A converse inequality of higher order weighted arithmetic and geometric means of positive definite operators

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Abstract

In this paper we consider weighted arithmetic and geometric means of higher orders constructed by the symmetrization method appeared in Ando–Li–Mathias’s definition of multi-variable geometric means and the arithmetic–geometric mean inequality of higher order weighted version. We establish a converse inequality of higher order weighed arithmetic and geometric means via Specht ratio.

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1. Introduction

In [2], Ando–Li–Mathias proposed a successful definition for geometric mean of several positive definite matrices by the method of the symmetrization procedure. Most properties of geometric mean of two positive definite matrices are extended to higher order geometric means, for instance, the arithmetic–geometric mean inequality: \( G(A_1, A_2, \ldots, A_n) \leq \frac{1}{n}(A_1 + A_2 + \cdots + A_n) \). For positive real numbers \( a_i \) with \( 0 < m \leq a_i \leq M \), a converse of arithmetic–geometric means inequality via Specht ratio is well-known.

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\[
\frac{a_1 + a_2 + \cdots + a_n}{n} \leq S_h(a_1^a \cdots a_n)^{1/n}, \tag{1.1}
\]

where \( h = \frac{M}{m} \) and \( S_h = \frac{(h-1)h^{(h-1)-1}}{e \log h} \) is the optimal constant. In [12] Yamazaki derived a converse inequality (1.1) of positive definite operators via Kantorovich constant.

The main purpose of this paper is to establish a converse inequality (1.1) in the context of higher order weighted arithmetic and geometric means via Specht ratio. Higher order weighted arithmetic and geometric means can be obtained from the symmetrization method [6], but not directly even for positive scalars (see Remark 2.2). In fact, the resulting weighted means are not invariant under permutation unless \( t = 1/2 \). Let \( A, B \) and \( C \) be positive definite operators in a Hilbert space and \( 0 < t < 1 \). Then the symmetrization procedures of \( t \)-weighted arithmetic and geometric means

\[
\alpha(A, B, C) = ((1 - t)B + tC, (1 - t)A + tC, (1 - t)A + tB),
\]

\[
\beta(A, B, C) = (B^{#t}C, A^{#t}C, A^{#t}B), \quad A^{#t}B = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2},
\]

are power convergent under the Thompson metric, existing an (congruence transformations and inversion) invariant metric \( d(A, B) \) on the convex cone of positive definite operators. That is, there exist positive definite operators \( A(t : A, B, C), G(t : A, B, C) \) such that

\[
\lim_{n \to \infty} \alpha^n(A, B, C) = (A(t : A, B, C), A(t : A, B, C), A(t : A, B, C)),
\]

\[
\lim_{n \to \infty} \beta^n(A, B, C) = (G(t : A, B, C), G(t : A, B, C), G(t : A, B, C)),
\]

respectively. Inductively the weighted arithmetic and geometric means are extended to all orders satisfying \( G(t : A_1, A_2, \ldots, A_n) \leq A(t : A_1, A_2, \ldots, A_n) \). In this paper we prove a converse inequality

\[
A(t : A_1, A_2, \ldots, A_n) \leq \left( S_h(n-1)! \right)^{1/2} G(t : A_1, A_2, A_n),
\]

where \( \rho = \max \{t, 1 - t\} \) and \( \log h = A(\Omega) \), the diameter of \( \Omega = (A_1, A_2, \ldots, A_n) \) for the Thompson metric \( d \). This is an extension of the converse inequality of order 2 [8]: \((1 - t)A + tB \leq S_h(A^{#t}B)\), where \( h = e^{d(A, B)} \).

Throughout this paper, we assume that \( \mathcal{H} \) is a Hilbert space and \( \Omega \) is the convex cone of positive definite operators on \( \mathcal{H} \). For Hermitian operators \( X \) and \( Y \), we write that \( X \preceq Y \) if \( Y - X \) is positive semidefinite, and \( X < Y \) if \( Y - X \) is positive definite (positive semidefinite and invertible).

