h\bar{h}$ we obtain that $\bar{h} \in C_C(S_1)$ for any $h \in H$ (here $C_C(S_1)$ is the centralizer of S_1 in C). Obviously, $\alpha_{h,g} \in K = Z(S_1)$ for every $h, g \in H$ and it follows that $C_C(S_1)$ is isomorphic to a crossed product of K and H in a natural way [5, Proposition 2.4.1]. In the sequel, when we write $K * H$ we refer to this crossed product. It is then straightforward to check that $g_1 \approx g_2$ if and only if $g_1g_2^{-1} \in H$.

Let $v \in C$. Write v in the form $v = \sum_i s_i\bar{g}_i$, where $s_i \in S_1$. Set $T_v = \sum_{i, s_i \neq 0} S_1\bar{g}_i$, $K' = S_1 \cap Z(C)$, and let N_v be the left module over $S_1 \otimes_{K'} S_1^{\circ}$

$$\Big(\sum_i s_i \otimes p_i\Big)v = \sum_i s_ivp_i.$$

(Here S_1° denotes the opposite ring of S_1.) It can be easily seen that $N_v = S_1vS_1$.

LEMMA 2.10. *Let $0 \neq s \in S_1$ and $t \in S$. Then $N_{sv} = N_v$ and $vt \in SN_v$.*

Proof. Since S_1 is simple, then $S_1\bar{g}$ is a simple $S_1 \otimes_{K'} S_1^{\circ}$-module for any $g \in G$ and so T_v is a semisimple left $S_1 \otimes_{K'} S_1^{\circ}$-module. Suppose that there exists an isomorphism ϕ of $S_1 \otimes_{K'} S_1^{\circ}$-modules between $S_1\bar{g}_i$ and $S_1\bar{g}_j$. Then we have

$$f\phi(\bar{g}_i) = \phi(f\bar{g}_i) = \phi\Big(\bar{g}_i\tau_{g_i}^{-1}(f)\Big) = \phi(\bar{g}_i)\tau_{g_i}^{-1}(f)$$

for all $f \in S_1$. If $\phi(\bar{g}_i) = t\bar{g}_j$, it follows that $ft = t\tau_{g_j}(\tau_{g_i}^{-1}(f))$, that is, $g_j \approx g_i$. Also it can be easily checked that if $g_j \approx g_i$ then $S_1\bar{g}_i \cong S_1\bar{g}_j$.

Consequently, by changing the numbering in $\{\bar{g}_i\}$ we can write

$$T_v = \bigoplus_{i=1}^{k} T_i, \qquad \text{where } T_i = \bigoplus_{g_{i,j} \approx g_{i,1}} S_1 \bar{g}_{i,j}.$$

Thus

$$\operatorname{End}_{S_1 \otimes_K S_1^\circ} T_v = \bigoplus_{i=1}^{k} \operatorname{End}_{S_1 \otimes_K S_1^\circ} T_i.$$

Set $H = \operatorname{Fix}(S_1)$. Since $g_{i,j} g_{i,1}^{-1} \in H$, we can write v in the form $v = \sum_{i=1}^{k} v_i \bar{g}_{i,1}$, where $v_i \bar{g}_{i,1} \in T_i$ and $v_i \in S_1 * H$. According to Proposition 1.3, for any i there exists $p_i \in S_1 \otimes_{K'} S_1^\circ$ such that $p_i v = v_i \bar{g}_{i,1}$ and consequently $v_i \bar{g}_{i,1} \in V_v$.

We can represent each v_i in the form $\sum_{j=1}^{k_i} s_{i,j} l_{i,j}$, with $l_{i,j} \in K * H$ and $\{s_{i,j} \in S_1 \mid j = 1, \ldots, k_i\}$ an independent set over K. We now apply the density theorem to $S_1 \bar{g}_{i,1}$, which is a simple $S_1 \otimes_{K'} S_1^\circ$-module, to assure the existence of elements $q_{i,j} \in S_1 \otimes_{K'} S_1^\circ$ such that

$$q_{i,j}(s_{i,k} \bar{g}_{i,1}) = \begin{cases} \bar{g}_{i,1}, & \text{if } k = j; \\ 0, & \text{if } k \neq j. \end{cases}$$

It then follows that

$$q_{i,j}(v_i \bar{g}_{i,1}) = q_{i,j}\left(\sum_k s_{i,k} l_{i,k} \bar{g}_{i,1} \right) = \sum_k l_{i,k}\big(q_{i,j}(s_{i,k} \bar{g}_{i,1}) \big) = l_{i,j} \bar{g}_{i,1},$$

since $l_{i,j} \in C_C(S_1)$. Hence N_v contains the elements $l_{i,j} \bar{g}_{i,1}$ and is generated by them as an $S_1 \otimes_{K'} S_1^\circ$-module. Thus

$$N_v = \sum_{i,j} S_1 l_{i,j} \bar{g}_{i,1} S_1 = \sum_{i,j} S_1 l_{i,j} \bar{g}_{i,1}.$$

