The Structurally Stable Linear Systems on the Half-Line Are Those with Exponential Dichotomies

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1. INTRODUCTION

We define a notion of topological equivalence between bounded time-dependent systems of linear differential equations on the positive half-line. A system is then said to be structurally stable if it is topologically equivalent to all systems sufficiently near it. It is shown that a system has this property if and only if it has an exponential dichotomy.

2. EXPONENTIAL DICHOTOMY AND TOPOLOGICAL EQUIVALENCE

Let $A(t)$ be a continuous real $n \times n$ matrix function defined on $[0, \infty)$. We say that the linear system,

$$\dot{x} = A(t)x,$$

has bounded growth (cf. Coppel [1, p. 8]) if there are constants $M > 0, L \geq 0$ such that

$$|X(t)X^{-1}(s)| \leq Me^{L(t-s)}$$

for $0 \leq s \leq t$, where $X(t)$ is a fundamental matrix for (1). [$|\cdot|$ means the Euclidean norm when the argument is a vector and the corresponding operator norm when the argument is a matrix.]

(1) has an exponential dichotomy if there are constants $K > 0, \gamma > 0$ and a projection $P$ such that

$$|X(t)PX^{-1}(s)| \leq Ke^{-\gamma(t-s)} \quad \text{for} \quad 0 \leq s \leq t$$

and

$$|X(t)(I - P)X^{-1}(s)| \leq Ke^{-\gamma(t-s)} \quad \text{for} \quad 0 \leq t \leq s.$$
Let $B(t)$ be another continuous real $n \times n$ matrix function defined on $[0, \infty)$. We say that (1) and
\begin{equation}
\dot{x} = B(t)x
\end{equation}
are *kinematically similar* if there exists a bounded continuously differentiable $n \times n$ matrix function $S(t)$ with bounded inverse $S^{-1}(t)$ such that if $x(t)$ is a solution of (1) then $S(t)x(t)$ is a solution of (2). By differentiation, we find that
\begin{equation}
B(t) = S(t)A(t)S^{-1}(t) + S(t)\dot{S}(t)S^{-1}(t).
\end{equation}

Clearly, kinematic similarity is an equivalence relation.

(1) and (2) are said to be *topologically equivalent* if there exists a function $h: [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:

(i) $|h(t, x)| \to \infty$ as $|x| \to \infty$, uniformly with respect to $t$;

(ii) $h_t: \mathbb{R}^n \to \mathbb{R}^n$, defined by
\[ h_t(x) = h(t, x), \]

is a homeomorphism for each fixed $t$;

(iii) $g: [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, defined by
\[ g(t, x) = (h_t)^{-1}(x), \]

has property (i) also;

(iv) if $x(t)$ is a solution of (1) then $h(t, x(t))$ is a solution of (2).

(iv) implies the identity
\begin{equation}
h(t, X(t)X^{-1}(s)x) = Y(t)Y^{-1}(s)h(s, x),
\end{equation}
where $Y(t)$ is a fundamental matrix for (2).

Note that without (i) and (iii) the concept of topological equivalence would be trivial for without them $h(t, x) = Y(t)X^{-1}(t)x$ would give a topological equivalence between any two systems (1) and (2).

It is a consequence of (4) and (ii) that $h$ is continuous. Really we only needed to assume that $h_t$ is a homeomorphism since (4) then implies that $h_t$ is also.

Note also it follows from (4) that if $y(t)$ is a solution of (2) then
\[ g(t, y(t)) = (h_t)^{-1}(Y(t)Y^{-1}(0)y(0)) = X(t)X^{-1}(0)g(0, y(0)) \]
and so is a solution of (1). This means that $g$ gives a topological equivalence between (2) and (1). It follows that topological equivalence is an equivalence relation.

We now show that topological equivalence preserves exponential dichotomy.
LEMMA 1. Suppose (1) has an exponential dichotomy and (2) has bounded growth. Then if (1) and (2) are topologically equivalent, (2) also has an exponential dichotomy.

