



The $D^{10}\mathcal{R}^4$ term in type IIB string theory

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Abstract

The modular invariant coefficient of the $D^{2k}\mathcal{R}^4$ term in the effective action of type IIB superstring theory is expected to satisfy Poisson equation on the fundamental domain of $SL(2, \mathbb{Z})$. Under certain assumptions, we obtain the equation satisfied by $D^{10}\mathcal{R}^4$ using the tree level and one loop results for four graviton scattering in type II string theory. This leads to the conclusion that the perturbative contributions to $D^{10}\mathcal{R}^4$ vanish above three loops, and also predicts the coefficients at two and three loops.

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1. Introduction

Understanding higher derivative corrections to the supergravity action is an important problem in string theory as well as in M theory. In particular, the large amount of supersymmetry and the exact $SL(2, \mathbb{Z})$ invariance of type IIB superstring theory allows us to study in detail certain higher derivative corrections to the type IIB supergravity action. Considering configurations where the axion-dilaton is constant, the effective action of type IIB superstring theory can be schematically written as

$$S = \frac{S^{(0)}}{\alpha'^4} + \frac{S^{(3)}}{\alpha'} + \sum_{n=1}^{\infty} \alpha'^n S^{(n+4)} + \dots, \quad (1)$$

where \dots represents the terms that are non-perturbative in α' . In the expression in (1), $S^{(0)}$ is the type IIB supergravity action, and the first corrections to it are at $O(1/\alpha')$. (Unlike the usual treatments, if the axion-dilaton is not constant, the structure of the terms in the effective action is different [1].) There are certain terms in the effective action that are tractable because they satisfy various conjectured (which have been proven in some cases) non-renormalization theorems. In fact, the axion-dilaton dependence of these terms can be completely determined in some cases, as we briefly review below (see [2] for various details). In the discussion below, we shall denote the type IIB

axion-dilaton by the complexified coupling

$$\tau \equiv \tau_1 + i\tau_2 = C^0 + \frac{i}{e^\phi}, \quad (2)$$

where ϕ is the dilaton and C^0 is the Ramond–Ramond pseudoscalar field. At $O(1/\alpha')$, one of the protected terms in the effective action is given in the string frame by¹

$$\frac{1}{\alpha'} \int d^{10}x \sqrt{-g} e^{-\phi/2} Z_{3/2}(\tau, \bar{\tau}) \mathcal{R}^4, \quad (3)$$

where \mathcal{R}^4 involves four powers of the Weyl curvature tensor. Its coupling dependence is given by the non-holomorphic Eisenstein series of modular weight $(0, 0)$,

$$Z_s(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}}, \quad (4)$$

for the value $s = 3/2$. Expanding $Z_{3/2}$ at weak coupling, one can show that it receives only two perturbative contributions at tree level and at one loop, as well as an infinite number of non-perturbative contributions due to D-instantons [3–8]. This kind of dramatic non-renormalization is a generic property which characterizes these protected terms. There are other terms at $O(1/\alpha')$ in the effective action which are related to (3) by supersymmetry, which also satisfy non-renormalization theorems.

¹ Note that \mathcal{R}^4 is modular invariant only after transforming to the Einstein frame.

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For example, one such term is given by

$$\frac{1}{\alpha'} \int d^{10}x \sqrt{-g} e^{-\phi/2} f^{(12,-12)}(\tau, \bar{\tau}) \lambda^{16}, \tag{5}$$

where λ is the complex dilatino of type IIB string theory and $f^{(12,-12)}$ has modular weight $(12, -12)$ [9,10]. In fact, the higher derivative terms in the effective action which are of the form $D^{2k} \mathcal{R}^4$ satisfy conjectured non-renormalization theorems, and so it is an interesting problem to determine their coupling dependence. Also the terms in the effective action which are related to $D^{2k} \mathcal{R}^4$ by supersymmetry, for example $\hat{G}^{2k} \lambda^{16}$ (\hat{G} involves the three-form field strength and certain fermion bilinears), are not renormalized (see [11] for the case when $k = 2$).