From this we can deduce the lemma. First, $N_{sv} = N_v$ for any $0 \neq s \in S_1$. This follows from the equalities $sv = \sum_{i=1}^{k} s v_i \bar{g}_{i,1}$ and $s v_i = \sum_{j=1}^{k_i} s s_{i,j} l_{i,j}$, where $\{s s_{i,j} \mid j = 1, \ldots, k_i\}$ is also an independent set over K because S_1 has no zero-divisors. Hence arguing as before we obtain

$$N_{sv} = \sum_{i,j} S_1 l_{i,j} \bar{g}_{i,1} = N_v.$$

Second, $vt \in SN_v$ for any $t \in S$, since

$$N_v t = \sum_{i,j} S_1 l_{i,j} \bar{g}_{i,1} t = \sum_{i,j} S_1 \tau_{g_{i,1}}(t) l_{i,j} \bar{g}_{i,1} \subseteq SN_v.$$

∎

Note that in the proof of the last lemma we have shown that $(CvC) \cap (K * H) \neq \{0\}$, because $l_{i,j} \in (CvC) \cap (K * H)$. Using this fact we can prove

LEMMA 2.11. *Let S_1 be a simple ring, G a finite group, and $C = S_1 * G$ a crossed product of S_1 and G. We suppose that C has no zero-divisors. Then C is simple.*

Proof. Let I be a non-zero ideal of C. Let $K = Z(S_1)$ and $H = \mathrm{Fix}(S_1)$. As H is finite and C has no zero-divisors, $K * H$ is a division ring. Then $I = K$. ∎

Suppose now in addition that $\mathscr{K}_l(S) > 0$.

LEMMA 2.12. *Let M be a submodule of the S-module ST_v such that ST_v/M has finite length. Then there exists $f \in S_1$ such that $M + SN_v = M + Svf$.*

Proof. We argue by induction on the length of the S-module $(M + SN_v)/M$. If it is not zero, then there exists $f_1 \in S_1$ such that $Svf_1 \not\subseteq M$. Choose M' such that $M \subset M' \subseteq M + Svf_1$ and M'/M is a simple S-module. As $\mathscr{K}_l(S) > 0$ we can find $0 \neq t \in S_1$ with $tvf_1 \in M$. Since $N_v = N_{tv}$, from the inductive hypothesis there exists $f_2 \in S_1$ such that

$$M + SN_v = M' + SN_v = M' + SN_{tv} = M' + Stvf_2.$$

If $M + Svf_2 = M + SN$, we are done. Otherwise, we have

$$M + Stvf_2 \subseteq M + Svf_2 \subset M' + Stvf_2$$

and

$$(M' + Stvf_2)/(M + Stvf_2) \cong M'/((M + Stvf_2) \cap M')$$

is simple. Consequently $M + Stvf_2 = M + Svf_2$. Set $N = M + Sv(f_1 + f_2) \subseteq M + SN_v$. From $tvf_1 \in M$ it follows that $tvf_2 \in N$. We then have the following chain of implications:

$$Stvf_2 \subseteq N \Rightarrow M + Stvf_2 \subseteq N \Rightarrow Svf_2 \subseteq N \Rightarrow Svf_1 \subseteq N \Rightarrow M' \subseteq N.$$

Then $M + SN_v = M' + Stvf_2 \subseteq N$. ∎

Proof of Step 2. As in Lemma 2.9, we can suppose $u = 0$ and $v \in C$. From Proposition 2.3 we derive the existence of $f = f_1 u^{-1} \in A$ with $f_1 \in C$ and $u \in U$ such that $AI + Avf_1 u^{-1} = A$. We can use the previous lemma with $vf_1 \in C$ in place of v and $M = AI \cap ST_{vf_1}$. Note that since $\mathscr{K}_l(S) = 1$ and S has no zero-divisors, ST_{vf_1}/M has finite length. Then there exists $f_2 \in S_1$ such that $SN_{vf_1} \subseteq AI + Avf_1 f_2$. But we know, by Lemma 2.10, that $vf_1 u^{-1} \in SN_{vf_1}$. Then $AI + Avf_1 f_2 = A$. ∎

2.3. *Step* 3

Proof of Proposition 2.2. We can suppose $u = 0$ and $v \in C$.

We first consider the case when I is a maximal left ideal of S. Set $K = Z(S_1)$. If $I \cap K$ contains an element $q \neq 0$, then Proposition 2.3 gives $f = f_1 s^{-1} \in A$ with $f_1 \in C$, $s \in U$ such that $Aq + Avf = A$. Since q and s commute, we have $Aq + Avf_1 = A$.

So we can suppose $I \cap K = \{0\}$. Denote $\overline{S}_1 = (K^*)^{-1} S_1$, $\overline{C} = \overline{S}_1 * G$, $\overline{S} = U^{-1}\overline{S}_1$, and $\overline{A} = \overline{S} * G$. $\overline{S}_1$ is a simple ring and, according to Step 2, there exists $f = f_1 m^{-1} \in \overline{C}$ with $f_1 \in C$, $m \in K^*$ such that $\overline{A} = \overline{A}I + \overline{A}vf$. Since I commutes with m we have $\overline{A} = \overline{A}I + \overline{A}vf_1$ and $(AI + Avf_1) \cap K \neq \{0\}$. As I is a maximal left ideal of S and $I \cap K = \{0\}$, then $AI + Avf_1 = A$.