Proof. Let \( h(t, x) \) and \( g(t, x) \) be the functions giving the topological equivalence between (1) and (2) and define for \( r \geq 0 \) the nondecreasing function,  
\[
b(r) = \inf\{\min(|h(t, x)|, |g(t, x)|): t \geq 0, |x| \geq r\}.
\]
Then \( b(r) \to \infty \) as \( r \to \infty \). In particular, there exists \( A > 0 \) such that \( b(A) > 0 \).

Choose \( \theta > 0 \) sufficiently small such that \( b(\theta^{-1}b(A)) \geq 2A \). Using the result given before Proposition 1 in [1, p. 141, there exists \( T > 0 \) such that if \( x(t) \) is a solution of (1),
\[
|x(t)| \leq \theta \sup\{|x(u)|: |u - t| \leq T\}
\]
for \( t \geq T \).

Let \( y(t) \) be any solution of (2) and fix \( t_0 \geq T \). Then, with \( \sigma = \Delta \cdot |y(t_0)|^{-1} \), \( x(t) = g(t, \sigma y(t)) \) is a solution of (1) and there exists \( u \), with \( |u - t_0| \leq T \), such that \( |x(t_0)| \leq \theta |x(u)| \). So
\[
|x(u)| \geq \theta^{-1} |x(t_0)| = \theta^{-1} |g(t_0, \sigma y(t_0))| \geq \theta^{-1} b(A)
\]
and thus
\[
|y(u)| = \sigma^{-1} |h(u, x(u))| \geq \sigma^{-1} b(\theta^{-1} b(A)) \geq 2 |y(t_0)|.
\]
This means that
\[
|y(t_0)| \leq \frac{1}{2} \sup\{|y(u)|: |u - t_0| \leq T\}
\]
and so, by Proposition 1 in [1, p. 14], it follows that (2) has an exponential dichotomy.

Remarks. We indicate an alternative proof of Lemma 1. It is sufficient to find a continuous function \( V: [0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) such that
\[
\begin{align*}
& (i) \sup\{|V(t, x)|: t \geq 0, |x| \leq r\} < \infty \text{ for } r \geq 0; \\
& (ii) \text{ if } x(t) \text{ is a solution of (2), } V(t, x(t)) \text{ is continuously differentiable and } \\
& \quad \frac{d}{dt} V(t, x(t)) \leq -b(|x(t)|),
\end{align*}
\]
where \( b(r) \) is a nonnegative nondecreasing function for \( r \geq 0 \) with \( b(r) > 0 \) if \( r \) is large enough. (This result almost follows from Theorem 3.1 in Massera and Schäffer [2, p. 546] but a simpler proof can be given using the method in the proof of Proposition 2 in [1, p. 61–63].)
By Proposition 1 in [1, p. 59], there exists a bounded continuously differentiable symmetric matrix function $H(t)$ such that if $x(t)$ is a solution of (1),

$$\frac{d}{dt} (x(t), H(t) x(t)) \leq - |x(t)|^2,$$

where $( , )$ denotes the inner product. Then the function $V(t, x) = (g(t, x), H(t) g(t, x))$ will do the job.

We remark also that by using Lemma 1 we can replace (i) in the statement of the theorem in Palmer [3] by: $|h(t, x)| \to \infty$ as $|x| \to \infty$, uniformly with respect to $t$.

3. STRUCTURAL STABILITY AND STATEMENT OF THE THEOREM

Now we restrict ourselves to systems (1) where $A(t)$ is continuous and bounded on $[0, \infty)$; then (1) has bounded growth. We identify a system (1) with its coefficient matrix $A$ and write

$$\| A \| = \sup_{t \geq 0} |A(t)|.$$

With this as norm the set of systems (1) becomes a Banach space $\mathcal{B}$. If (1) and (2) are topologically equivalent, we write $A \simeq B$.

$A \in \mathcal{B}$ is said to be structurally stable if there exists $\delta > 0$ such that $B \in \mathcal{B}$, $\| B - A \| < \delta$ implies that $B \simeq A$.

**Lemma 2.** (a) The set of structurally stable systems is an open subset of $\mathcal{B}$.

(b) If $A \in \mathcal{B}$ is structurally stable and $A$ is kinematically similar to $B \in \mathcal{B}$, then $B$ is structurally stable.