2. Higher order weighted operator means

Let \( t \) be a positive real number with \( 0 < t < 1 \), and let \( A, B > 0 \). The \( t \)-weighted (power) arithmetic, harmonic and geometric means are defined by

\[
A(t : A, B) = (1 - t)A + tB,
\]

\[
H(t : A, B) = ((1 - t)A^{-1} + tB^{-1})^{-1} = A(t : A^{-1}, B^{-1})^{-1},
\]

\[
G(t : A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2} = A^{#t}B.
\]
The weighted arithmetic–geometric–harmonic mean inequalities hold
\[ H(t : A, B) \leq G(t : A, B) \leq A(t : A, B). \]

In [6] and [7], Lawson and Lim showed that the weighted arithmetic, harmonic and geometric means have extensions of all higher orders using the symmetrization method appeared in Ando–Li–Mathias’s definition of multi-variable geometric means. Let \( M(t : A, B) \) be one of the weighted mean of arithmetic, harmonic and geometric means. Then the barycentric operator \( \beta_3 \) on \( \Omega^3 \) defined by
\[
\beta_3(A, B, C) = (M(t : B, C), M(t : A, C), M(t : A, B))
\]
is power convergent, that is, there exists a positive definite \( X \) such that
\[
\lim_{n \to \infty} \beta_n^3(A, B, C) = (X, X, X).
\]
Denote \( X = M(t : A, B, C) \). Inductively, if \( M(t : \cdot) \) is defined on \( \Omega^n \) by the symmetrization, then on \( \Omega^{n+1} \) the barycentric operator
\[
\beta_{n+1}(A) = (M(t : \pi_{\neq 1} \bar{A}), M(t : \pi_{\neq 2} \bar{A}), \ldots, M(t : \pi_{\neq n+1} \bar{A}))
\]
is power convergent, where \( \bar{A} = (A_1, A_2, \ldots, A_{n+1}) \in \Omega^{n+1} \) and
\[
\pi_{\neq j} \bar{A} := (A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_{n+1}) \in \Omega^n,
\]
resulting the \((n+1)\)-variable weighted mean \( M(t : \bar{A}) \) of \( \bar{A} \in \Omega^{n+1} \). The metric used in convergence is the Thompson metric on the open convex cone \( \Omega \) which is defined by
\[
d(A, B) = \max\{\log M(A/B), \log M(B/A)\},
\]
where \( M(A/B) := \inf\{\lambda > 0 : A \leq \lambda B\} \). Thompson [11] has shown that \( \Omega \) is a complete metric space with respect to this metric and the corresponding metric topology on \( \Omega \) agrees with the relative norm topology. The convergence of these symmetrizations for the common metric distance has played a key role in extending inequalities and invariant properties of means of two positive definite operators.

We observe that the barycentric operator \( \beta_n \) is monotone: \( A \leq B \) implies \( \beta_n(A) \leq \beta_n(B) \) and \( \beta_n(sA) = s\beta_n(A) \). Thus, resulting weighted means satisfy the monotone property and are invariant under scalar multiplications.

For \( t = 1/2 \) and \( M(1/2 : A, B) = G(1/2 : A, B) \), the higher order geometric mean extensions are discovered by Ando–Li–Mathias [2] and Petz–Temesi [10,9].

**Definition 2.1.** We denote the \( n \)-variable \( t \)-weighted arithmetic, harmonic and geometric means by
\[
A(t : \bar{A}), \quad G(t : \bar{A}), \quad H(t : \bar{A}), \quad \bar{A} \in \Omega^n, \quad 0 < t < 1.
\]

**Remark 2.2.** The higher order weighted arithmetic–harmonic–geometric mean inequalities \( H(t : \bar{A}) \leq G(t : \bar{A}) \leq A(t : \bar{A}) \) are easily derived by the construction and by the inequalities of two-variable ones. The geometric mean is invariant under the congruence transformations and inversion. For any invertible operator \( M \) on \( \mathcal{H} \),
\[
G(t : M \bar{A} M^*) = MG(t : \bar{A}) M^*, \quad G(t : \bar{A})^{-1} = G(t : \bar{A}^{-1}),
\]
where
\[
M(A_1, A_2, \ldots, A_n) M^* = (MA_1 M^*, MA_2 M^*, \ldots, MA_n M^*),
\]
\[
(A_1, A_2, \ldots, A_n)^{-1} = (A_1^{-1}, A_2^{-1}, \ldots, A_n^{-1}).
\]
For \( n = 3 \), \( A(t : A, B, C) \) and \( H(t : A, B, C) \) are explicitly presented by [7]

\[
A(t : A, B, C) = \frac{1 - t}{2 - t} A + \frac{1 - t + t^2}{2 + t - t^2} B + \frac{t}{1 + t} C,
\]

\[
H(t : A, B, C) = \left[ \frac{1 - t}{2 - t} A^{-1} + \frac{1 - t + t^2}{2 + t - t^2} B^{-1} + \frac{t}{1 + t} C^{-1} \right]^{-1}.
\]