If I is an arbitrary left ideal of S, we argue by induction on the length of the module S/I, as in Proposition 2.3. ∎

2.4. *Almost Simple Rings*

In this section we give some examples of almost simple rings. In particular, the example of Proposition 2.17, part (iii), will be needed in the proof of Theorem 2.1.

LEMMA 2.13. *Let G be a finite group, S a ring without zero-divisors, and $A = S * G$. Set $K = Z(S)$ and suppose* $\mathrm{Fix}(K) = \{e\}$. *Let I be an ideal of A such that $I \cap K \neq \{0\}$. If $\{h_i \in G\}_{i=1}^n$ is a family of distinct elements, $\{c_i \in S\}_{i=1}^n$ and $\sum_i c_i \tau_{h_i}(a) = 0$ for all $a \in I \cap K$, then $c_i = 0$ for all i.*

Proof. We argue by way of contradiction. From all possible families $\{h_i\}$ giving a counterexample, we choose one with minimum cardinality. We can suppose $h_1 = e$. Then $c_1 a + \cdots + c_n \tau_{h_n}(a) = 0$ for all $a \in I \cap K$ and consequently,

$$c_1 ab + \cdots + c_n \tau_{h_n}(ab) = bc_1 a + \cdots + \tau_{h_n}(b) c_n \tau_{h_n}(a) = 0$$

for any $a \in I \cap K$ and for any $b \in K$. From the minimality of the family $\{h_i\}$ it follows that $\tau_{h_n}(b) c_n = bc_n$ for any $b \in K$. Since S has no zero-divisors and $\mathrm{Fix}(K) = \{e\}$, we must have $h_n = e$, a contradiction. ∎

If $A = S * G$ is a crossed product, for any $g \in G$ we have defined the automorphism τ_g of S by means of $\tau_g(s) = \bar{g} s \bar{g}^{-1}$. Clearly this definition can be extended to the whole of A and we can then consider τ_g as an inner automorphism of A.

LEMMA 2.14. *Let S be a simple ring, G a supersolvable group, and suppose that $A = S * G$ has no zero-divisors. Set $K = Z(S)$, $G_0 = \langle g^2 \mid g \in G \rangle$ and $H \subseteq$ $\mathrm{Fix}(S)$ a normal subgroup of G. Then for any two-sided ideal $I \neq \{0\}$ of the ring $K * H$ such that $\tau_g(I) = I$ for all $g \in G_0$, there exists $0 \neq a \in I$ such that $\tau_g(a) = a$ for any $g \in G_0$.*

Proof. Since H is a normal subgroup of G and G is a supersolvable group, there is a series $\{e\} = H_0 \subset \cdots \subset H_k = H$ of normal subgroups of G such that H_i/H_{i-1} is cyclic for each i. We prove the lemma for $K * H_m$ by induction on m. Suppose it is true for $K * H_{i-1}$. Let $I \neq \{0\}$ be a two-sided ideal of the ring $K * H_i$ such that $\tau_g(I) = I$ for all $g \in G_0$. Suppose first that H_i/H_{i-1} is finite. Since $K * H_i = (K * H_{i-1}) * (H_i/H_{i-1})$, it follows that

$$\left((K * H_{i-1})^*\right)^{-1}(K * H_i)$$

is a division ring and consequently $K * H_{i-1} \cap I \neq \{0\}$. Then it suffices to apply the inductive hypothesis. If $H_i/H_{i-1} = \langle hH_{i-1} \rangle$ is infinite then $K * H_i = K * H_{i-1}[z, z^{-1}, \alpha]$, where $z = \overline{h}$. Consider the set of elements $c = c_0 + \cdots + c_n z^n$ of I, with $c_j \in K * H_{i-1}$, for which n is minimal. The coefficients of degree 0 in z of these elements form a non-zero ideal $J = \{c_0 \mid c = c_0 + \cdots + c_n z^n \in I\}$ of $K * H_{i-1}$. Since H_{i-1} and H_i are normal subgroups of G, $\tau_g(z) = p_g z^{\pm 1}$, where $p_g \in K * H_{i-1}$ for any $g \in G$ and then $\tau_g(z) = p_g z$ for any $g \in G_0$. We deduce that $\tau_g(J) = J$ for any $g \in G_0$. By the inductive hypothesis, there is c_0 with $c = c_0 + \cdots + c_n z^n \in I$ such that $\tau_g(c_0) = c_0$ for all $g \in G_0$. Now, the minimality of n yields $\tau_g(c) - c = 0$ for all $g \in G_0$. ∎

LEMMA 2.15. *Let A_i $(1 \leq i \leq n)$ be simple rings and K a ring which can be embedded in every $Z(A_i)$. Then the tensor product $A = (\otimes_K)_{i=1}^n A_i$ is almost simple.*

Proof. We will prove this lemma in the case $n = 2$. The proof of the general case is similar. Let v be a non-zero element of $A_1 \otimes_K A_2$. We can write v in the form

$$v = \sum_i (c_i \otimes 1) d_i,$$

where $\{c_i \in A_1\}$ is an independent set over $Z(A_1)$ and $d_i \in Z(A_1) \otimes_K A_2$. If we consider A_1 as an $A_1 \otimes_{Z(A_1)} A_1^\circ$-module, then by the density theorem, we can find $t = \sum_j e_j \otimes f_j \in A_1 \otimes_{Z(A_1)} A_1^\circ$ such that $tc_1 = 1$ and $tc_i = 0$ for $i \neq 1$. Therefore

$$d_1 = \sum_j e_j v f_j \in A_1 v A_1.$$

(Here we identify A_1 with $A_1 \otimes_K 1$.)