**Proof.** (a) Let $A$ be structurally stable and suppose $B \in \mathcal{B}$, $\| B - A \| < \delta$ where $\delta$ is the number in the definition of structural stability. Then, if $C \in \mathcal{B}$ and

$$\| C - B \| < \delta - \| B - A \|,$$

$$\| C - A \| \leq \| C - B \| + \| B - A \| < \delta$$

and so $C \simeq A$. Hence $B \simeq A \simeq C$ and we have proved that $B$ is structurally stable.

(b) Let $A$ and $\delta$ be as in the first part of this proof and suppose $A$ is kinematically similar to $B$. Let $S(t)$ be the matrix function giving the kinematic similarity. Suppose $C \in \mathcal{B}$ and

$$\| C - B \| < \delta \| S \| \| S^{-1} \|.$$
Define

\[ D(t) = S^{-1}(t) C(t) S(t) - S^{-1}(t) S(t). \]

Then, using (3),

\[
|D(t) - A(t)| = |S^{-1}(t)[C(t) - B(t)] S(t)| \\
\leq \|S^{-1}\|\|C - B\|\|S\| < \delta.
\]

So \( D \in \mathcal{B} \) and \( D \cong A \). But, by definition of \( D \), \( C \) and \( D \) are kinematically similar and hence \( C \cong D \). Thus \( B \cong A \cong D \cong C \). This means that \( B \) is structurally stable and the proof of the lemma is complete.

In [4] Čeban has considered almost periodic systems on \((-\infty, \infty)\) and with a similar (but only local) definition of topological equivalence has shown that an almost periodic system is structurally stable if and only if it has an exponential dichotomy. Our theorem was suggested by his.

**THEOREM.** The system (1), with \( A(t) \) bounded and continuous on \([0, \infty)\), is structurally stable if and only if it has an exponential dichotomy.

4. **PROOF OF THE SUFFICIENCY**

So we suppose that \( A \in \mathcal{B} \) and (1) has an exponential dichotomy. We define \( A(t) \) on \((-\infty, 0]\) in such a way that the extended function is bounded and continuous on \((-\infty, \infty)\) and (1) has an exponential dichotomy on \((-\infty, 0]\) with projection having nullspace \( V_+ \) supplementary to the range \( V_- \) of the projection \( P \) in (1)'s dichotomy on \([0, \infty)\). Then, by the theory in Lecture 2 of [1], the extended equation has an exponential dichotomy on \((-\infty, \infty)\) with projection \( \tilde{P} \) having range \( V_- \) and nullspace \( V_+ \).

The theorem in Palmer [3] (see also Remarks 1 and 2 after the statement) then tells us that the extended equation is topologically equivalent on \((-\infty, \infty)\) to a system,

\[ \dot{x}_i = e_i x_i \quad (i = 1, \ldots, n), \tag{5} \]

where \( e_i = 1 \) or \(-1\) and the number of \( i \)'s with \( e_i = -1 \) is equal to the rank of \( \tilde{P} (\equiv \text{the rank of } P) \). Restricting to \([0, \infty)\) we see that (1) \( \cong (5) \).

Now by Proposition 1 in [1, p. 34], there exists \( \delta > 0 \) such that if \( B \in \mathcal{B} \), \( \|B - A\| < \delta \) (2) has an exponential dichotomy with projection having the same rank as \( P \). Then (2) \( \cong (5) \cong (1) \) and so (1) is structurally stable.
5. PROOF OF THE NECESSITY

In this section we use results of Bylov and Millionščikov. (1) is said to be a system with integral division if \( A(t) \in \mathcal{B} \) and it has a basis of solutions \( x_1(t), \ldots, x_n(t) \) satisfying
\[
| x_{i+1}(t) | | x_i(s) | | x_{i+1}(s) | | x_i(t) | \geq e^{\beta + \alpha(t-s)}
\]
for \( 0 \leq s \leq t, \ i = 1, \ldots, n - 1 \) where \( \beta \) and \( \alpha > 0 \) are constants.