The conjectured non-renormalization theorem for the $D^{2k} \mathcal{R}^4$ term ($k > 0$) in the type IIB effective action predicts that this term does not receive perturbative corrections above k string loops [12,13]. In fact, this has been proven for $0 < k < 6$ [12]. The structure of these terms has been worked out for some values of k . It turns out that some of these terms actually receive even fewer perturbative contributions, as some of the string loop coefficients vanish. For example at $O(\alpha')$, the relevant term in the effective action is given by

$$\alpha' \int d^{10}x \sqrt{-g} e^{\phi/2} Z_{5/2}(\tau, \bar{\tau}) D^4 \mathcal{R}^4, \tag{6}$$

which receives perturbative contributions only at tree level and at two loops [14–17] (also see [18,19]). At $O(\alpha'^2)$, we have the term

$$\alpha'^2 \int d^{10}x \sqrt{-g} e^{\phi} \mathcal{E}_{(3/2,3/2)}(\tau, \bar{\tau}) D^6 \mathcal{R}^4, \tag{7}$$

where $\mathcal{E}_{(3/2,3/2)}$ receives contributions from 0, 1, 2, and 3 loops [20]. Similar is the analysis at the next order, where we have the term

$$\alpha'^3 \int d^{10}x \sqrt{-g} e^{3\phi/2} Z_{7/2}(\tau, \bar{\tau}) D^8 \mathcal{R}^4 \tag{8}$$

which receives perturbative contributions only at tree level and at three loops [21]. Based on the conjectures stated above, this pattern of non-renormalization is believed to persist at higher orders in α' (see [21] for example, for a series of such conjectures).

Note that the coupling dependence of all the terms in the effective action described above (except (7)) is given by the Eisenstein series Z_s for specific values of s . It is easy to see that Z_s satisfies the differential equation

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} Z_s(\tau, \bar{\tau}) = s(s-1) Z_s(\tau, \bar{\tau}). \tag{9}$$

Thus Z_s is an eigenfunction of the Laplace operator on the fundamental domain of $SL(2, \mathbb{Z})$. However, the coefficient of the $D^6 \mathcal{R}^4$ term satisfies the differential equation [20]

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} \mathcal{E}_{(3/2,3/2)} = 12\mathcal{E}_{(3/2,3/2)} - 6Z_{3/2}^2. \tag{10}$$

Thus $\mathcal{E}_{(3/2,3/2)}$ satisfies the Laplace equation in the presence of a source term on the fundamental domain of $SL(2, \mathbb{Z})$, which

can be understood heuristically by considering the supersymmetry transformations at higher orders in α' . Clearly as we go to higher and higher orders in the derivative expansion of the effective action, we expect the coupling dependent coefficients to satisfy the Laplace equation in presence of the source terms [20, 22]. Thus the coupling dependent coefficients of the $D^{2k} \mathcal{R}^4$ terms for all k generically satisfy Poisson equations on the fundamental domain of $SL(2, \mathbb{Z})$ (for low values of k , the source terms vanish).

Based on heuristic arguments of supersymmetry and the structure of the three loop four graviton amplitude of eleven dimensional supergravity compactified on S^1 and T^2 , it seems natural to assume that the coefficient of the $D^{10} \mathcal{R}^4$ term in the type IIB effective action is given by

$$\alpha'^4 \int d^{10}x \sqrt{-g} e^{2\phi} \mathcal{E}_{(3/2,5/2)}(\tau, \bar{\tau}) D^{10} \mathcal{R}^4, \tag{11}$$

which satisfies the Poisson equation

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} \mathcal{E}_{(3/2,5/2)} = \lambda_1 \mathcal{E}_{(3/2,5/2)} + \lambda_2 Z_{3/2} Z_{5/2}, \tag{12}$$

on the fundamental domain of $SL(2, \mathbb{Z})$ [20], where λ_1 and λ_2 are numerical factors. In this Letter, we shall solve for λ_1 and λ_2 based on known results about the four graviton scattering amplitude in type IIB superstring theory at tree level and one loop. Apart from completely specifying the equation in (12), this will automatically lead to predictions for the two loop and three loop coefficients of the four graviton amplitude in superstring perturbation theory, as well as equations that give the contributions due to D-instantons.

We would like to stress that the ansatz suggested in [20] for the coefficient of the $D^{10} \mathcal{R}^4$ term does not give the complete picture, in particular, it does not satisfy various constraints imposed by unitarity [1]. However, our analysis does illustrate some of the features of the exact answer.