In the same way we can prove that

$$\left(Z(A_1) \otimes_K Z(A_2)\right) \cap A_2 d_1 A_2 \neq \{0\}.$$

Since $Z(A_1) \otimes_K Z(A_2) \subseteq Z(A_1 \otimes_K A_2)$, then

$$(A_1 \otimes_K A_2) v (A_1 \otimes_K A_2) \cap Z(A_1 \otimes_K A_2) \neq \{0\}.$$

Then $(A_1 \otimes_K A_2)$ is almost simple. ∎

LEMMA 2.16. *Let R be a ring and let $U \subseteq Z(R)^*$ be a multiplicative subset of regular elements. Suppose that $U^{-1}R$ is almost simple. Then R is almost simple.*

Proof. Let I be a non-zero ideal of R. Then we have

$$\{0\} \neq U^{-1}I \cap Z(U^{-1}R) = U^{-1}I \cap U^{-1}Z(R).$$

Hence $I \cap Z(R) \neq \{0\}$. ∎

PROPOSITION 2.17. *The following rings are almost simple*:

(i) *Any commutative ring.*

(ii) *The ring $A = S * G$, where S is an almost simple ring, G is a finite group, and A has no zero-divisors.*

(iii) *The ring $A = S * G$, where S is simple, G is a supersolvable group, and A has no zero-divisors.*

(iv) *The ring $A = (\otimes_K)_{i=1}^n A_i$, where A_i is an almost simple ring without zero-divisors and $K \subseteq Z(A_i)$.*

Proof. (i) It is immediate.

(ii) Let $I \neq \{0\}$ be a two-sided ideal of A. Set $K = Z(S)$, $F = (K^*)^{-1}S$, and $T = (K^*)^{-1}A$, which is a crossed product of F and G. Since F is simple, then from Lemma 2.11 we obtain that $TI = T$, and so $K \cap I \neq \{0\}$.

We proceed now by induction on the order of G. Set $H = \mathrm{Fix}(K)$. We first see that $S * H$ is almost simple. Since $K \subseteq Z(S * H)$, we have $Z(S * H) \cap I \neq \{0\}$ for any two-sided ideal I of $S * H$.

Since H is a normal subgroup of G then $A = S * G \cong (S * H) * (G/H)$. If $H \neq \{e\}$ we can apply the inductive hypothesis. Thus we can suppose $\mathrm{Fix}(K) = \{e\}$. From Lemma 2.13 we deduce that there is $a \in I \cap K$ such that $c = \sum_{g \in G} \tau_g(a) \neq 0$. It is clear that $c \in I \cap Z(A)$.

(iii) Set $K = Z(S)$, $G_0 = \langle g^2 \mid g \in G \rangle$, and $H = \mathrm{Fix}(S) \cap G_0$. Let $I \neq \{0\}$ be any two-sided ideal of $S * G_0$. Since S is simple, we have $K * H \cap I \neq \{0\}$. According to Lemma 2.14 there is $0 \neq a \in Z(S * G_0) \cap I$. Hence $S * G_0$ is almost simple. Since G_0 is a normal subgroup of G of finite index, it follows from part (ii) of this proposition that $A = S * G$ is almost simple.

(iv) Set $\bar{A}_i = (Z(A_i)^*)^{-1}A_i$, $Q = (K^*)^{-1}K$ and

$$\bar{A} = \left((Z(A_1)^*)^{-1} \cdots (Z(A_n)^*)^{-1}\right)A \cong (\otimes_Q)_{i=1}^n \bar{A}_i.$$

Since $\overline{A_i}$ is simple then by Lemma 2.15, $\overline{A}$ is almost simple. As $Z(A_i) \subseteq Z(A)$, by Lemma 2.15, $\overline{A}$ is almost simple. As $Z(A_i) \subseteq Z(A)$, by Lemma 2.16, A is also almost simple. ∎

COROLLARY 2.18. *Let $A = S * G$ be a crossed product without zero-divisors, where S is a simple ring and G is a supersolvable group. Then A is simple if and only if $Z(A)$ is a field.*

Proof. We prove sufficiency. According to Proposition 2.17, A is almost simple. So if $Z(A)$ is a field, A is simple. ∎

2.5. *Main Theorem*

We can now proceed to prove our main result, Theorem 2.1, following the lines of the proof of Stafford for the analogous result in the case of Weyl algebras [9, Theorem 3.1].

Proof of Theorem 2.1. The case $c = 0$ is immediate, so we suppose $c \neq 0$. We can also assume that $a, b \neq 0$ after replacing if necessary a by $a + s_1 c$ and b by $b + s_2 c$.

Since G is a supersolvable group, there is a series of normal subgroups of G, $\{e\} = G_0 \subset \cdots \subset G_n = G$ such that G_{k+1}/G_k is cyclic. Define $T_i = T * G_i$, so that $T_0 = T$ and $T_n = C$, and set $T_{-1} = \{1\}$. We prove by induction on k the existence of elements $d_k, e_k \in C$ and non-zero $q \in T_k$ such that $qc \in C(a + s_1 d_k c) + C(b + s_2 e_k c)$ for $k = n, \dots, -1$. Observe that this result for $k = -1$ proves the theorem.