In [5] Bylov shows that such a system is kinematically similar to the diagonal system,
\[
\dot{x}_i = \phi_i(t) x_i \quad (i = 1, \ldots, n),
\]
where
\[
\phi_i(t) = \frac{d}{dt} \log | x_i(t) |.
\]
Writing \( \phi_i(t) = \int_0^t \phi_i(s) \, ds \), (6) implies that
\[
[\phi_{i+1}(t) - \phi_{i+1}(s)] - [\phi_i(t) - \phi_i(s)] \geq \beta + \alpha(t - s)
\]
for \( 0 \leq s \leq t, \ i = 1, \ldots, n - 1 \). Note also that (7) is in \( \mathcal{B} \).

In [6] Millionščikov shows that the systems with integral division are dense in \( \mathcal{B} \). [Actually, Millionščikov considers systems (1) where \( A(t) \) is bounded and piecewise continuous and shows that for arbitrary \( \delta > 0 \) there exists a bounded and piecewise continuous matrix function \( B(t) \) such that \( \| B - A \| < \delta \) and (2) is a system with integral division.

Suppose that \( A(t) \) is in fact continuous and let \( \epsilon > 0 \) be given. By altering \( B(t) \) near its points of discontinuity, we can find a bounded and continuous \( \tilde{B}(t) \) such that \( \| A - \tilde{B} \| < \delta \) and \( \sup_{t \geq 0} \int_t^{t+1} | \tilde{B}(s) - B(s) | \, ds < \epsilon \).

It follows from Theorem 15.2.1 in [7, p. 208] that the systems with integral division form an open subset of \( \mathcal{B} \). Let \( \mathcal{B}_1 \) be the set of systems (1), where \( A(t) \) is bounded and piecewise continuous, and make \( \mathcal{B}_1 \) into a normed space by defining
\[
\| A \|_1 = \sup_{t \geq 0} \int_t^{t+1} | A(s) | \, ds.
\]
Then, by modifying the arguments used in [7, p. 210–223], it is not hard to show that the systems with integral division form an open subset of \( \mathcal{B}_1 \). This means that there exists \( \epsilon > 0 \) such that if \( \tilde{B} \in \mathcal{B}_1 \) and \( \| \tilde{B} - B \|_1 < \epsilon \) then
\[
\dot{x} = \tilde{B}(t)x
\]
is a system with integral division. This is what we want.]

Suppose now we can show that a structurally stable system of the form (7), satisfying (8), has an exponential dichotomy. Then, by Bylov’s result and
Lemmas 2(b) and 1, any structurally stable system with integral division must have an exponential dichotomy. But given any structurally stable system \( A \) in \( \mathfrak{B} \), by Lemma 2(a) there exists \( \delta > 0 \) such that if \( B \in \mathfrak{B} \) and \( \| B - A \| < \delta \) then \( B \) is also structurally stable and \( B \simeq A \). Now, by Millionščikov’s result, we can choose \( B \) to be a system with integral division. Then \( B \) has an exponential dichotomy and so, by Lemma 1, \( A \) also has an exponential dichotomy.

So we need only consider structurally stable systems of the form (7), satisfying (8). We show that such a system has an exponential dichotomy.

First of all, assume there exists an \( i \) such that \( \phi_i(t) \) does not tend to \( \infty \) as \( t \to \infty \) and let \( k \) be the largest such \( i \).

By hypothesis, if the real number \( \alpha \) is sufficiently small, (7) and the system,

\[
\begin{align*}
\dot{x}_i &= p_i(t) x_i & (i = 1, \ldots, k - 1) \\
\dot{x}_k &= [p_k(t) + \alpha] x_k \\
\dot{x}_i &= p_i(t) x_i & (i = k + 1, \ldots, n),
\end{align*}
\]

are topologically equivalent. Let \( h(t, x) \) be the function giving the equivalence. Then, by the identity (4) and with \( e_k \) as the vector with \( k \)th component 1 and all others zero,

\[
h_i(t, e^{\phi_2(t) - \phi_k(t)} x_k e_k) = e^{\phi_i(t) - \phi_k(t)} h_i(s, x_k e_k)
\]