One can directly see that the weighted means are not invariant under permutation unless \( t = 1/2 \).

**Definition 2.3.** For \( \mathbb{A} = (A_1, A_2, \ldots, A_n) \in \Omega^n \), we denote \( \Delta(\mathbb{A}) \) by the diameter of \( \mathbb{A} \) for the Thompson metric \( d(\mathbb{A}) = \max\{d(A_i, A_j) : 1 \leq i, j \leq n\} \).

**Proposition 2.4.** Let \( \beta_n : \Omega^n \to \Omega^n \) be the barycentric operator for the \( t \)-weighted geometric mean on \( \Omega^n \). Let \( \rho = \max\{t, 1-t\} \). Then for \( k \geq 1 \)

1. \( \Delta(\beta_n(\mathbb{A})) \leq \Delta(\mathbb{A}) \) for all \( \mathbb{A} \in \Omega^n \);
2. \( \Delta(\beta_n^k(\mathbb{A})) \leq (n-1)^k \Delta(\mathbb{A}) \).

**Proof.** The nonpositive curvature of the Thompson metric (cf. [3, 5])

\[
d(X_1 \#_t X_2, Y_1 \#_t Y_2) \leq (1 - t)d(X_1, Y_1) + td(X_2, Y_2), \quad t \in [0, 1]
\]

yields that the weighted geometric mean \( A \#_t B \) is a nonexpansive and coordinatewise \( \rho \)-contractive 2-mean following notions from [6]. By Proposition 3.13 of [6], the higher order extension mean \( G(t : \mathbb{A}) \) on \( \Omega^n \) is also nonexpansive and coordinatewise \( \rho \)-contractive. The nonexpansive property implies that

\[
d(G(t : \pi_{\neq i} \mathbb{A}), G(t : \pi_{\neq j} \mathbb{A})) \leq \Delta(\mathbb{A}),
\]

where \( \mathbb{A} \in \Omega^n \). Thus \( \Delta(\beta_n(\mathbb{A})) \leq \Delta(\mathbb{A}) \). The coordinatewise \( \rho \)-contractive and the triangle inequality imply \( \Delta(\beta_n^k(\mathbb{A})) \leq (n-1)^k \Delta(\mathbb{A}) \). See the proof of Lemma 3.8 of [6]. \( \square \)

**Remark 2.5.** For \( t = 1/2 \), \( \Delta(\beta_n^k(\mathbb{A})) \leq \rho^k \Delta(\mathbb{A}) \) holds true. For \( t \neq 1/2 \), it does not hold due to the nonsymmetric property of \( G(t : \mathbb{A}) \).

**Lemma 2.6.** Let \( 0 < A \leq B \) and let \( A \leq X, Y \leq B \). Then \( d(X, Y) \leq d(A, B) \).

**Proof.** From \( Y \leq B \leq M(B/A)A \leq M(B/A)X \) and \( X \leq B \leq M(B/A)A \leq M(B/A)Y, M(Y/X), M(X/Y) \leq M(B/A) \). \( \square \)

### 3. A converse inequality

**Definition 3.1.** For \( h, s \geq 1 \), we define the Specht ratio by

\[
S_h(s) = \frac{(h^s - 1)h^{s(h^s-1)-1}}{e \log h^s}, \quad S_1(s) = 1.
\]

(3.3)
Lemma 3.2. The Specht ratio $S_h := S_h(1)$ has the following properties.

(1) $s \mapsto S_h(s)^{1/s}$ and $h \mapsto S_h$ are increasing functions for $s \geq 1$ and $h \geq 1$, respectively.
(2) $S_{h^{ho}} \leq S_h^{ho}$ for $0 < \rho \leq 1$.