If $k = n, d_n, e_n$ and q exist because C has the left Ore condition (for example, $Ca \cap Cc \neq \{0\}$). Suppose now they exist for some $k \geq 0$. We simplify the notation by writing a and b instead of $a + s_1 d_k c$ and $b + s_2 e_k c$.

Then $qc \in Ca + Cb$ and $q \in T_k$. First suppose that $k > 0$. As G_{k-1} is normal in G, we can consider the ring $A = (T_{k-1}^*)^{-1} C$. Set $S = (T_{k-1}^*)^{-1} T_k$. We have that $S = F_{k-1} * (G_k/G_{k-1})$, where $F_{k-1} = (T_{k-1}^*)^{-1} T_{k-1}$ is the division ring of quotients of T_{k-1}. Since $C \cong T_k * (G/G_k)$, we have $A \cong S * (G/G_k)$.

If G_k/G_{k-1} is finite, then S is a division ring. Hence there exists $q' \in T_k$ such that $q'q \in T_{k-1}$ and $(q'q)c \in Ca + Cb$, as desired.

So we can suppose that G_k/G_{k-1} is infinite cyclic, so that $\mathscr{K}_l(S) = 1$. Write $qc = h_1 a + h_2 b$, where we can suppose $h_1, h_2 \neq 0$ because $Ca \cap Cb \neq \{0\}$. Since T satisfies the right Ore condition, then C satisfies the right Ore condition [7, Lemma 9.3.8] and there are $g_1 \in C$ and $g_2 \in C$, different from zero, such that $h_1 s_1 g_1 + h_2 s_2 g_2 = 0$. Also, we can find non-zero elements $s \in C$ and $t \in C$ such that $sqc = tb$. Note that, by Proposition 2.17, T_k is almost simple and therefore we can apply Proposition 2.2. So there is $f \in C$ such that $Aq + A(sq + ts_2 g_2 f) = A$. Set $L = A(a + s_1 g_1 fc) + A(b + s_2 g_2 fc)$. Then

$$\begin{aligned} qc &= h_1 a + h_2 b = h_1 a + h_2 b + (h_1 s_1 g_1 + h_2 s_2 g_2) fc \\ &= h_1(a + s_1 g_1 fc) + h_2(b + s_2 g_2 fc) \in L, \end{aligned}$$

and

$$sqc + ts_2 g_2 fc = t(b + s_2 g_2 fc) \in L.$$

Hence $c \in L$ and there are $q' \in T_{k-1}$ and $e_{k-1}, d_{k-1} \in C$ such that

$$q'c \in C(a + s_1 d_{k-1} c) + C(b + s_2 e_{k-1} c).$$

If $k = 0$ we can put $A = C$, $S = T$, and use the same arguments. ∎

Let R be a left Noetherian ring without zero-divisors and Q its left quotient ring. Let M be a finitely generated left R-module. Then we define rank M as the integer k such that $Q \otimes_R M \cong Q^k$.

DEFINITION. Define a left R-module M over a ring R to be *stably m-generated* if for any $r \geq m$ and $a_1, \ldots, a_{r+1} \in M$ such that $M = \sum_{i=1}^{r+1} Ra_i$ there exist $f_i \in R$ such that $M = \sum_{i=1}^{r} R(a_i + f_i a_{r+1})$.

The results of the following corollary can be proved as in [9] by using Theorem 2.1.

COROLLARY 2.19. *Let T be either a division ring or a left Noetherian simple ring with the right Ore condition and $\mathscr{K}_l(T) = 1$, G a supersolvable group, and $C = T * G$ a crossed product of T and G. We suppose that C is simple and has no zero-divisors. Then the next propositions hold.*

(i) *Any left ideal of C is stably two-generated.*

(ii) *Any left stably free module P with* rank $P \geq 2$ *is free.*

(iii) *Let M be a finitely generated left C-module. Then $M \cong N \oplus C^k$, where N is a module with* rank $N \leq 1$. *If N is torsion-free, then N is isomorphic to a left ideal of C.*

(iv) *Let M be a finitely generated torsion left C-module. Then $M \cong I/J$, where I is a projective left ideal of A. In particular, M is stably two-generated.*

Remark 2.20. All proofs remain true if we suppose that G has a series $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ of normal subgroups of G such that G_k/G_{k-1} is infinite cyclic or finite. Note that a policyclic-by-finite group does not necessarily have a series like this. Most of the propositions remain true if we suppose that G is a policyclic-by-finite group. Nevertheless, for example, in Proposition 2.17(iii) and Theorem 2.1 we cannot do this substitution.

3. GENERALIZED WEYL ALGEBRAS

In this section we obtain the same results as in the preceding one, for tensor products of generalized Weyl algebras of degree 1.

DEFINITION 2. A *generalized Weyl algebra* of degree 1 is a ring $A = D(\alpha, a) = D[x, y, \alpha, a]$ generated by a commutative Noetherian ring D without zero-divisors with Krull dimension 1, and elements x, y such that

$$yx = a \in D, \qquad xy = \alpha(a), \qquad xd = \alpha(d)x, \qquad \text{and} \qquad yd = \alpha^{-1}(d)y$$

for all $d \in D$, where α is a ring automorphism of D.