We also discuss certain generalizations for terms at higher order in the α' expansion to understand some features of the exact solution. In particular, we consider the $D^{12} \mathcal{R}^4$ and $D^{14} \mathcal{R}^4$ terms in the effective action. We propose that the coupling dependences of these terms also satisfy Poisson equation on the fundamental domain of $SL(2, \mathbb{Z})$. (For $D^{12} \mathcal{R}^4$, there is another possibility where the coupling dependence satisfies the Laplace equation.) Simply based on the assumed structure of these equations, we obtain simple vanishing theorems for the perturbative contributions to these terms.

2. Some features of the $D^{10} \mathcal{R}^4$ coupling dependence

In order to determine the differential equation satisfied by $\mathcal{E}_{(3/2,5/2)}$, we write it as

$$\mathcal{E}_{(3/2,5/2)}(\tau, \bar{\tau}) = \mathcal{E}_{(3/2,5/2)}^{(0)}(\tau_2) + \sum_{k \neq 0} \mathcal{E}_{(3/2,5/2)}^{(k)}(\tau_2) e^{2\pi i k \tau_1}. \tag{13}$$

The “zero-mode” piece $\mathcal{E}_{(3/2,5/2)}^{(0)}$ is independent of τ_1 , and receives two kinds of contributions:

- (i) the perturbative string loop contributions which involve power law behavior in τ_2 ,

(ii) the non-perturbative contributions due to D-instantons and anti-D-instantons carrying equal and opposite charges. Thus in these terms which receive contributions from double instantons, the $e^{2\pi i k \tau_1}$ factor from the D-instanton of charge k cancels the $e^{-2\pi i k \tau_1}$ factor from the anti-D-instanton of charge $-k$. So at weak coupling, the leading behavior of this part of the zero-mode should be given by

$$\sum_{k \neq 0} f_k \tau_2^{w_k} e^{-4\pi |k| \tau_2}, \tag{14}$$

and thus these contributions are exponentially suppressed.

The “non-zero mode” part of $\mathcal{E}_{(3/2,5/2)}$ which contains the entire τ_1 dependence, receives contributions from $\mathcal{E}_{(3/2,5/2)}^{(k)}$ for all non-zero values of k . From the form of (13), we see that $\mathcal{E}_{(3/2,5/2)}^{(k)}$ gives the non-perturbative contribution from the charge k sector. In fact, this can arise from two sources: the charge k single D-instanton contribution, or the double D-instanton contribution from two D-instantons of charges k_1 and k_2 such that $k = k_1 + k_2 \neq 0$.

We first determine the two numerical constants λ_1 and λ_2 in (12) which completely specify the differential equation that $\mathcal{E}_{(3/2,5/2)}$ satisfies. Note that we define the $D^{10}\mathcal{R}^4$ term in the type IIB effective action to be given by the specific structure of index contractions such that it leads to a contribution proportional to

$$\sigma_2 \sigma_3 \equiv \left(\frac{\alpha'}{4}\right)^5 (s^2 + t^2 + u^2)(s^3 + t^3 + u^3) \tag{15}$$

in string amplitudes, where s, t , and u are the Mandelstam variables satisfying $s + t + u = 0$. In order to obtain λ_1 and λ_2 , we shall make use of two pieces of information about the four graviton scattering amplitude in type II superstring theory; namely, the coefficients of (15) at tree level and at one loop in superstring perturbation theory.²

The relevant term at tree level is given by [23,24]

$$A^{\text{tree}} = \kappa_{10}^2 e^{-2\phi} \hat{K} \left(\dots + \frac{2}{3} \zeta(3) \zeta(5) \sigma_2 \sigma_3 + \dots \right), \tag{16}$$

where \hat{K} is the linearized approximation to \mathcal{R}^4 [23,25]. Also, the relevant term at one loop is given by [13,15]

$$A^{\text{one loop}} = 4\zeta(2) \kappa_{10}^2 \hat{K} \left(\dots + \frac{29\zeta(5)}{960} \cdot \frac{5 \cdot 4^5}{6 \cdot 5!} \sigma_2 \sigma_3 + \dots \right). \tag{17}$$