Proof

(1) Lemma 5 and Lemma 9 of [4].
(2) By (1), $S_h(1) \leq S_h(\rho^{-1})^\rho$ for all $0 < \rho \leq 1$. That is,

$$
\frac{(h - 1)h^{(h-1)^{-1}}}{e \log h} \leq \left\{ \frac{(h^{-1})h^{-1} - 1}{e \log h} \right\}^\rho.
$$

Replacing $h$ as $h^\rho(\geq 1)$, we obtain

$$
S_{h^\rho} = \frac{(h^\rho - 1)h^{\rho(h-1)^{-1}}}{e \log h^\rho} \leq \left\{ \frac{(h - 1)h^{(h-1)^{-1}}}{e \log h} \right\}^\rho = S_h^\rho.
$$

A converse inequality of weighted arithmetic and geometric means of two positive definite matrices (operators) by the Specht ratio appears in [8,1].

Theorem 3.3. Let $A$ and $B$ be two positive definite operators and let $0 < t < 1$. Then

$$
A(t : A, B) \leq S_h \cdot G(t : A, B),
$$
where $h = e^{d(A, B)}$ for the Thompson metric $d$.

Definition 3.4. Let $\beta_n : \Omega^p \to \Omega^p$ be the barycentric operator for the $t$-weighted geometric mean. For $A \in \Omega_n$, we define

$$
h_k(A) = e^{A^\beta_k(A)}, \quad h_0(A) = e^{A(A)}.
$$

Proposition 3.5. Let $0 < t < 1$ and $A \in \Omega^p$. Then

$$
limit sup_k S_{h_0(A)} S_{h_1(A)} \cdots S_{h_k(A)} \leq \left( S_{h_0(A)^{n-1}} \right)^{1/\rho}, \quad \rho = \max\{t, 1 - t\}.
$$

Proof. Let $A \in \Omega^n$. Set $h_i = h_i(A)$. By Proposition 2.4, $\Delta \left( \beta_n^k(A) \right) \leq (n-1)\rho^k A(A)$ and hence

$$
1 \leq h_k = e^{A(\beta_n^k(A))} \leq e^{(n-1)\rho^k A(A)} = h_0^{(n-1)\rho^k}.
$$

By Lemma 3.2, $S_{h_0} \leq S_{h_0^{n-1}}$ and $S_{h_k} \leq S_{h_0^{(n-1)\rho^k}} \leq \left( S_{h_0^{n-1}} \right)^{\rho^k}$ and hence

$$
S_{h_0} S_{h_1} \cdots S_{h_k} \leq S_{h_0^{(1+\rho+\rho^2+\cdots+\rho^k)}} \to S_{h_0^{1/\rho}}.
$$

Theorem 3.6. Let $A = (A_1, A_2, \ldots, A_n) \in \Omega^p$ and let $h_0 = e^{A(A)}$. Then

$$
A(t : A) \leq \left( S_{h_0^{(n-1)}} \right)^{1/(n-2)} G(t : A), \quad \rho = \max\{t, 1 - t\}.
$$
Moreover, if $A_i$ satisfies $0 < mI \leq A_i \leq MI$ with $m < M$, then

$$A(t : \mathcal{A}) \leq \left\{ \frac{(M^n! - m^n!)}{M^{n!-m!}} \left( \frac{1}{\frac{1}{1-p}} \right)^{n-2} \left\{ \frac{e\alpha^n}{n!(\log M - \log m)m^{n!-m!}} \right\} \right\} G(t : \mathcal{A}). \tag{3.6}$$

Proof. It holds true for $n = 2$ by Theorem 3.3. Suppose that (3.5) holds true for $n \geq 2$. Let $\alpha_{n+1}$ denote the barycentric operator for the $t$-weighted arithmetic mean on $Q^{n+1}$. From $\Delta(\pi_{\neq i}\mathcal{A}) \leq \Delta(\mathcal{A})$, Lemma 3.2 (1) and by induction argument,

$$A(t : \pi_{\neq i}\mathcal{A}) \leq \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} G(t : \pi_{\neq i}\mathcal{A}) \quad (h' = e^{A(\pi_{\neq i}\mathcal{A})} = h_0) \leq \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} \beta_{n+1}(\mathcal{A}), \quad i = 1, 2, \ldots, n + 1.$$