Many examples of generalized Weyl algebras can be found in [3].

Note that A is a $\mathbb{Z}$-graded ring, where the k-component of A is

$$(A)_k = \begin{cases} Dx^k & \text{for } k > 0, \\ D & \text{for } k = 0, \\ Dy^{-k} & \text{for } k < 0. \end{cases}$$

We shall use the following two propositions, whose proofs can be found in [2].

PROPOSITION 3.1 [2, Theorem 3]. *Let $A = D(\alpha, a)$ be a generalized Weyl algebra of degree* 1. *Then A is simple if and only if for any maximal ideal P of D and any $0 \neq n \in \mathbb{Z}$ the following conditions hold*:

(i) $\alpha^n(P) \neq P$,

(ii) $a \notin \alpha^n(P) \cap P$.

PROPOSITION 3.2 [2, Theorem 2]. *Let $A = D(\alpha, a)$ be a generalized Weyl algebra of degree* 1. *Suppose that $\alpha^n(P) \neq P$ for all $0 \neq n \in \mathbb{Z}$ and any maximal ideal P of D. Then $\mathscr{K}_l(A) = 1$.*

If $A = D(\alpha, a) = D[x, y, \alpha, a]$ is a generalized Weyl algebra of degree 1, we set $a_k = y^k x^k$ and $A^{(k)} = D(\alpha^k, a_k) = D[x^k, y^k, \alpha^k, a_k]$, which is the algebra generated by D and the elements x^k and y^k.

LEMMA 3.3. *Let $A = D(\alpha, a) = D[x, y, \alpha, a]$ be a simple generalized Weyl algebra of degree* 1. *Then $A^{(k)}$ is simple for any $k > 0$.*

Proof. We have to verify the hypotheses of Proposition 3.1. The first condition is obvious. Suppose that $a_k \in \alpha^{nk}(P) \cap P$ for some maximal ideal P of D and some $n \neq 0$. Then a_k and $\alpha^{-nk}(a_k)$ belong to P. Since

$$a_k = \alpha^{-k+1}(a) \cdots \alpha^{-1}(a)a \in P,$$
$$\alpha^{-nk}(a_k) = \alpha^{-(n+1)k+1}(a) \cdots \alpha^{-nk}(a) \in P$$

and the sets $\{-k+1,\dots,-1,0\}$ and $\{-(n+1)k+1,\dots,-nk\}$ are disjoint, there exist $m_1 \neq m_2$ such that $\alpha^{m_1}(a), \alpha^{m_2}(a) \in P$. This is a contradiction because A is simple. ∎

Let $A = D(\alpha, a) = D[x, y, \alpha, a]$ be a simple generalized Weyl algebra of degree 1, where D is finitely generated over a field $K \subseteq Z(A)$. Let $z \in Z(A)^*$. Note that z is invertible. Since D has no zero-divisors and A is $\mathbb{Z}$-graded, then $z \in D$. As D is finitely generated over K, we have that z is an algebraic element over K. Then $Z(A)$ is a subset of algebraic elements of D. We will apply several times this fact, sometimes without further mention.

LEMMA 3.4. *Let $A = D(\alpha, a)$ be a simple generalized Weyl algebra of degree* 1 *and suppose D is finitely generated over $K = Z(A)$ and $U(D) = K$. Then, for any field extension Q of K the tensor product $Q \otimes_K D$ has no zero-divisors and its Krull dimension is* 1.

Proof. Since $D \cong K[x_1,\dots,x_p]/J$, then $Q \otimes_K D \cong Q[x_1,\dots,x_p]/QJ$. The Krull dimension of D is 1, so there exists $z \in D$ such that D is finitely generated as a $K[z]$-module. Hence $Q \otimes_K D$ is finitely generated as a $Q[z]$-module and therefore $\mathscr{K}_l(Q \otimes_K D) = 1$.

Let Q_1 be a maximal subfield of Q such that $Q_1 \otimes_K D$ has no zero-divisors and take an element $t \in Q \setminus Q_1$. If t is not an algebraic element over Q_1, then $D \otimes_K Q_1(t) \cong (D \otimes_K Q_1)[t](Q_1[t]^*)^{-1}$ has no zero-divisors.

We suppose now that t is an algebraic element over Q_1. Let h be an irreducible polynomial over Q_1 such that t is root of h. Set $D_1 = Q_1 \otimes_K D$ and $D_2 = Q_1[t] \otimes_K D \cong Q_1[t] \otimes_{Q_1} D_1$. We want to prove that D_2 has no zero-divisors. Let P be the field of quotients of the ring D_1 and $T = D_2(D_1^*)^{-1}$. It is clear that $T \cong P[z]/P[z]h$. The automorphism α can be extended to an automorphism of the ring T. Let $h = h_1 \cdots h_s$ be the decomposition of the polynomial h, where the factors h_i are irreducible polynomials over P. Since $D_2[x, y, \alpha, a] = Q_1[t] \otimes_K A$ is a simple ring, then $T[x, x^{-1}, \alpha]$ is also a simple ring and hence, the Jacobson radical of T is zero. So, all the polynomials h_i are different. Then $T \cong \oplus_{i=1}^{s} T_i$, where $T_i \cong P[z]/P[z]h_i$ is a field. Let p be the identity element of T_1. The number of idempotent elements of T is finite. Then there exists k such that $\alpha^k(p) = p$. By Lemma 3.3, $D(\alpha^k, y^k x^k)$ is simple. Since $U(D) = K$, then $Z(D(\alpha^k, y^k x^k)) = K$. Hence $Q_1[t] \otimes_K D(\alpha^k, y^k x^k)$ is simple and we deduce that $T[x, x^{-1}, \alpha^k]$ is also simple. Then $T_1 = T$, and this means that D_2 does not contain any zero-divisors. ∎