2.1. The perturbative contribution to $D^{10}\mathcal{R}^4$

The perturbative part of the zero-mode piece $\mathcal{E}_{(3/2,5/2)}^{(0)}(\tau_2)$ can receive contributions up to five string loops [12,13]. Thus using (16) and (17) we have that

$$\mathcal{E}_{(3/2,5/2)}^{(0)}(\tau_2) = \frac{2}{3} \zeta(3) \zeta(5) \tau_2^4 + \frac{29 \cdot 5 \cdot 4^6 \zeta(2) \zeta(5)}{960 \cdot 6 \cdot 5!} \tau_2^2 + A + \frac{B}{\tau_2^2} + \frac{C}{\tau_2^4} + \frac{D}{\tau_2^6} + \dots, \tag{18}$$

² At one loop, the four graviton scattering amplitude is the same in type IIA and type IIB string theories.

where the \dots denotes the non-perturbative terms involving the instanton–anti-instanton contributions discussed above. In (18), A, B, C , and D are the coefficients at 2, 3, 4, and 5 string loops, respectively. The basic idea is now to use (12) to write down the differential equation satisfied by the perturbative piece of $\mathcal{E}_{(3/2,5/2)}^{(0)}$. In order to do so, we shall need the expression for Z_s given by

$$Z_s(\tau, \bar{\tau}) = 2\zeta(2s) \tau_2^s + 2\sqrt{\pi} \tau_2^{1-s} \frac{\Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s)} + \frac{4\pi^s \sqrt{\tau_2}}{\Gamma(s)} \sum_{k \neq 0} |k|^{s-1/2} \mu(k, s) K_{1/2-s}(2\pi |k| \tau_2) e^{2\pi i k \tau_1}, \tag{19}$$

where

$$\mu(k, s) = \sum_{m|k} \frac{1}{m^{2s-1}}. \tag{20}$$

Note the perturbative contributions to Z_s are given by the first two terms in (19). Plugging in the perturbative contributions from the various terms into (12), and equating the coefficients of different powers of τ_2 , we get the system of equations

$$\begin{aligned} 8 - \frac{2}{3} \lambda_1 &= 4\lambda_2, \\ \lambda_2 &= \frac{29(2-\lambda_1)}{270}, \\ \lambda_1 A &= -\frac{16}{3} \lambda_2 \zeta(3) \zeta(4), \\ (6-\lambda_1) B &= \frac{32}{3} \lambda_2 \zeta(2) \zeta(4), \\ (20-\lambda_1) C &= (42-\lambda_1) D = 0. \end{aligned} \tag{21}$$

The solution to (21) is given by

$$\begin{aligned} \lambda_1 &= \frac{241}{8}, & \lambda_2 &= -\frac{145}{48}, & A &= \frac{8 \cdot 145}{9 \cdot 241} \zeta(3) \zeta(4), \\ B &= \frac{16 \cdot 145}{9 \cdot 193} \zeta(2) \zeta(4), & C &= D = 0. \end{aligned} \tag{22}$$

Thus we see that $\mathcal{E}_{(3/2,5/2)}$ satisfies the Poisson equation

$$4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} \mathcal{E}_{(3/2,5/2)} = \frac{241}{8} \mathcal{E}_{(3/2,5/2)} - \frac{145}{48} Z_{3/2} Z_{5/2} \tag{23}$$

on the fundamental domain of $SL(2, \mathbb{Z})$.

Also from (22), we see that the four and five loop coefficients vanish, and A and B give predictions for the two loop and three loop amplitudes, respectively. Thus the $D^{10}\mathcal{R}^4$ term in the type IIB effective action receives perturbative contributions only up to three loops.

2.2. The non-perturbative contribution to $D^{10}\mathcal{R}^4$

We now consider the non-perturbative part of $\mathcal{E}_{(3/2,5/2)}$, which receives contributions both from the zero-mode as well as the non-zero-mode terms in (13). Let us call $\tilde{\mathcal{E}}_{(3/2,5/2)}^{(0)}$ the non-perturbative part of $\mathcal{E}_{(3/2,5/2)}^{(0)}$. Then using (19) we see that

$\tilde{\mathcal{E}}_{(3/2,5/2)}^{(0)}$ satisfies the differential equation

$$\begin{aligned} & \left(\tau_2^2 \frac{\partial^2}{\partial \tau_2^2} - \frac{241}{8} \right) \tilde{\mathcal{E}}_{(3/2,5/2)}^{(0)} \\ &= -\frac{16 \cdot 145\pi^3 \tau_2}{3} \sum_{k \neq 0} |k|^3 \mu(k, 3/2) \mu(k, 5/2) \\ & \quad \times K_{-1}(2\pi |k| \tau_2) K_{-2}(2\pi |k| \tau_2). \end{aligned} \quad (24)$$