for all \( t, s \), scalar \( x_k \) and \( i = k + 1, \ldots, n \). There exists a sequence \( t_m \to \infty \) such that \( \phi_k(t_m) \) is bounded above. Then, using (iii) in the definition of topological equivalence,

\[
e^{\phi_i(t_m) - \phi_k(t_m)} h_i(s, x_k e_k) = h_i(t_m, e^{\phi_k(t_m) - \phi_k(t_m)} x_k e_k)
\]

is bounded as \( m \to \infty \). Since \( \phi_i(t_m) \to \infty \) as \( m \to \infty \) if \( i > k \), we conclude that

\[
h_i(s, x_k e_k) = 0 \quad \text{for all } s, x_k \text{ and } i = k + 1, \ldots, n.
\]

We now show that there exists a positive integer \( m \) such that

\[
| \phi_k(t + m) - \phi_k(t) | \geq 1 \quad \text{for all } t \geq m. \tag{9}
\]

If not, then there exists a sequence \( t_m \to \infty \) such that

\[
| \phi_k(t_m + m) - \phi_k(t_m) | \leq 1.
\]

By the identity (4),

\[
h_i(t_m + m, \Delta e_k) = e^{\phi_k(t_m + m) - \phi_k(t_m)} h_i(t_m, \Delta e_k) e^{\phi_k(t_m + m) - \phi_k(t_m + m)} e_k)
\]

\[
(i = 1, \ldots, k - 1),
\]

where \( \Delta > 0 \) is such that \( | h(t, x) | \geq 1 \) if \( | x | \geq \Delta \). Also

\[
h_k(t_m + m, \Delta e_k) = e^{\phi_k(t_m + m) - \phi_k(t_m) + \alpha \Delta} h_k(t_m, \Delta e_k) e^{\phi_k(t_m + m) - \phi_k(t_m + m)} e_k).
\]
Using (8),
\[
\phi_{k-1}(t_m + m) - \phi_{k-1}(t_m) \leq \phi_k(t_m + m) - \phi_k(t_m) - \beta - am
\]
\[
\leq 1 - \beta - am \to -\infty \quad \text{as} \quad m \to \infty.
\]
Repeating this argument, we conclude that \(\phi_i(t_m + m) - \phi_i(t_m) \to -\infty\) for \(i = 1, \ldots, k - 1\). Also note that by property (iii) of topological equivalence, \(h(t_m, \Delta e_k(t_m - \phi_k(t_m + m))\epsilon_k)\) is bounded as \(m \to \infty\).

Hence, if we choose \(\alpha < 0\), \(h_i(t_m + m, \Delta e_k) \to 0\) as \(m \to \infty\) for \(i = 1, \ldots, k\). But we already know that \(h_i(t_m + m, \Delta e_k) = 0\) if \(i = k + 1, \ldots, n\) and so we deduce that \(h(t_m + m, \Delta e_k) \to 0\). This contradicts the fact that \(|h(t_m + m, \Delta e_k)| \geq 1\).

So (9) does hold for some \(m\) and hence, by Proposition 1 in [1, p. 143], the scalar equation,
\[
\dot{x}_k = p_k(t) x_k,
\]
has an exponential dichotomy. Moreover, since \(\phi_k(t)\) does not tend to \(\infty\) as \(t \to \infty\) there must exist constants \(K > 0\), \(\gamma > 0\) such that for \(0 \leq s \leq t\),
\[
e^{\phi_k(t) - \phi_k(s)} \leq Ke^{-\gamma(t-s)}.
\]
Then, by (8),
\[
e^{\phi_{k-1}(t) - \phi_{k-1}(s)} \leq e^{\phi_k(t) - \phi_k(s) - \beta - \alpha(t-s)}
\]
\[
\leq Ke^{-\beta e^{-(\gamma + \alpha)(t-s)}} \quad \text{if} \quad 0 \leq s \leq t.
\]
Repeating this argument we see that each of the first \(k\) equations in (7) has an exponential dichotomy.