Now we can prove a criterion for $C = (\otimes_K)_{i=1}^{n} A_i$ to be a simple ring without zero-divisors.

PROPOSITION 3.5. *For each* $1 \le i \le n$, *let* $A_i = D_i(\alpha_i, a_i) = D_i[x_i, y_i, \alpha_i, a_i]$ *be a simple generalized Weyl algebra of degree* 1, *where* D_i *is a finitely generated ring over a field* $K \subseteq Z(A_i)$. *Then the following conditions are equivalent*:

(i) $C = (\otimes_K)_{i=1}^n A_i$ *has no zero-divisors.*

(ii) $D = (\otimes_K)_{i=1}^n D_i$ *has no zero-divisors.*

(iii) $U = (\otimes_K)_{i=1}^n U_i$ *has no zero-divisors, where* U_i *is the set of algebraic elements of* D_i *over* K.

Also, if any of these conditions holds, then C *is simple.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are obvious.

Let us prove (ii) ⇒ (i). Each A_i is a $\mathbb{Z}$-graded ring. Then C is a $\mathbb{Z}^n$-graded ring with

$$C_{(k_1,\dots,k_n)} = (A_1)_{k_1} \otimes_K \cdots \otimes_K (A_n)_{k_n}.$$

Let $k = (k_1, \dots, k_n)$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$. We will write $k < m$ if there is an integer $l \in [1, n]$ such that $k_i = m_i$ for $i = 1, \dots, l-1$ and $k_l < m_l$. Suppose that $k_1, k_2, k_3, k_4 \in \mathbb{Z}^n$ and $k_1 \le k_2$, $k_3 < k_4$. Then it is clear that $k_1 + k_3 < k_2 + k_4$.

Let $a, b \in C$. We can write a and b according to the graduation of C,

$$a = a_k + \sum_{i<k} a_i, \qquad b = b_m + \sum_{i<m} b_i,$$

where $a_i, b_i \in C_i$ and $a_k \neq 0$, $b_m \neq 0$. Then

$$ab = a_k b_m + \sum_{i<k+m} c_i, \qquad \text{for some } c_i \in C_1.$$

If $ab = 0$ then $a_k b_m = 0$, which is impossible, because $D = (\otimes_K)_{i=1}^n D_i$ has no zero-divisors. Then (ii) ⇒ (i).

Now we prove the implication (iii) ⇒ (ii). First note that $U_i = U(D_i)$, because D_i is finitely generated over K. Set

$$M_i = \left((\otimes_K)_{j=1}^{i} D_j\right) \otimes_K \left((\otimes_K)_{j=i+1}^{n} U_j\right).$$

Arguing by induction on i we prove that M_i has no zero-divisors. Suppose that M_{i-1} has no zero-divisors. Since D_i is finitely generated over K then there exists m such that $\alpha_i^m(u) = u$ for any $u \in U_i$. Hence $Z(A_i^{(m)}) = U_i = U(D_i)$ and $A_i^{(m)}$ satisfies the conditions of Lemma 3.4. Therefore $M_i \cong D_i \otimes_{U_i} M_{i-1}$ has no zero-divisors.

Now suppose that one of the conditions (i), (ii), or (iii) holds. Since D_i is finitely generated over K, by the remark before Lemma 3.4 we have $Z(A_i) \subseteq U_i$. Hence $Z(C) = (\otimes_K)_{i=1}^n Z(A_i)$ is a field. By Proposition 2.17, C is almost simple and consequently it is also simple. ∎

PROPOSITION 3.6. *Let $C = (\otimes_K)_{i=1}^n A_i$ be a ring without zero-divisors, where $A_i = D_i(\alpha_i, a_i) = D_i[x_i, y_i, \alpha_i, a_i]$ is a simple generalized Weyl algebra of degree 1 and D_i is finitely generated over a field $K \subseteq Z(A_i)$. Set $T_m = ((\otimes_K)_{i=1}^m A_i) \otimes_K ((\otimes_K)_{i=m+1}^n D_i)$ for $0 \le m < n$, $S = T_{m+1}(T_m^*)^{-1}$, and $A = C(T_m^*)^{-1}$. Then for any left ideal $I \neq \{0\}$ of S and elements $u \in A$, $0 \neq v \in A$, there exists an element $f \in C$ such that $AI + A(u + vf) = A$.*