The term on the right-hand side of (24) contains the total contribution from instanton–anti-instanton configurations which carry total charge zero. Similarly it is easy to see that $\mathcal{E}_{(3/2,5/2)}^{(k)}$ satisfies the differential equation

$$\begin{aligned} & \left(\tau_2^2 \frac{\partial^2}{\partial \tau_2^2} - 4\pi^2 k^2 - \frac{241}{8} \right) \mathcal{E}_{(3/2,5/2)}^{(k)} \\ &= -290\pi \left(\frac{2\pi}{3} \{ \zeta(3) \tau_2^2 + 2\zeta(2) \} k^2 \mu(k, 5/2) K_{-2}(2\pi |k| \tau_2) \right. \\ & \quad + \left\{ \zeta(5) \tau_2^3 + \frac{4}{3} \zeta(4) \tau_2^{-1} \right\} |k| \mu(k, 3/2) K_{-1}(2\pi |k| \tau_2) \\ & \quad + \frac{8\pi^2 \tau_2}{3} \sum_{k_1 \neq 0, k_2 \neq 0, k_1+k_2=k} |k_1| |k_2|^2 \mu(k_1, 3/2) \mu(k_2, 5/2) \\ & \quad \left. \times K_{-1}(2\pi |k_1| \tau_2) K_{-2}(2\pi |k_2| \tau_2) \right). \end{aligned} \quad (25)$$

The first two terms on the right-hand side of (25) give the contributions due to single instantons of charge k , while the last term gives the double instanton contributions with total charge $k_1 + k_2 = k$. One can obtain the D-instanton contributions to $\mathcal{E}_{(3/2,5/2)}$ from the differential equations above.

For example, using the asymptotic expansion

$$K_s(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad (26)$$

for large z , the leading contribution to $\tilde{\mathcal{E}}_{(3/2,5/2)}^{(0)}$ at weak coupling is given by

$$\tilde{\mathcal{E}}_{(3/2,5/2)}^{(0)}(\tau_2) \approx -\frac{145\pi}{12\tau_2^2} \sum_{k \neq 0} \mu(k, 3/2) \mu(k, 5/2) e^{-4\pi |k| \tau_2}, \quad (27)$$

which is of the form (14).

Thus we see that the coupling dependence of the $D^{10}\mathcal{R}^4$ term in the effective action of type IIB superstring theory is given by the Poisson equation (23), which leads to predictions for the two loop and three loop scattering amplitude of four gravitons in type IIB superstring theory.³ It should be possible to verify the prediction for the two loop amplitude along the lines of [16,17].

³ At two loops, the amplitude is the same in type IIB and type IIA string theories. It has been shown [12] that the perturbative contributions are the same to all loop orders for $D^{2k}\mathcal{R}^4$ for $k \leq 4$.

3. Some further generalizations

In order to illustrate some features of the coupling dependence of the $D^{2k}\mathcal{R}^4$ terms in the type IIB effective action for higher values of k using the method described above, one has to know the four graviton amplitude in type IIB superstring theory to sufficiently high order in the genus expansion, which is a difficult problem. However, as we now show, it is easy to obtain certain vanishing theorems for the perturbative contributions which we illustrate below with two examples.

3.1. The $D^{12}\mathcal{R}^4$ term

The $D^{12}\mathcal{R}^4$ term arises at $O(\alpha'^5)$ in the effective action of type IIB string theory. From the tree level amplitude, one can easily see that there are two independent ways of contracting the various indices, which lead to contributions proportional to σ_2^3 and σ_3^2 . In fact the tree level contributions are proportional to $\zeta(9)$ and $\zeta(3)^3$. Thus one linear combination of σ_2^3 and σ_3^2 should lead to the term in the effective action (where in $D^{12}\mathcal{R}^4$ the indices are to be contracted appropriately) [21]

$$\alpha'^5 \int d^{10}x \sqrt{-g} e^{5\phi/2} Z_{9/2}(\tau, \bar{\tau}) D^{12}\mathcal{R}^4, \quad (28)$$

which gives $\zeta(9)$ at tree level. So there are only two perturbative contributions at tree level and at four loops. Another linear combination of σ_2^3 and σ_3^2 yields