Therefore,

$$\alpha_{n+1}(\mathcal{A}) = (A(t : \pi_{\neq 1}\mathcal{A}), A(t : \pi_{\neq 2}\mathcal{A}), \ldots, A(t : \pi_{\neq n+1}\mathcal{A})) \leq \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} G(t : \pi_{\neq 1}\mathcal{A}), G(t : \pi_{\neq 2}\mathcal{A}), \ldots, G(t : \pi_{\neq n+1}\mathcal{A})) \leq \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} \beta_{n+1}(\mathcal{A}), \quad h_0 = e^{A(\mathcal{A})},$$

and by replacing $\mathcal{A}$ to $\beta_{n+1}(\mathcal{A})$,

$$\alpha_{n+1}(\beta_{n+1}(\mathcal{A})) \leq \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} \beta_{n+1}^2(\mathcal{A}), \quad h_1 = e^{A(\beta_{n+1}(\mathcal{A}))}.$$

By monotonicity of $\alpha_{n+1}$, we have

$$\alpha_{n+1}^2(\mathcal{A}) \leq \alpha_{n+1} \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} \beta_{n+1}(\mathcal{A})) \leq \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} \alpha_{n+1}(\beta_{n+1}(\mathcal{A})) \leq \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} \left( S_{h_{1}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} \beta_{n+1}^2(\mathcal{A}).$$

Inductively, we have

$$\alpha_{n+1}^r(\mathcal{A}) \leq \left( \prod_{k=0}^{r-1} S_{h_{k}^{(n-1)!}} \right) \frac{1}{(1-\rho)^{n-2}} \beta_{n+1}^r(\mathcal{A}), \quad h_k = e^{A(\beta_{n+1}^k(\mathcal{A}))}.$$

Since (3.4), $h_{k}^{(n-1)!} \leq (h_{0}^{n!})^\rho$ for $k \geq 1$, and

$$\prod_{k=0}^{r-1} S_{h_{k}^{(n-1)!}} \leq \prod_{k=0}^{r-1} \left( S_{h_{0}^{n!}} \right) \rho^k \leq \left( S_{h_{0}^{n!}} \right) \frac{1}{(1-\rho)^{n-1}}$$

and hence

$$\alpha_{n+1}^r(\mathcal{A}) \leq \left( S_{h_{0}^{n!}} \right) \frac{1}{(1-\rho)^{n-1}} \beta_{n+1}^r(\mathcal{A}).$$
Taking the limit (for the Thompson metric) of both sides as $r \to \infty$ and projecting into the first coordinate yield
\[
A(t : \mathbb{A}) = \pi_1 \left( \lim_{r \to \infty} \alpha_{n+1}^r (\mathbb{A}) \right) \\
\leq \left( S_{h_0^2} \right)^{\frac{1}{1-\rho}} \pi_1 \left( \lim_{r \to \infty} \beta_{n+1}^r (\mathbb{A}) \right) \\
= \left( S_{h_0^2} \right)^{\frac{1}{1-\rho}} G(t : \mathbb{A}).
\]
Suppose that $A_i$ satisfies $0 < mI \leq A_i \leq MI$ for $i = 1, 2, \ldots, n$. Then by Lemma 2.6, $A(\mathbb{A}) \leq d(mI, MI) = \log(M/m)$ and hence $h \leq M/m$, so $S_{h_0} \leq S_{\frac{M}{m}}$. □

**Remark 3.7.** For $t = 1/2$, Yamazaki [12] proved that
\[
\frac{A_1 + A_2 + \cdots + A_n}{n} \leq \left( \frac{1 + h}{2\sqrt{h}} \right)^{n-1} G(1/2 : A_1, A_2, \ldots, A_n),
\]
where $h = e^{d(\mathbb{A})}$ and the result follows from the inequality $\frac{A_1+A_2}{2} \leq \frac{1+h}{2\sqrt{h}} (A_1 \# A_2)$, $h = e^{d(A_1, A_2)}$.
From $\frac{1+h}{2\sqrt{h}} \leq S_h$, the inequality obtained by Yamazaki is sharper than ours for $t = 1/2$. However for general $t$, our result is new (from the construction) even for positive scalars.

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