If \(k = n\) we are finished. So suppose that \(k < n\) or that \(\phi_i(t) \to \infty\) as \(t \to \infty\) for all \(i\) (in the latter case, take \(k = 0\) in the following).

Suppose that
\[
\dot{x}_{k+1} = p_{k+1}(t) x_{k+1}
\]
does not have an exponential dichotomy. Then, as above, there exists a sequence \(t_m \to \infty\) such that
\[
|\phi_{k+1}(t_m + m) - \phi_{k+1}(t_m)| \leq 1.
\]
Now (7) and
\[
\dot{x}_i = p_i(t) x_i \quad (i = 1, \ldots, k)
\]
\[
\dot{x}_{k+1} = [p_{k+1}(t) + \alpha] x_{k+1}
\]
\[
\dot{x}_i = p_i(t) x_i \quad (i = k + 2, \ldots, n)
\]
are topologically equivalent if \( \alpha > 0 \) is small enough. Let \( h(t, x) \) be the function giving the equivalence. Then, by the identity (4) with \( \Delta \) chosen as above,

\[
h_i(t_m, \Delta e_{k+1}) = e^\phi(t_m)h_i(0, \Delta e_{k+1}) (i = 1, \ldots, k),
\]

\[
h_{k+1}(t_m, \Delta e_{k+1}) = e^{\phi_{k+1}(t_m)} - \phi_{k+1}(t_m + m) - \beta am \quad \Delta e_{k+1} (t_m + m) \quad \phi_{k+1}(t_m) e_{k+1} (i = k + 2, \ldots, n).
\]

Note that \( \phi_{k+1}(t_m) \to \infty \) as \( m \to \infty \). Also \( \phi_i(t_m) \to -\infty \) for \( i = 1, \ldots, k \), and by (8),

\[
\phi_{k+1}(t_m) - \phi_{k+2}(t_m + m) \leq \phi_{k+1}(t_m) - \phi_{k+1}(t_m + m) - \beta - am

\leq 1 - \beta - am \to -\infty \quad \text{as} \quad m \to \infty.
\]

Repeating this argument, we deduce that \( \phi_i(t_m) - \phi_i(t_m + m) \to -\infty \) if \( i = k + 2, \ldots, n \).

So we conclude that \( h(t_m, \Delta e_{k+1}) \to 0 \) as \( m \to \infty \). This leads to a contradiction, as before.

Hence (10) does have an exponential dichotomy and since \( \phi_{k+1}(t) \to \infty \) as \( t \to \infty \) there must exist constants \( K > 0, \gamma > 0 \) such that for \( 0 \leq t \leq s \)

\[
e^{\phi_{k+1}(t) - \phi_{k+1}(s)} \leq Ke^{-\gamma(s-t)}.
\]

Then, by (8),

\[
e^{\phi_{k+2}(t) - \phi_{k+1}(s)} \leq e^{\phi_{k+1}(t) - \phi_{k+1}(s) - \beta - a(s-t)}

\leq Ke^{\beta a(t+1)} \quad \text{if} \quad 0 \leq t \leq s.
\]

Repeating this argument we deduce that each of the last \( (n - k) \) equations in (7) has an exponential dichotomy. This means that the whole system (7) has an exponential dichotomy and so we are finished.

6. Final Remarks

If \( \alpha(t) \) is a real continuous function with

\[
1 < \inf_{t \geq 0} \alpha(t) \leq \sup_{t \geq 0} \alpha(t) < \sqrt{2},
\]

then it can be shown, as in Coppel [8, p. 52–53], that the scalar equation,

\[
\dot{x} = [\sin \log(t + 1) + \cos \log(t + 1) - \alpha(t)]x,
\]

is asymptotically stable but not uniformly stable on \([0, \infty)\). This shows that the systems (1) with exponential dichotomy are not dense in \( \mathcal{B} \) and hence also that the structurally stable systems are not dense.
Note also that it is a consequence of our theorem that in \( S \) structural stability is preserved by topological equivalence.

Finally, it is clear that the theorem will still hold if in (ii) in the definition of topological equivalence it is only assumed that \( h_t \) is a bijection for each \( t \).

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**References**