$$\alpha'^5 \int d^{10}x \sqrt{-g} e^{5\phi/2} \mathcal{E}_{(3/2,3/2,3/2)}(\tau, \bar{\tau}) D^{12}\mathcal{R}^4, \quad (29)$$

which must give $\zeta(3)^3$ at tree level. This suggests that $\mathcal{E}_{(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})}$ should satisfy Poisson equation of the form

$$\begin{aligned} & 4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} \mathcal{E}_{(3/2,3/2,3/2)} \\ &= \lambda_1 \mathcal{E}_{(3/2,3/2,3/2)} + \lambda_2 \mathcal{E}_{(3/2,3/2)} Z_{3/2} + \lambda_3 Z_{3/2}^3. \end{aligned} \quad (30)$$

One should be able to fix the three undetermined coefficients in (30) if the four point graviton amplitude is known up to two loop level at this order in the derivative expansion. However, simply based on the structure of (30) without any additional information, we can obtain a constraint on the perturbative contributions as we now explain. Assuming that $D^{12}\mathcal{R}^4$ can receive perturbative contributions only up to six loops, we can write the perturbative part of $\mathcal{E}_{(3/2,3/2,3/2)}$ as

$$\mathcal{E}_{(3/2,3/2,3/2)}^{\text{pert}}(\tau_2) = \dots + \frac{A}{\tau_2^{11/2}} + \frac{B}{\tau_2^{15/2}}, \quad (31)$$

where A and B are the five and six loop contributions, respectively, and the \dots stands for the other lower loop perturbative contributions. Then (30) implies that

$$\left(\frac{11 \cdot 13}{4} - \lambda_1 \right) A = \left(\frac{15 \cdot 17}{4} - \lambda_1 \right) B = 0, \quad (32)$$

and so A and B cannot be both non-vanishing. Thus at least one of the two highest loop contributions to $\mathcal{E}_{(3/2,3/2,3/2)}$ must vanish.

3.2. The $D^{14}\mathcal{R}^4$ term

Proceeding exactly along the same lines as before, we note that the $\sigma_2^2\sigma_3$ contribution to the tree level amplitude is proportional to $\zeta(5)^2$ as well as $\zeta(3)\zeta(7)$. Thus the $D^{14}\mathcal{R}^4$ term in the effective action is given by

$$\alpha'^6 \int d^{10}x \sqrt{-g} e^{3\phi} \mathcal{E}_{(3/2,5/2,7/2)}(\tau, \bar{\tau}) D^{12}\mathcal{R}^4, \quad (33)$$

where $\mathcal{E}_{(3/2,5/2,7/2)}$ should satisfy the Poisson equation

$$4\tau_2^2 \frac{\partial^2}{\partial\tau\partial\bar{\tau}} \mathcal{E}_{(3/2,5/2,7/2)} = \lambda_1 \mathcal{E}_{(3/2,5/2,7/2)} + \lambda_2 Z_{3/2} Z_{7/2} + \lambda_3 Z_{5/2}^2. \quad (34)$$

Again assuming that $D^{14}\mathcal{R}^4$ can receive perturbative contributions only up to seven loops, we see that

$$\mathcal{E}_{(3/2,5/2,7/2)}^{\text{pert}}(\tau_2) = \dots + \frac{A}{\tau_2^5} + \frac{B}{\tau_2^7} + \frac{C}{\tau_2^9}, \quad (35)$$

where A , B and C are the five, six and seven loop contributions, respectively, and the \dots stands for the lower loop perturbative contributions. Then (34) implies that

$$(30 - \lambda_1)A = (56 - \lambda_1)B = (90 - \lambda_1)C = 0, \quad (36)$$

and so at least two of A , B , and C must vanish.

Clearly this kind of analysis can be carried out for the $D^{2k}\mathcal{R}^4$ terms for higher values of k . This will give vanishing theorems for the perturbative contributions at high orders in the string loop expansion. It would be interesting to use such constraints along with the explicit coefficients of the four graviton scattering amplitude at low string loops to try to constrain the coupling dependence of the various protected higher derivative terms in type IIB superstring theory. However, as mentioned before, this kind of analysis does not give the complete structure of the coupling dependence satisfied by the coefficients of the higher derivative terms, although it does illustrate some of the general features.

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