Proof. We can suppose that $u = 0$. It is clear that $A \cong S * \mathbb{Z}^{n-m-1}$. Set

$$\overline{C} = C\Big(\big((\otimes_K)_{i=m+2}^n D_i\big)^*\Big)^{-1}.$$

Then $\overline{C} \cong S_1 * \mathbb{Z}^{n-m-1}$, where $S_1 = T_{m+1}(((\otimes_K)_{i=m+2}^n D_i)^*)^{-1}$. By Proposition 2.17, T_{m+1} is almost simple, and so S_1 is almost simple. Then by Proposition 2.2, there exists $f \in \overline{C}$ such that $AI + Avf = A$, where $f = cd^{-1}$, $c \in C$ and $d \in (\otimes_K)_{i=m+2}^n D_i$. As d commutes with S_1 then $AI + Avc = A$. ∎

PROPOSITION 3.7. *Let $C = (\otimes_K)_{i=1}^n A_i$ be a ring without zero-divisors, where $A_i = D_i(\alpha_i, a_i) = D_i[x_i, y_i, \alpha_i, a_i]$ is a simple generalized Weyl algebra of degree 1 and D_i is finitely generated over a field $K \subseteq Z(A_i)$. Set $U_m = (\otimes_K)_{i=1}^m D_i$ with $1 \le m < n$, $Q = U_m(U_m^*)^{-1}$, $S = Q \otimes_K A_{m+1}$, $D = Q \otimes_K D_{m+1}$, and $A = C(U_m^*)^{-1}$. Then for any left ideal I of S such that $D \cap I \neq \{0\}$ and for any elements $u \in A$, $0 \neq v \in A$ there exists an element $f \in C$ such that $AI + A(u + vf) = A$.*

Proof. First of all we prove that $\mathscr{K}_l(S) = 1$. We have $Z(S) = Q \otimes_K Z(A_{m+1})$ and $Z(A_{m+1})$ is finitely generated and algebraic over K. Hence $Z(S)$ is finitely generated and algebraic over Q and it must be a field, because it has no zero-divisors. Since S is almost simple, it follows that S is simple. On the other hand $D = Q \otimes_K D_{m+1}$ has Krull dimension 1 and consequently $S = D[x_{m+1}, y_{m+1}, 1 \otimes \alpha_{m+1}, 1 \otimes a_{m+1}]$ is a generalized Weyl algebra of degree 1. We conclude now from Propositions 3.1 and 3.2 that $\mathscr{K}_l(S) = 1$.

Again we suppose that $u = 0$ and I is a maximal left ideal of S. It is clear that $A \cong S * \mathbb{Z}^m \otimes_K ((\otimes_K)_{i=m+2}^n A_i)$. As in Lemma 2.5 we have

$$\operatorname{End}_A(A/AI) \cong (F * H) \otimes_K \big((\otimes_K)_{i=m+2}^n A_i\big),$$

where $F \cong \operatorname{End}_S(S/I)$ and $H \subseteq \mathbb{Z}^m$. Hence $\operatorname{End}_A(A/AI)$ is left and right Noetherian.

Reasoning as in the proofs of Lemma 2.9 and Proposition 2.3, it follows that for any left ideal $I \neq \{0\}$ of S we can find an element $f \in A$ such that $AI + Avf = A$.

Set $J = S \cdot (I \cap D)$. Then there exists $f \in A$ such that $AJ + Avf = A$, where $f = cd^{-1}, c \in C, d \in U_m$. As d commutes with D then $AJ + Avc = A$ and so $AI + Avc = A$. ∎

Note that the conclusion in the previous proposition is also true for $U_0 = \{1\}$ if we define $S = A_1$, $D = D_1$, and $A = C$.

THEOREM 3.8. *Let $C = (\otimes_K)_{i=1}^n A_i$ be such that C has no zero-divisors, where $A_i = D_i(\alpha_i, a_i) = D_i[x_i, y_i, \alpha_i, a_i]$ is a simple generalized Weyl algebra of degree 1 and D_i is a finitely generated ring over a field $K \subseteq Z(A_i)$. Then for any $a, b, c \in C$ and $s_1, s_2 \in C^*$ there are $f_1, f_2 \in C$ such that $a, b, c \in C(a + s_1 f_1 c) + C(b + s_2 f_2 c)$.*

Proof. The proof is as in Theorem 2.1, working with the series

$$\{1\} = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{n-1} \subseteq T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n = C$$

and using both Propositions 3.6 and 3.7. ∎

The next corollary can be proved in a similar way to Corollary 2.19.

COROLLARY 3.9. *Let $C = (\otimes_K)_{i=1}^n A_i$ be such that C has no zero-divisors, where $A_i = D_i(\alpha_i, a_i) = D_i[x_i, y_i, \alpha_i, a_i]$ is a simple generalized Weyl algebra of degree 1 and D_i is a finitely generated ring over a field $K \subseteq Z(A_i)$. Then the next propositions hold.*

(i) *Any left ideal of C is stably two-generated.*

(ii) *Any left stably free module P with $\operatorname{rank} P \geq 2$ is free.*

(iii) *Let M be a finitely generated left C-module. Then $M \cong N \oplus C^k$, where N is a module with $\operatorname{rank} N \leq 1$. If N is torsion-free, then N is isomorphic to a left ideal of C.*

(iv) *Let M be a finitely generated torsion left C-module. Then $M \cong I/J$, where I is a projective left ideal of A. In particular, M is stably two-generated.*

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